Differential Equations for Engineers
Professor Dr. Srinivasa Rao Manam
Department of Mathematics
Indian Institute of Technology Madras
Lecture 50
Finite length string vibrations (continued)

Welcome back in the last video we have seen vibration of a finite string when when both the ends of the strings are fixed that means the displacement is 0 at both the ends, we have seen how find the solution by separation of variables method. So in this video we will just see we will try to do one more problem something similar problem where the finite string is one end is attached and one end is fixed one end is kind of free and other end is fixed. So we will see this how what is the vibration of the string for all the times we will just find solutions by the separation of variables taking.

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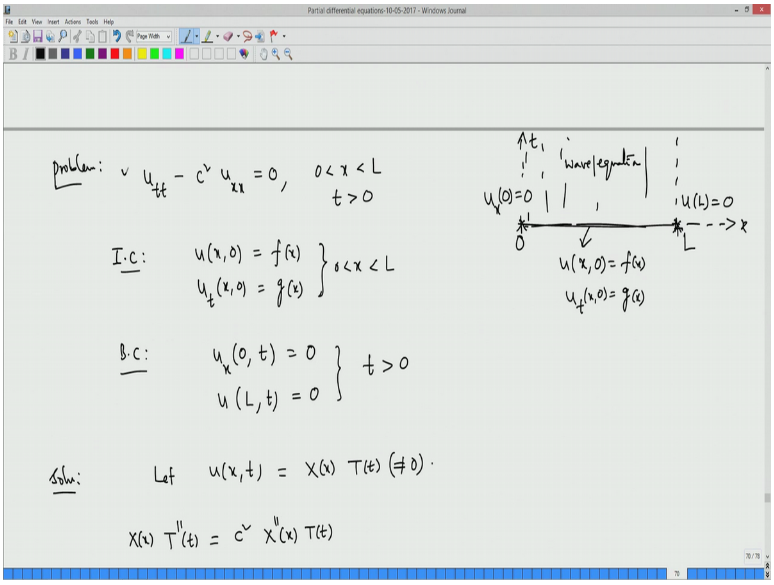

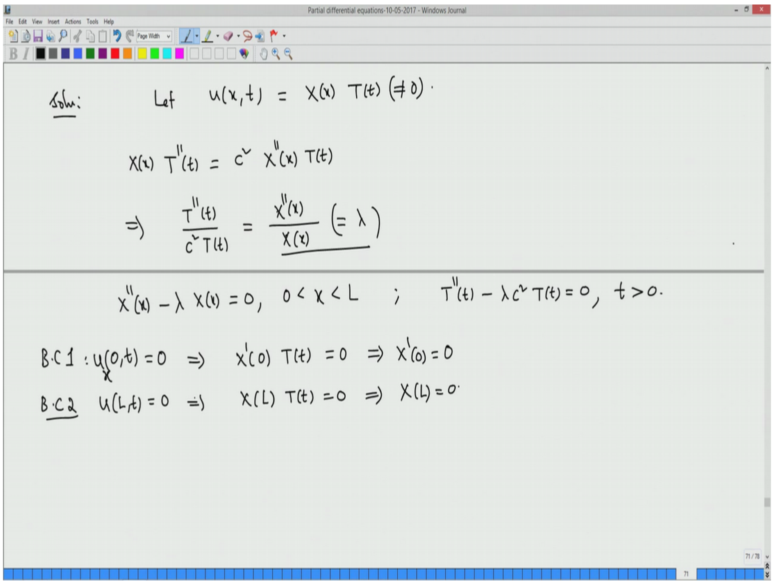

So this video we will start with this initial boundary value problem so that is consider problem we have this consider this finite string so let us today let us chose from 0 to L, so a string of length L which is one end is at 0, other end is at L you consider this one end you fix it as u at L equal to 0 so you fix it this end you make it free so that is u x at 0 equal to 0. So at this this is the boundary you have solution and you can see that this is your time t and this is your x axis so you have wave equations satisfied here wave equation is satisfied here so within this strip so within this strip infinite strip you have wave equations satisfied initial conditions are u at x at t equal to 0 is given f x and u t at velocity of this string at t equal to 0 that is also given so these are the this is the complete well defined problem.

