Course on Differential Equations for Engineers
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Lecture 52
Vibration of a circular drum

Welcome back, last video we have seen vibration of a string in one dimension. So basically it is a one dimension way equation we solved when the string is infinite length or semi-infinite length or finite length with boundary conditions and (())(0:38) conditions we have solved the vibration of the string problem. So we can also solve second order I mean we can also solve two dimension wave equation that is actually that models the vibration of a membrane or vibration of a drum. So we will just demonstrate this vibration of a drum problem over a circular domain.

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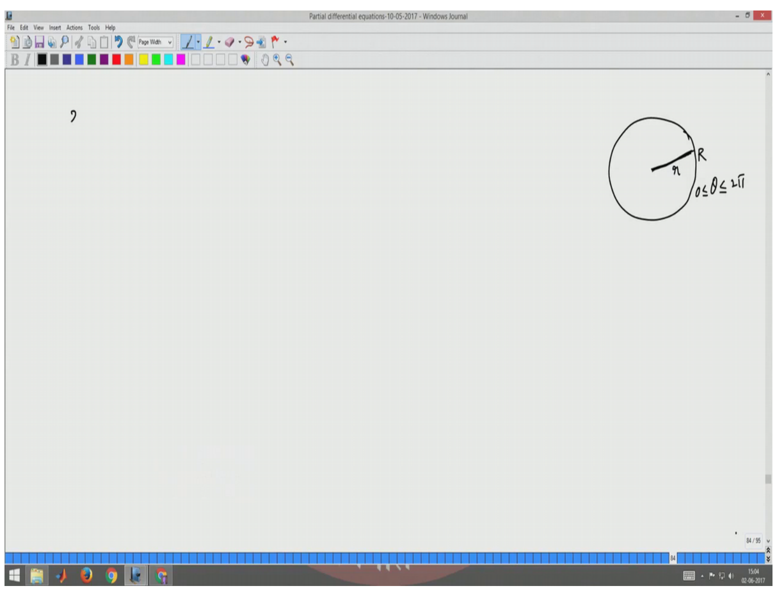

So let us consider a drum which is like this which is circular so that we can use the polar coordinates polar coordinates r and theta and also when you have the vibration so you can assume that is a radially symmetric that means so you have to give r and theta, so this is a plane domain in this disk you have r and theta so r is between 0 to big R and theta is between 0 to 2 pi so theta is between 0 to 2 pi so you can think of solutions that are not depending on theta.

So that means if you know the solution on this line that means you know is very where it is same, so all along theta that means so whatever the curve you have on this it will be repeated everywhere for every theta. So that means something like circular shape of solutions so if you look for when you when the drum is in the vibrating mode that is what actually (())(1:59) assume that you can the vibration so that the displacement of a drum along is radially only it depends only radially.

So that means is a symmetric radially symmetric solutions its reasonable assumption so we will try to look solve second order wave equation over this domain circular domain 0 to circle circle of radius r that is the domain spatial domain and t is a time t is greater than 0 so we will try to see first what is the second order two dimensional wave equation.

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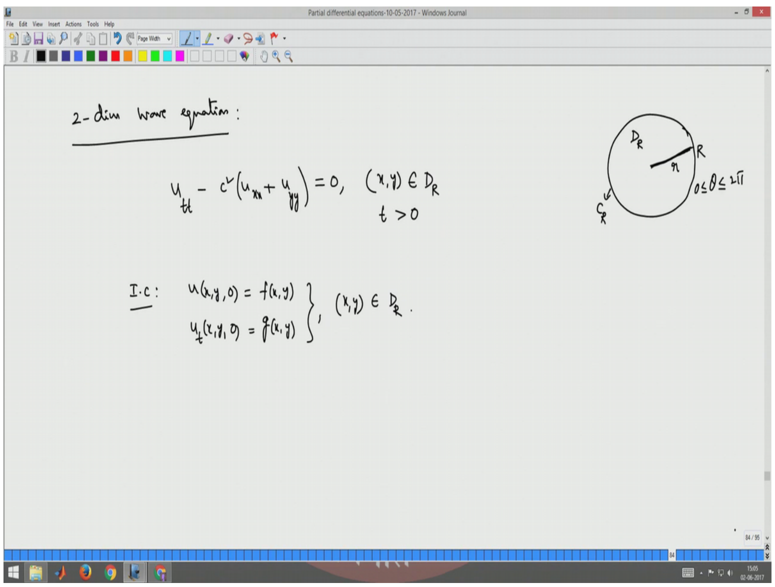

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$u_{tt} - c\,(u_{xx} + u_{yy})$ ... $t > 0$

I.C: $\left.\begin{aligned} u(x,y,0) &= f(x,y) \\ u_t(x,y,0) &= g(x,y) \end{aligned}\right\}, \ (x,y) \in D_R$

B.C: $u(x,y,t) = 0, \ \forall \ (x,y) \in C_R$.

$x = r\cos\theta, \quad y = r\sin\theta \quad , \quad r>0, \ 0 \le \theta \le 2\pi$.

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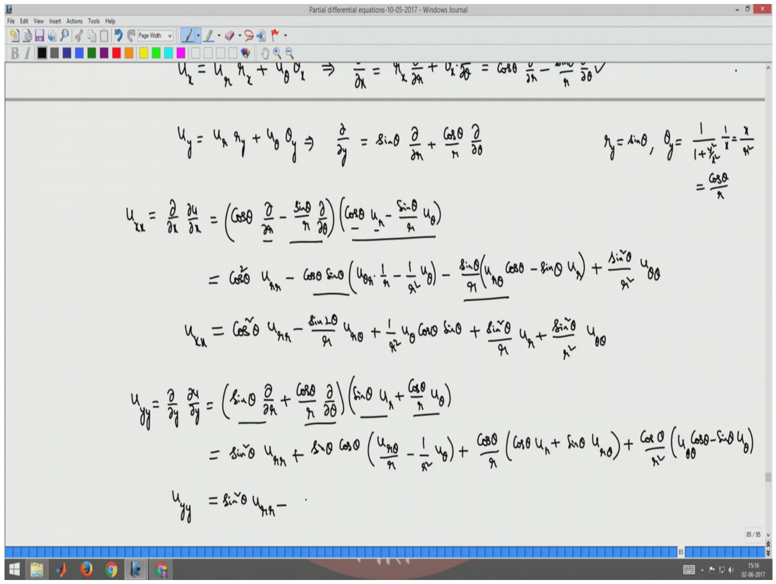
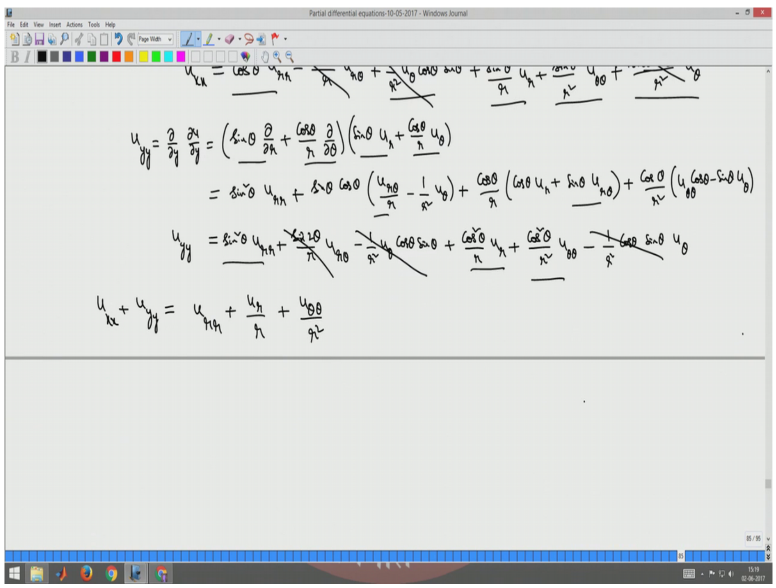
B.C: $u(x,y,t) = 0, \ \forall \ (x,y) \in C_R$.