So we will just write it u tt minus c square u xx equal to 0 for x is between 0 to L t is positive, so this is your differential equation and your initial data initial data is initial conditions are u at x equal to 0 x t equal to 0 is f x and u t at x, 0 is g x for x is between 0 to L so this is your data for 0 to L that means you have a at initial time you have displacement and velocity of the string is known.

Boundary conditions one end is fixed, other end is free so you have u x partial derivative with respect to x at x equal to 0 for all times you make it 0 and u is fixed at displacement is fixed displacement is 0 at the other end that is at x equal to L for all times equal to 0. So this is true for every t positive, okay for all times this is true. So this is the problem so we need to find a solution so u of x, t for every x and t.

So solution as usual just look for solution because it is a finite domain so you see that if you look at the domain of x t domain so x domain is bounded 0 to L, t domain is actually positive so you have a this actually t is only positive so this is your domain this is your domain is not this (())(4:02) than 0, okay so this is within this this kind of this is the kind of domain so you have this is your domain within this wave equation is true and we have the initial data is here and we have boundary data.

So look for solution let u of x, t be solution in a separable form, so that is X of x, T of t which is nonzero we look for solution in this form then you substitute into the equation so wave equation so you have T double dash of t into X of x equal to c square X double dash of x into T of t that is what if you substitute to this equation. So divide because this is nonzero you divide with x u of x, t that is X of x into T of t if we divide both sides you see that T double dash of t divided by bring this c square here you have T of t so X of x cancels I brought this c square here so you have X double dash of x divided by X of x divided X of x, T of t, T of t T of t goes so this is what you have.

So variables are separated so should be that means and both are same functions of x left hand side functions of t. So that means it has to be constant so call this the parameter lambda. And now from this x x is ordinary differential equation so if you consider this you have X double dash of x minus lambda X of x equal to 0, x is the domain 0 to L, okay and what happens to the other one.

So for t you have other ODE that is T double dash of t minus lambda c square T of t equal to 0 this is for t positive. So these are the two ODE's you have, clearly you see that this is in the self adjoint form so we can easily see that so this is equation in the self adjoint form so we will see so you apply u x, t so boundary conditions u x at 0, t equal to 0 boundary condition 1 implies this so this implies boundary condition 1 is this.

So this implies if you apply what you get is X dash of x so that is what is your X dash of 0 into T of t that is what is u x at 0, t if you differentiate this is what you get. So this is equal to 0, T of t cannot be 0 as a function of t implies X dash of 0 is 0. Now you apply the boundary condition 2 will give me u at L, t equal to 0 this will give me X at L T of t has to be 0 that means T of t cannot be 0 so you have X at L equal to 0.

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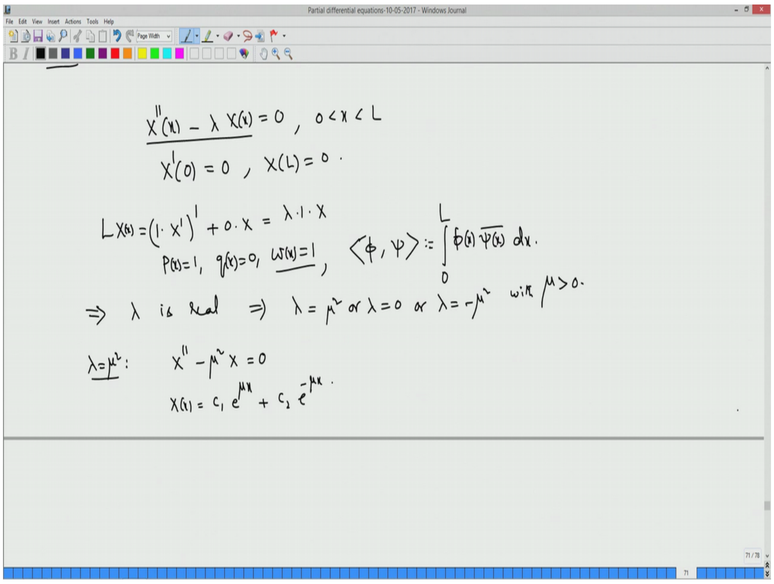

So extracted Sturm-Liouville problem that is for X of x where is lambda X of x equal to 0 is between 0 to L and you have X dash at 0 equal to 0 which is equal to X at L, so X L equal to 0 so you can write separately so these are the boundary conditions. So we can see that this is clearly in the self adjoint form like P is 1 X dash of whole dash that is the first term so minus plus q 0 into X q 0 equal to lambda into w of x that is 1 into X. So this is in the self adjoint form so P x is 1, q x is 0, w x equal to 1, so this is what you need.

So w x equal to 1 implies the dot product you can define the dot product call this Phi n or Phi and Psi dot product we can define it as integral domain is 0 to L Phi x Psi x bar dx, okay. So these are the steps so now consider so implies now because this is in the self adjoint form (()) (8:54) form this is L y equal to this is my L X of X, okay so this is my L X so there is L is in the self adjoint form.

So implies all lambdas are real implies a lambda is a real that means lambda is Mu square lambda or lambda is equal to 0 or lambda is equal to minus Mu square with Mu positives, okay. So look at all these three cases consider lambda equal to Mu square first you what is the equation, equation becomes X double dash minus Mu square X equal to 0 and so the general solution is c 1 Mu x e power Mu x plus c 2 e power minus Mu x.

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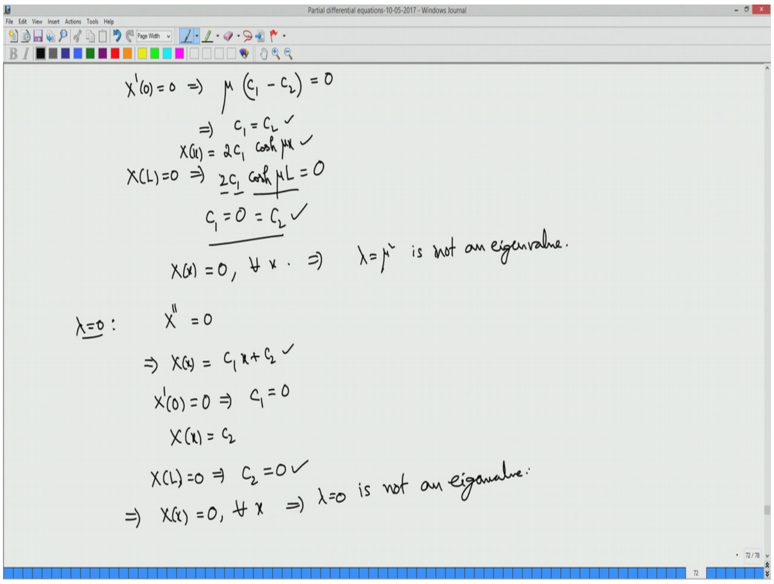

Now you apply boundary conditions X dash of 0 is 0 will give me Mu times c 1 minus c 2 equal to 0 so that is what is the first equation. So it will give me Mu cannot be 0, Mu is positive so c 1 has to be equal to 0. Apply X L equal to 0 then what happens once you get c 1 equal to 0 what happens to the general solution general solution is simply c 1 equal to c 2. So you have c 1 (())(10:21) c 1 by we can make it 2 c 1 so divide by 2 so that will be Cos hyperbolic Mu x that is what it becomes. So X L now if you apply this other boundary condition for this general solution of you have X L so 2 c 1 into Cos hyperbolic Mu L equal to 0.

So Cos hyperbolic Mu L can never be 0 this is always positive because exponential some of the exponential functions never be 0. So that means this cannot be 0, 2 cannot be 0 that means c 1 has to be 0 if c 1 is 0 from c 1 equal to c 2 which is equal to c 2 so both are 0. So my general solution becomes 0 completely, okay for every x. So implies lambda equal to Mu square is not an eigenvalue.