$x = r\cos\theta, \quad y = r\sin\theta \Rightarrow r>0, \ 0 \le \theta \le 2\pi$. and $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1} y/x$

$r_x = \cos\theta$, $\theta_x = \dfrac{1}{1+y^2/x^2} \cdot \dfrac{-y}{x^2} = -\dfrac{y}{r^2} = -\dfrac{\sin\theta}{r}$

$u_x = u_r r_x + u_\theta \theta_x \Rightarrow \dfrac{\partial}{\partial x} = r_x \dfrac{\partial}{\partial r} + \theta_x \cdot \dfrac{\partial}{\partial \theta} = \cos\theta \dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta}$.





So two dimensional so wave equation so this is as usual is same so utt minus C square uxx so this is for the string now because it is involved two spatial dimensions so what you have is to uyy equal to 0 so xy belongs to this is if I include every region inside this disk that is we call this Dr as a disk and this circle is called Cr I call it Cr so you have this belongs to Dr, okay.

So this is what is this and t belongs to t greater than 0, so this is the domain so this is your two dimensional wave equation. Now if you give the velocity of this drum vibration at t equal to 0 so that is velocity and displacement of the drum that is our initial conditions initial conditions are u at x0 x, y, 0 this is equal to some function of x, y. Similarly the velocity of the drum x, y, 0 is some g of x, y so this is what you have for all x, y belongs to x, y belongs to Dr so this is the initial condition that is what is known, so t equal to 0 you have the initial data and you have the boundary data that is on the boundary you have this drum that is that means normally so u at x equal to so what happens on the Cr, so u at x, y and t t equal to 0 for every x, y belongs to Cr so this is what it means.

So once you fix the all along the boundary so assume that the displacement of the drum or membrane is not is 0 so that means it is not vibrating it is fixed. So this is the problem so this is a boundary value problem you want to solve because you are solving over this circular domain we can reduce this equation into polar coordinates in polar coordinates so when you have the polar coordinates x, r cos theta y equal to r sin theta so we want to reduce the spatial variables into this polar variables r and theta. So you have this you have r is positive and theta is between 0 to 2 pi so that is what we have. So once you have this x and y you have this is the one and what you need is uxx and uyy so to start with ux, so ux is so u is a function of xyt so for the time being you ignore t so all you are changing only xy to r and theta.

So x, y if you replace r cos theta r sin theta u of x, y, t becomes u of r theta and t so if you want ux then you have to differentiate with this respect to r and rx plus u theta theta x that is what is your ux, okay. And also there is another variable that is ut into t of x so t is independent from the spatial variables so that is 0, so that is why this is ux is this. So immediately you have the domain so dough dough x is equal to r x times dough dough r plus dough dough theta x times dough dough theta so dough dough theta so if you apply u dough u by dough x is that is what it is (()) (6:31) you can get this.

So from this we can also get r is this and you can get r equal to square root of x square plus y square under root and theta is tan inverse y by x, okay so simply divide it y by x will give you r tan theta so r r goes so that is simply tan theta so tan inverse of y by x is theta. So we can calculate your rx and t so rx and theta x so this will give you rx as what we get 1 by r to x 1 by 2 r into rx with respect to x you have 2x 2, 2 goes so basically x by rx is r cos theta so is basically r cos theta by r that is cos theta, and you can also get theta x as 1 by 1 plus y square by x square d dx of y by x is minus y by x square.

So this x square x square goes that means you have minus y by r square. So y is r sin theta so you have minus sin theta by r, so you can write this as cos theta dough dough r minus sin theta by r dough dough theta so that is what is your dough dough x. So now you can write your uxx in the same way you can also write uy, uy is ur ry plus u theta theta y so this gives me dough dough y as r y, what is ry? ry is you can also get ry so ry is r is square root of x square plus y square so ry is again you can see that r that will be sin theta.

So 1 by 2 square root of x square plus y square into 2y so that is 2, 2 goes y by r so y is our sin theta by r so that is sin theta. So you have ry is sin theta dough dough r and you can also get theta y theta y is simply 1 by 1 plus pi square by x square d dx of d dy of y by x is simply 1 by x so that will give me x by r square, so we have cos theta by r so you have cos theta by r into dough d theta dough dough theta is my dough dough y.

So now you can get your uxx that is dough dough x of dough u by dough x, so dough dough x now I can get it from here, okay so this is from dough dough x you have cos theta dough dough r minus sin theta by r dough dough theta into dough u by dough x is from here so you can get it from so dough that is also simply so the same thing you operate on u you get cos theta ur minus sin theta by r u theta.

So now cos theta is common so cos theta now dough dough r you operate on to this function, so you have so cos theta cos square theta so dough dough r of cos theta this is our two functions, one function, second functions so this is a function of theta so this is 0 plus so cos theta into urr so cos theta cos theta cos square theta urr, then minus you have a cos theta sin theta cos theta into sin theta what you have is you have to differentiate with respect to r which is 1 by r into u theta so you have u theta is also common so this is a function of theta so u is a function of r also so you have you have to differentiate with respect to r as u theta by r so you have u theta r into 1 by r minus 1 by r square that is a derivative of 1 by r, so minus minus plus so that will be plus anyway so that minus I have already taken so your only differentiating with respect to 1 by r u theta by r so and you do that this what we have so into you theta, okay.

Then now you consider this part so this you add theta on here so you have a minus sin theta by r dough dough theta of so ur so now you have to differentiate with respect to theta for this function of xy so you have a ur theta cos theta plus cos theta is if you differentiate that will be minus sin theta and ur into ur, so that is what is this one and minus minus plus now you are acting on this this one so you have sin square theta by r square u theta theta.

Now that is my uxx so you have uxx as cos square theta urr minus 1 by r u theta r and 1 by r u theta r so this one this one together so you have so that will be sin theta cos theta by r into u theta r, u theta r and ur theta both are same so you have sin 2 theta 2 sin theta cos theta becomes that by r ur theta is what is become these two terms and what you get is this one plus 1 by r square u theta cos theta sin theta, okay cos theta sin theta and that is what is this one 1 by r square minus minus plus now this one you have plus sin square theta by r ur plus sin square theta by r square u theta theta this is my uxx you calculate similarly uyy so that is dough dough y of dough you by dough y so again you do the same thing so if you do this one here dough dough y is sin theta dough dough r plus cos theta by r dough dough theta acting on sin theta you have ur plus cos theta by r u theta.

So if you actually do this sin square theta urr and this will be if you that is what is the if you add this on this term, so and this term this term if you do plus sin theta cos theta that is the common and you are differentiating this u theta theta with theta r. So ur theta by r minus 1 by r square you theta and that is what is the this derivative now you add this one on each of this you have sin theta cos theta so you have cos theta by r and if you differentiate this this will be cos theta ur plus sin theta ur theta.

And now if you do this one here so you have cos square theta by r square u so cos theta by r, r comes out r square dough dough theta of cos theta u theta theta into cos theta minus sin theta into u theta, so this is what you have so you finally get uyy as sin square theta urr minus look at this sin theta cos theta and sin theta cos theta this term this mixed derivatives if you add this will be simply plus sin 2 theta by r ur theta and what you have is minus 1 by r square u theta cos theta sin theta and here cos square theta by r ur and this will be u theta (())(15:40) so you have a cos square theta by r square you theta theta and here you have a minus sin theta u theta by minus sin theta u theta by r square.

So this one and this one so you have a minus 1 by r square u theta cos theta sin theta so I miss something so you have one more term here so you have this one if you dough dough x here when you operate this on to this you have one more term so that is minus minus so what you have here is sin theta by r square dough dough theta so you have a u theta theta sin theta plus cos theta u theta so this is what you have. So you have extra term that is sin theta by cos theta by r square u theta, so you have two terms here so this one and this one.

So similarly here you have our last term is here 1 by r square cos theta sin theta u theta, so you can see now uxx plus uyy is actually now if you combine this one and this one together you have urr now this gets cancelled if you add this and this gets cancelled and this one this together will give me plus ur by r and this will give me simply u theta theta by r square, okay and this gets cancelled, so you have this gets cancelled, okay so these are the things will remain so this is what you have.

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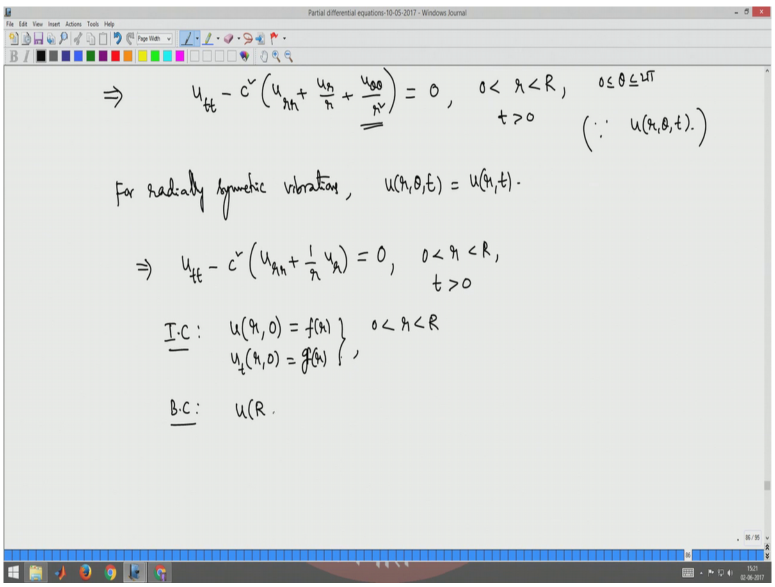

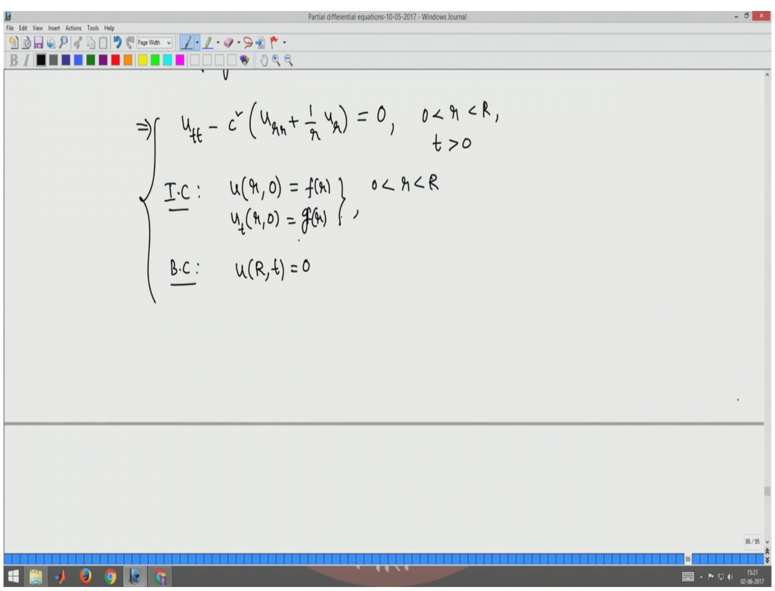

So now the wave equation becomes utt as C square urr plus ur by r plus u theta theta by r square equal to 0 this is now from r is now if you take xy belongs to r so 0 is between r to r theta so theta is theta is between 0 to 2 pi and t is positive so these are the three variable you have u is a function of u is a function of r theta and t since u is this one so, okay so is function of r theta t. So now if you consider drum will radially symmetric for radially symmetric for radially symmetric vibrations u is u of r theta t should be it depend on theta so that means u depends only on r and t.

So if you do that this is 0 so what you have is the equation becomes utt minus C square urr plus 1 by r ur equal to 0. So now you have the domain is between 0 to r and theta is anyway so it is independent so you have theta t is positive. So you have two variables u is only a function of r and t now what happens to the initial conditions now this is a equation you have the initial conditions u at x, y, 0 so at t equal to 0 for all r so ur 0 equal to f of r and r is between 0 to r.

Similarly ut at r0 is g of r, okay so this is what you have for 0 r is between 0 to r. And boundary conditions of the drum so you have you fixed it so this becomes this is a capital R that is the boundary of the drum so that is for all times 0 so it is fixed. So this is the boundary value problem you want to solve by separation of variables. So let so let us let us use a separation of variables like we did earlier so the main purpose of main purpose of solving this two dimensional wave equation in this domain is to extract this Sturm liouville problem for the Bessel equation that you have learnt Sturm liouville theory in the ordinary differential equations in earlier videos, okay.

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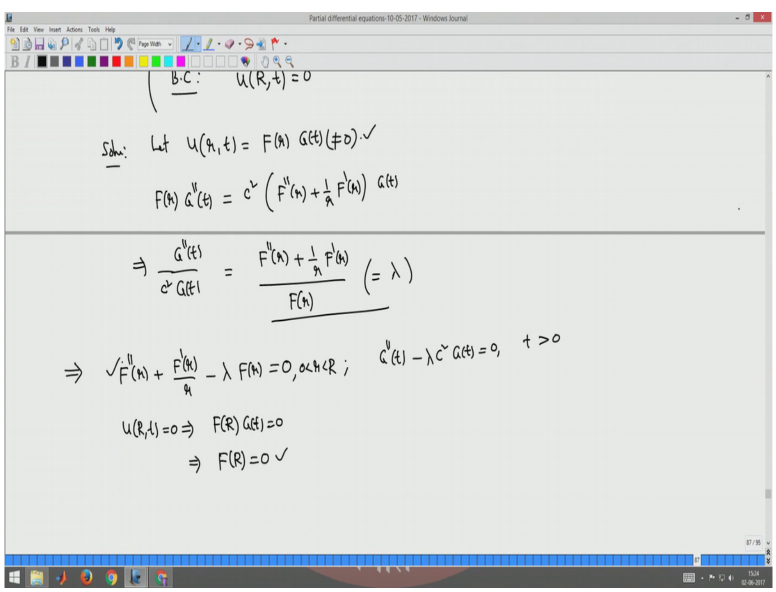

$$\Rightarrow \checkmark F''(r) + \frac{F'(r)}{r} - \lambda F(r) = 0, \; 0<r<R \; ; \qquad G''(t) - \lambda c^2 G(t) = 0, \quad t>0$$

$$u(R,t) = 0 \Rightarrow F(R)\,G(t) = 0$$

$$\Rightarrow F(R) = 0 \; \checkmark$$

$$r F''(r) + F'(r) - \lambda r F(r) = 0, \quad 0 < r < R$$

$$\left(r F'(r)\right)' = \lambda r F(r)$$

$$p(r) = r, \quad q(r) = 0, \quad \underline{w(r) = r}, \qquad \langle \phi, \psi \rangle := \int_0^R r\, \phi(r)$$

$$u(R,t) = 0 \Rightarrow F(R)\,G(t) = 0$$

$$\Rightarrow F(R) = 0 \; \checkmark \quad \underline{\text{B.C 1}}$$

$F(0), F'(0)$ are finite $\underline{\text{B.C 2}}$:

$$\checkmark\; r F''(r) + F'(r) - \lambda r F(r) = 0, \quad 0 < r < R$$

$$\underline{\left(r F'(r)\right)' = \lambda r F(r)}$$

$$p(r) = r, \quad q(r) = 0, \quad \underline{w(r) = r}, \qquad \langle \phi, \psi \rangle := \int_0^R r\, \phi(r)\, \overline{\psi(r)}\, dr.$$

$\Rightarrow$ λ is real. $\lambda = 0$ or $\lambda = -\mu^2$ or $\lambda = \mu^2$ with $\mu > 0$.

$\underline{\lambda = 0}$: $\quad r^2 F''(r) + r F'(r) = 0 \; \checkmark$

Let $F(r) = r^m$, $\quad m(m-1) + m = 0 \Rightarrow m^2 = 0$

$m = 0, 0$

$\underline{\lambda = 0}$: $\quad r^2 F''(r) + r F'(r) = 0$ ✓

Let $F(r) = r^m$, $\quad m(m-1) + m = 0 \Rightarrow m^2 = 0$

$m = 0, 0$

$F(r) = C_1 + C_2 \log r.$

$F(R) = 0 \Rightarrow C_1 + C_2 \log R = 0 \Rightarrow C_1 = -C_2 \log R$ ✓

$u(r,t) = F(r)\, G(t)$

$u(0,t) = \underline{F(0)\, G(t) < \infty}$ ✓

Since $F(0) < \infty$; $C_2 = 0 \Rightarrow C_1 = 0$

$\Rightarrow F(r) = 0, \quad \forall\, 0 < r < R.$

$\Rightarrow \lambda = 0$ is not an eigenvalue.

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$\underline{\lambda = \mu^2}$: $\quad r^2 F''(r) + r F'(r) - r^2 \mu^2 F(r) = 0.$

$\underline{r = \frac{k}{\mu}}$ ✓

Let $k = r\mu.$

$\frac{dF}{dr} = \frac{dF}{dk} \cdot \frac{dk}{dr} = \mu \cdot \frac{dF}{dk}. \quad$ lly, $\frac{d^2F}{dr^2} = \mu^2 \frac{d^2F}{dk^2}$

$\underline{k^2 \frac{d^2F}{dk^2} + k \frac{dF}{dk} - k^2 F = 0}, \qquad \underline{F(r) = F\left(\frac{k}{\mu}\right)}$ ✓

$x^2 y'' + x y' + x^2 y = 0$

Bessel eqn with $\alpha = 0$

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$$\checkmark\quad k^2 \frac{d^2F}{dk^2} + k\frac{dF}{dk} - k^2 F = 0, \qquad F(r) = F\left(\frac{k}{\mu}\right) \checkmark$$

$$\checkmark\quad x^2 y'' + x y' + x^2 y = 0$$

Bessel eqn with $\alpha = 0$

Let $z = ix$, $\frac{dy}{dx} = \frac{dy}{dz} i$

$x = -iz$ $\qquad \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} i^2$

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} - z^2 y = 0 \checkmark$$

Modified Besell eqn with $\alpha = 0$

$$y(x) = c_1 J_0(x) + \cancel{c_2 Y_0(x)} \checkmark$$

$$y(z) = c_1 J_0(-iz) \checkmark$$

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general soln is

$$F(k) = c_1 J_0(-ik) + \cancel{c_2 Y_0(-ik)}$$

Since $F(0) < \infty$, $\Rightarrow c_2 = 0$

$$\Rightarrow F(k) = c_1 J_0(-ik)$$

B.C 1 $F(R) = 0 \Rightarrow F(R) = 0 \Rightarrow c_1 J_0(-iR) = 0$

$$\Rightarrow c_1 \sum_{m=0}^{\infty} \frac{R^{2m}}{2^{2m} m!^2} = 0$$

Let $z = ix$, $\frac{dy}{dx} = \frac{dy}{dz} i$

$x = -iz$ $\qquad \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} i^2$

$$\checkmark\quad z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} - z^2 y = 0 \checkmark$$

Modified Besell eqn with $\alpha = 0$

$$y(x) = c_1 J_0(x) + \cancel{c_2 Y_0(x)} \checkmark$$

$$y(z) = c_1 J_0(-iz) \checkmark$$

$$= c_1 \sum_{m=0}^{\infty} \frac{(-1)^m (-iz)^{2m}}{2^{2m} m!^2} \checkmark$$

$$= c_1 \sum_{m=0}^{\infty} \frac{z^{2m}}{2^{2m} m!^2} \checkmark$$

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$F(k)=0 \Rightarrow F(r)=0, \ \forall\, 0<r<R$

$\Rightarrow \lambda=\mu^2$ is not an eigenvalue.

$\lambda=-\mu^2$: $\quad r^2F''(r)+rF'(r)+\mu^2 r^2 F(r)=0$

Let $k=\mu r$. Then

✓ $k^2F''(k)+kF'(k)+k^2F(k)=0, \quad 0<k<\mu R$

general soln: $F(k)=C_1J_0(k)+C_2Y_0(k)$

$F(0)<\infty \Rightarrow C_2$

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$\lambda=-\mu^2$: $\quad r^2F''(r)+rF'(r)+\mu^2 r^2 F(r)=0$

$F(r)=F\left(\frac{k}{\mu}\right)$

$F(R)=0 \Rightarrow F\left(\frac{\mu R}{\mu}\right)=F(R)=0$

Let $k=\mu r$. Then

✓ $k^2F''(k)+kF'(k)+k^2F(k)=0, \quad 0<k<\mu R$

general soln: $F(k)=C_1J_0(k)+C_2Y_0(k)$

$F(0)<\infty \Rightarrow C_2=0$

$F(k)=C_1J_0(k)$

$F(\mu R)=0 \Rightarrow C_1J_0($

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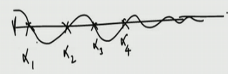
general soln: $F(\kappa) = c_1 J_0(\kappa) + c_2 Y_0(\kappa)$

$F(0) < \infty \Rightarrow c_2 = 0$

$F(\kappa) = c_1 J_0(\kappa)$



$F(\mu R) = 0 \Rightarrow c_1 J_0(\mu R) = 0$

Let $\alpha_1, \alpha_2, \dots$ are roots of $J_0(x) = 0$, $x > 0$,

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$\Rightarrow \mu_i R = \alpha_i \Rightarrow \mu_i = \frac{\alpha_i}{R}$, $i = 1, 2, 3, \dots$

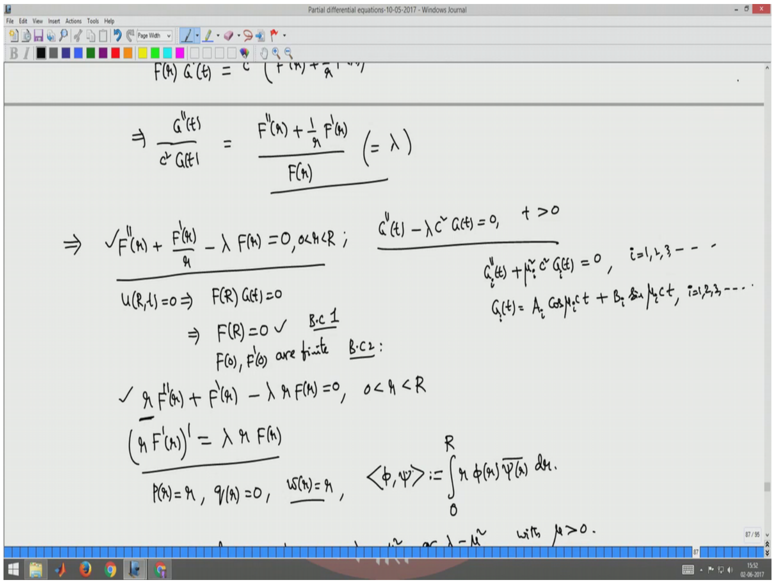
$\Rightarrow J(\alpha_i) = J_0(\mu_i R) = 0 \Rightarrow c_1$ is arbitrary constant.

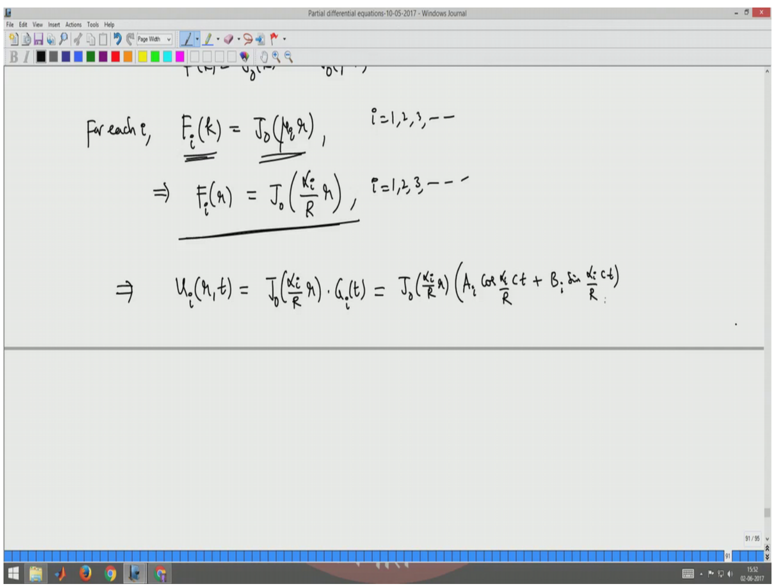
$\Rightarrow$ Nonzero solutions are eigenfunctions and they are

$F(\kappa) = J_0(\kappa) = J_0(\mu r)$

For each i, $F_i(\kappa) = J_0(\mu_i r)$, $i = 1, 2, 3, \dots$

$\Rightarrow F_i(r) = J_0\left(\frac{\alpha_i}{R} r\right)$, $i = 1, 2, 3, \dots$





So let us solve so solution is for this problem you consider u of let the solution be a separation of variables so you have r, t you consider like F of r capital F and G of t which is non-zero. So if you look for the solution as a non-zero solution you substitute the equations what we have is F of r G double dash of t equal to actually C square and urr that is F double dash of r plus 1 by r F dash of r into G of t so that is a common term both the cases because only r derivatives.

So this because this is non-zero you can divide it so that you can write G dash of t by G of t if you divide with f or G t and C square also you bring it here so you put it here which is equal to F dash of F double dash of r 1 by r F dash of r by Fr Gr so this is divide by F of r G of t G of t goes so this is a function of t left hand side and the right hand side is a function of r so this should be constant in the separation of variables method.

So you have this lambda is a parameter, okay so now we have so you if you consider this this is one ordinary differential equations so what you get is F double dash of r plus F dash of r by r minus lambda F of r equal to 0 this is one ODE another ODE is G double dash of t minus lambda c square G of t equal to 0 so this is for t positive and this is actually for r is between 0 to capital R, so there are the two things.

Now you apply the boundary conditions so if you apply the boundary condition what you get is u of R, t equal to 0 will give me basically so FR into F capital R G of t has to be 0 so G of t cannot be 0 so implies because t is varying so implies F or R has to be 0 so this is your boundary condition this is your ordinary differential equation which is n which is already in this in the self-adjoin form so how do we see that it is self-adjoin form you have r into F dash of r you can rewrite so r into F dash of r plus F dash of r minus lambda r F of r equal to 0, so this is the equation so this you can write it like F dash of r times r the old dash so p is r P of r equal to so implies p of r equal to r q of r what is that is so we have a q of r is 0 plus q of r into F of r that is 0 and you have lambda times w is w of r is r into F of r so this is what you have.

So immediately once you have the w in Sturm liouville you have we can define your phi and xi as a inner product as sorry dot product dot product you can write like this which is what is your domain, domain is between 0 to R capital R so you have 0 to capital R and you have the w is this that is the weight function r phi r xi r bar dr, so this is what is your dot product from the Sturm liouville (())(24:07). So because in already in the self-adjoin form so all lambda should be real so this implies a lambda is real, so lambda is real that means lambda equal to 0 lambda is minus mu square or lambda is mu square with mu is positive that is the meaning of real so you can choose any of this cases so start with the lambda equal to 0 case.

So if you choose the equation becomes this equation becomes r F double dash of r it is F dash of r equal to 0. So this is actually your if you actually multiply you can multiply both sides r so that you can write r into this one so you can write you can rewrite this just multiply r so that lambda equal to 0 this is your equation. So this is like x square y double dash plus xy dash equal to 0 this is a kind of Euler (())(25:07) type equation.

So you can look for solutions F of r as r power m, so if you look for solutions like this so if you what you get is m into m minus 1 plus m equal to 0 so that will give me m square equal to 0. So roots are m equal to 0 and 0 so you get general solution of that equation is F of r as c1 times r power 0 that is 1 plus c2 times because they are repeated roots r power 0 into log r so you have a r power 0 as 1 you have log r. So this is the solution if you apply the boundary condition that is F at capital R equal to 0 will give me c1 plus c2 log r equal to 0, okay.

So this will give me c1 equal to minus c2 log r and also what happens at F at r equal to 0 though it is point of the drum but it is not in the domain of the differential equations so that r as r goes to 0 that is Fr 0 has to be so you cannot have so u of since u of rt that is the displacement of this thing that is you look for Fr into G of t for all times you cannot have u at 0 t that is f of 0 that is the displacement, okay F of 0 into G of t it has to be finite.

So if this has to be finite so F of 0 has to be finite, okay. So F of 0 is finite so that is the actually another boundary condition so we have given only one boundary condition here, okay. So one boundary condition is this you can also say and because r is the so r is a similar point you can give other boundary conditions, so boundary condition this is one boundary condition 1 and the other boundary condition is F at 0 and F dash of 0 because r equal to 0 is singular point are finite this is my second boundary condition for the Sturm Liouville problem.

So you have since F of 0 is finite as a second boundary condition you can see that log r that is infinity so c2 has to be 0, once c2 is 0 from this you can see that c1 is also 0. So implies you have Fr is completely 0 for every r between 0 to r implies lambda equal to 0 is not a eigen value, okay. So we have to see what happens if lambda equal to mu square other things, so lambda equal to mu square other things, so lambda equal to mu square let us consider so the equation becomes what happens if lambda equal to mu square if you put r double r r F double dash the equation is this plus F dash of r and what you have is minus lambda equal to mu square minus r mu square, so minus r mu square F of r equal to 0, so this is what is the equation if you multiply r both sides you have r times this you have r mu square, okay.

So you bring in a new variable so this is actually something familiar equation but if you just use the if you change the independent variable r is the independent variable here so if you change this as a k so new independent variable k let k be r and mu, so you try to change this so if you do this F dash of r so that is dF by dr you can write it like dF by dk into dk by dr that is 1 by mu, okay. So dk by dr so this is equal to 1 by dk by dr is mu times d F by dk.

So similarly you can see that d square F by dfr square equal to mu square d square F by dk square, so that is how you see so you can substitute r square, so r square is k square by mu square that mu square mu square goes so what you get is k square d square F by dk square plus similarly so you have k times dF by dk minus k square F of now F is a function of k now, okay. So just write F, F is a function of k h is equal to 0 where F is function of k, okay.

So F of is a function of r after so once you replace r is actually so F is depending on r, r is actually k by mu so mu is fixed k by mu since r is k by mu so is F of r is actually F of k by mu so is basically some function of k mu is fixed, okay. So you can write if you want, okay so F is depending on k so you can say that F is as it is so F is depending on k. So this is the equation we are familiar with what is this equation this is the Bessel equation, okay. This is not Bessel equation this is something similar to Bessel equation Bessel equation is x square y double dash plus x y dash plus x square minus alpha square into y equal to 0.

So if alpha equal to 0 what you have is something similar but you have a minus x square, okay. So let us choose alpha equal to 0 in the Bessel equation Bessel equation with alpha equal to 0 Bessel equation with alpha equal to 0, okay if I want if I replace again do the same thing, so I tried to change let z be the new variable new independent variable instead of x here if you convert this as a z a new independent variable z is as ix then you see that d square y by dx square which is in this equation becomes d square y by dz square into so simply start with like earlier dy by dx, dy by dx is dy by dz into dz by dx that is i.

So you see that immediately d square by dx square as d square y by dz square into i square so that is minus i square, so that is what you get. So you again ix square is z square by i square, i square i square goes the equation becomes now simply z square y d square y by dz square plus z into dy by dz so these two terms will not change like here with this transformation plus x square now I can write, so where is the equation? Yes, you can now write the equation plus x square, x square becomes minus z square, right? x square is because x equal to minus iz so minus i square, right? So what happens to minus ix x equal to minus iz so x square is simply minus z square y of z now is a function of z equal to 0.

So this is the Bessel this is called modified Bessel equation with alpha equal to 0. So with this transformation you see that the equation is modified so this equation is modified to something similar here this is exactly what you have here you have there. So we know that Bessel solutions are y at so Bessel solutions general solution of Bessel equation is the c1 J0 of x plus c2 this is y0 of x when you have n equal to 0 when alpha is equal to 0 these are the two solutions y0 you will get a repeated roots if you use the (())(34:06) method you see that you get a repeated root if you look back into the videos of Bessel solutions when alpha is equal to 0 what you get is this is the general solution, okay.

And at x equal to 0 r equal to 0 you want to have your solution to be bounded because of this boundary condition this boundary condition you can see that this cannot be 0. So the solution of this general solution of this equation the actual Bessel equation is if you say that if you want your at r if you want mounded this is what is the solution this is what is the function of x if you want function of z what is this is constant multiply of J0 of instead of x that is minus iz this is what you have.

So yz is actually J0 of iz, so if you write this J0, J0 of iz if you write J0 is c1 times J0 is m is from 0 to infinity so instead of what you have is (())(35:12). So you know the formula J not so that is now you have iz power two m divided by 2 power 2 m into m factorial into m factorial actually square so that is what you get gamma of m plus 1 that is m factorial gamma of m plus 1 m square. So this is what you get so you can see that minus 1 power m and minus i power 2 m, okay that becomes 1.

So what you see is this is actually c1 times m is from 0 to infinity minus 1 power m into minus 1 power 2 m so that is minus i square is minus 1 so power m so the whole term this minus iz minus i power 2 m is actually equal to minus 1 power m this together with this becomes plus 1. So you have z power 2 m divided by 2 power 2 n by m factorial square so this is what is the equation. So the general solution of the Bessel equation and because with this transformation this is what you get. So that is this is exactly that modified Bessel equation is what we have here.

So the general solution is F of r instead of z you write F of instead of r you are writing a k, so you have a k so instead of z you are writing k so k is equal to c1 times this J0 so I will just simply write J0 of minus ik but J0 of minus k is this square and plus c2 of y0 of minus i of k, okay but at k equal to 0 this is not bounded, okay that we already know so has to be 0 so since y at F at 0 is bounded implies c2 is 0 so for that reason I cancel this.

So implies F of k is c1 times J0 of minus ik. Now we have another condition F of r equal to 0 so boundary condition 1 gives this implies F of r has to be 0. So that means c1 times J0 of minus i capital R has to be 0. So this implies c1 times what is J0 of minus iR you simply replace in this sum, so you have a summation m is from 0 to infinity z power that is R power 2 m divided by 2 power 2 m into m factorial square this has to be 0, but you see that each of this in this sum terms in this sum are positive R since capital R is positive so this implies c1 has to be 0 because this quantity is non-zero this is clearly term is positive this cannot be 0 so implies c1 is 0.

So you found c2 is 0 and c1 is 0 so finally you get F of k which is equal to completely 0, what is f of k? F of k is the solution if F of the k is solution F of r is also so you can go back to your new terms so F of you can write your F of r F of k is 0 implies F of r is also 0 for every r between 0 to R. So what is your domain F of k so once you have this because r is between 0 to r k is between 0 to mu R so that is what is the domain of this, okay.

So what you have to write is actually boundary condition becomes F of mu R F of mu R equal to 0 so you have F of mu R is 0, so F of mu R is instead of you simply replace with mu power 2 m, okay. So R mu R equal to 0 so mu R so this cannot be 0 because mu is positive R is positive so it implies this so you have Fk Fk is 0 and that implies F of r is also 0 in this domain F of k is 0 in the domain between 0 to mu R F of R is 0 that means F of r is 0 in this domain.

So that implies mu square lambda equal to mu square is not an eigen value, okay. So now we have we are left with only lambda equal to minus mu square minus mu square if you see the equation becomes F double dash of r into r plus F dash of r minus minus plus mu square into r F of r equal to 0, okay. So if you multiply r both sides you get this and you have this. Now you change the variable so again use the same argument k equal to mu r then change the independent where r to k so what you get is a k square F double dash of k plus k times F dash of k plus k square F of k equal to 0.

So this is what is the equation now k is between 0 to mu R, so this is what is the Bessel equation with Bessel equation from 0 to mu R that is the domain this is your domain and this is a Bessel equation know the solution F of k which is equal to c1 J0 of this is the Bessel equation with alpha equal to 0, so you have J0 of k plus c2 general solution is this. So the general solution of general solution because this is the Bessel equation y0 of k. So again you apply the boundary condition F at 0 has to be bounded implies c2 because y0 is unbounded at k equal to 0 so you have c2 has to be 0 so and now you apply now what happens your solution becomes c1 J0 of k.

Now you apply first other boundary condition that is F at R has to be 0, so this means c1 J0 of F at what is your boundary condition F at mu R equal to 0, right? F at mu R equal to 0 F at R in this domain and because F at R is k by mu, right? F at so mu R so F at R is actually equal to F at R is k by mu so F at capital R means equal to 0 implies F at k is actually k you have to replace with mu R by mu equal to FR which is 0.

So k the boundary is boundary condition becomes this one in this domain after the transformation. So c1 into J0 of mu R equal to 0 and we know that J0 function has infinitely many 0's on the positive x axis so you call them alpha 1 or because we use yeah we can call this alpha 1, alpha 2, alpha 3 and so on. So you denote them as alpha 3 alpha and so on. So the roots of let alpha 1, alpha 2, alpha 3 and so on are the roots of infinite roots of J0 of x equal to 0 for x positive, okay.

So this is what you alphas are all in the positive side of the positive roots because this is defined, okay. We defined between 0 to this thing, so implies so what is your mu? Mu is now varying mu is positive so mu into R equal to alpha 1 if you consider you call them mu 1. So you have so like that each mu i you denote that is corresponds to alpha i. So implies mu i equal to alpha i by R so i is running from 1, 2, 3 onwards whatever denoted alpha 1, alpha 2, alpha n and so on.

So this implies so for which so for these values of mu you have J0 of mu i into R equal to 0 because this is nothing but J of alpha a these are the roots, okay. So that means c1 is that implies c1 is arbitrary you can take as 1. So you have a non-zero solution so this implies a solution is F of r equal to so F of k so solution the solution is solutions are eigen functions and they are and the non-zero solutions are non-zero you have to say specifically non-zero solutions that is when c1 is you take it as 1 and you take c1 equal to you have J0 of k, so J0 of k k is what k is equal to mu R so mu i into R so mu i so corresponds to this same mu i into R are eigen functions i is running from 1, 2, 3 onwards.

So this is how you find your F of R, okay. So F of R is F of k by mu, okay F of k by mu so F of k is so let us write what is this what are they are what are the solutions you have F of k F of k equal to J not of k, F of k as J not of k, okay. What is k? k is mu R, okay so what is F of R? F of r you want F of r, right? So you want F of r, finally not F of k. So F of r is basically F of r equal to k by mu so this is J0 of k is actually mu into R so mu into r so these are there solutions.

So you have you call them Fi, Fi of r is okay so you write like this so F of k so these are your solutions F of k you call this Fi Fi J0 of i. So you have if you so far each alpha a for each mu i your Fi if you denote for each i you can define as Fi of k as J0 of mu i into r, okay. So J0 into r so this implies F so this is running i is running from 1, 2, 3 onwards. So and then you can write Fi of r as J0 of, what is mu i? Mu i is your alpha i by r into r so this is what you get 1, 2, 3 onwards. As a function of k they are J not of k that is k is mu r, so for each i so each i the solution is for each mu i you have J0 of mu r is the solution Fi of k.

So if you want in terms of r you have to write mu i so r is mu i is you can replace you can reduce k into r, okay is already in r so you have to replace mu i as alpha a bar so these are the solutions these are your non-zero solutions that implies Fi of r so they implies you have ui of r, t the solutions for i you have the solutions Fi r into that is J0 of alpha a by r into r this is one solution into, now what is other solution that is G into G you have G of t G call this Gi of t corresponds to for each i you call it G of t Gi is satisfying the second differential equation here.

So the second differential equation is this so if you consider put comeback and put G double dash of t so lambda is minus mu square so you have a plus mu square into so for each i so you have mu i square into c square G of t equal to 0. So you have G of t is c as you call this for each i you all this ai cos mu i ct plus bi sin mu i ct, okay i is running from 1, 2, 3 onwards. So because i for each is depending on see G of t I am denoting also G is depending on i so you have each i.

So Gi is actually this for each i is from 1, 2, 3 onwards. So you know what is your Gi of t and Fi of t, so go back and put it here so Gi of t is J0 of alpha I by R into r into Ai cos mu i ct plus bi sin mu i ct. Actually mu i is alpha a by r so you can replace this if you want so alpha I by r you can replace so alpha i by R alpha i by R. So implies so this is to for every i equal to 1, 2, 3 onwards so for each i u of ui of r, t that is the product of this solution and this solution is satisfying this two dimensional wave equation and the boundary conditions so far we have not used the initial conditions.

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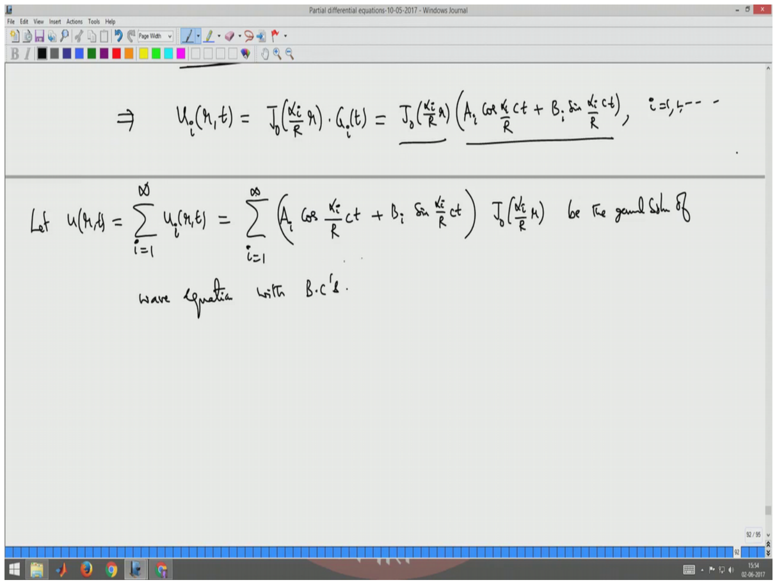

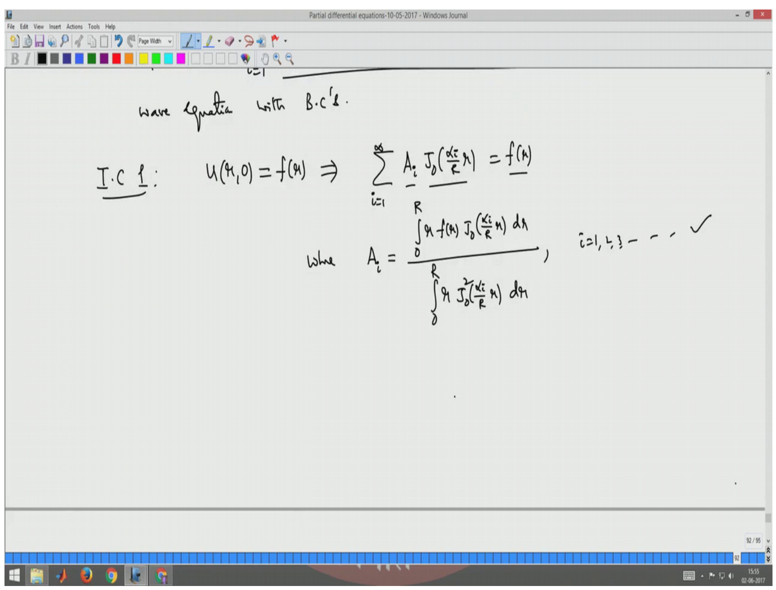

Now like earlier you make a superposition of this let u be the solution so u of r, t as a superposition of all this i is from 1 to infinity ui of r, t that is sigma i is from 1 to infinity so you have a Ai cos alpha i by R ct plus Bi sin alpha i by R ct into this J0 of alpha i by R capital R into small R so this is what you have be the solution the general solution of solution of wave equation wave equation with boundary condition, okay.

So if you assume this anyone can show that is actually uniform (())(51:59) like earlier that I am not proving so if you apply now apply the initial condition initial condition 1 will give you u at r, 0 equal to f of r implies this will give me u at t equal to 0 cos t that is one sin t will be 0 so you have i is from 1 to infinity Ai into J0 of alpha i by R into r equal to fr. Now I can get my Ai by simply integrate with use the dot product that we defined earlier so you get Ai you simply integrate both sides like this so you integrate from 0 to capital R and you have a weight function here r f of r times J0 all these functions are real valid functions so bar does not matter J0 of alpha i by capital R into small r into dr divided by now you are simply integrating this side from multiplying r J0 of alpha i by R into r this you are multiplying itself that is what only survives in this infinite sum so you have a square dr.

So this is what you get where i is this for each i is from 1, 2, 3 onwards. So if you apply initial condition 1 I can get my Ai's like this in the general solution here so Ai is known which is this.

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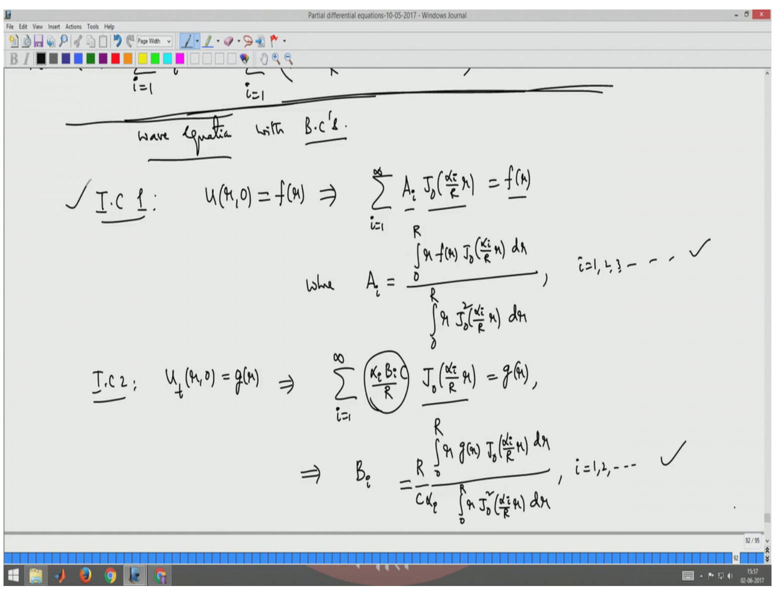

Now apply the initial condition 2 so you have that is ut at r, 0 is 0 is g of r that is ut ut at r, 0 if you differentiate this this is cos becomes and this becomes sin. So this will go of by an input t equal to 0 after differentiation with respect to t.

So what survives is di alpha i cr so you can see what you get is i is from 1 to infinity alpha i Bi by R c, okay so this is what you get into cos, cos alpha a by R ct t is equal to 0 that is 1 into J0 of alpha i by R into r equal to gr. So again now from this if you apply one can see that Bi so Bi you can get it as not Bi alpha i Bi c by R this whole constant I can get it as using the dot product that is 0 to capital R and you have a weight function r g of r into this eigen function that is J0 of alpha i by R into r dr and left hand side is survives only integral r to R r into J0 square of alpha i by capital R into small r dr, okay.

So this will determine you are what is you are Bi so this implies this is R by C into alpha i if you bring this R C alpha at other side this is what you get as Bi's so I know what is my Bi Ai in this solution. So that's solves now this is the solution as a superposition of all the solutions in this approval form but satisfies wave equation now boundary condition and the initial conditions so this is the solution that we are looking for, okay.

So this is the solution that is required here. So this is how you solve a second order this is how you solve two dimensional wave equation on a circular domain in the form of symmetrical radially symmetric solutions that satisfies the boundary conditions so that means the vibrations no vibrations at the boundary and you have the initial data that is you providing as a vibration at time t equal to 0 you have the displacement of the disk displacement of the drum or the membrane is 0 is given (())(56:12) and its velocity is also given.

That means vibrations so displacement and velocity of the vibrations of the drum are given, okay. Displacement and velocity of the drum at initially at time t equal to 0 is given. So this is how you solve so the general solution (())(56:34) in a separable form that you can see as a linear superposition of functions of r and functions of t, okay. So these are radially symmetric separable form of solutions for the vibrations of a drum, okay so which is a which so that actually is the solution of two dimensional wave equation, okay.

So we will see this is how we solve the wave equations and in one dimension and two dimension wave equation so far so we can pick up any other domain and make you so our intention is not just to solve two dimension wave equation in other domains we will main purpose is to make use of the Bessel equation that we have derived in the ordinary differential equations earlier. So This is how we can solve this is just to demonstrate the solutions of two dimensional wave equation in similar domain like circular domain, okay as an application.

So in the next video we will see how to solve we will move on to heat equation, so again like for wave equation like in the wave equation we try to solve the heat equation in the infinite case infinite spatial domain that is you consider an infinite rod infinite rod you give an initial condition that you because in the heat equation you have only time derivative is 1 you have to provide only one initial condition and you provide the initial data so with this initial data heat equation you trying to solve in the full domain and the you can also do the semi-infinite domain and then (())(58:08) finite domain that is you can think of finite rods. So we will just see that how the solutions of heat equation will be in the next few videos, thank you very much.