So look at the case now the lambda equal to 0, so you have x double dash minus lambda, lambda is 0 so equal to 0. So this now you get X of x equal to c 1 x plus c 2 so that is the general solution, you apply the boundary condition x dash of 0 equal to 0 will give me c 1 equal to 0. So you have the general solution becomes after applying the boundary condition 1 you get c 2.

Now if you apply the other boundary condition X at L (2) L equal to 0 will give me c 2 has to be 0. So implies X of x is completely 0, for every x, implies 0 is not lambda equal to 0 is not an eigenvalue.

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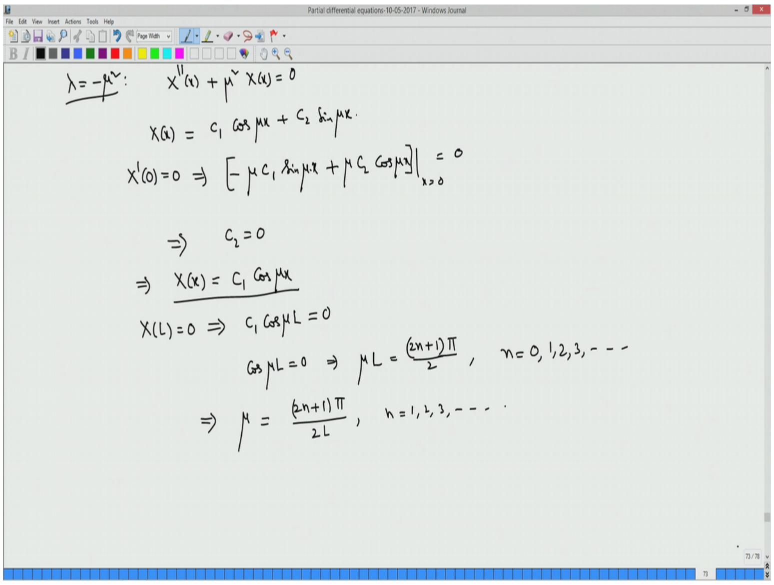

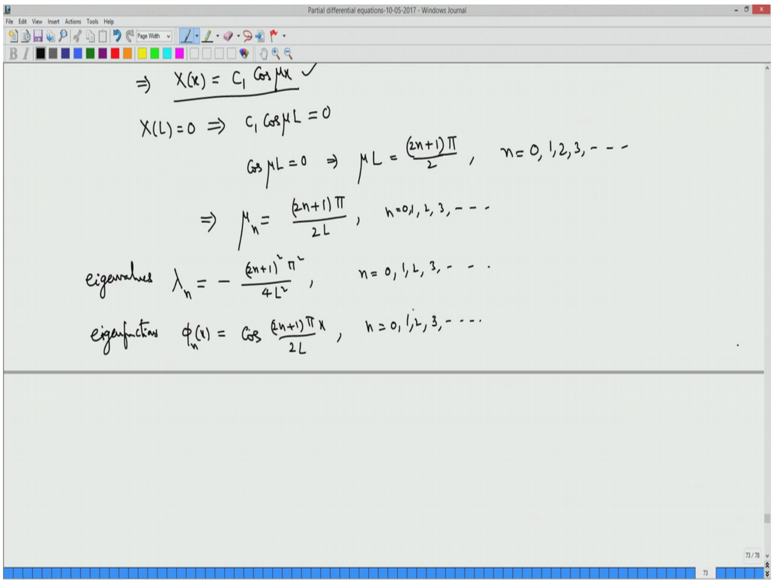

So the only option is lambda equal to minus Mu square this is what you can actually directly write so always you give Sin, Cosine, Sin solutions always give sometimes 0, okay. So if you do this you have X double dash of x minus minus plus Mu square X of x equal to 0. So this is the ODE now. So the solution is general solution is general solution becomes c 1 Cos Mu x plus c 2 Sin Mu x.

Apply the boundary conditions X dash 0 equal to 0 on this you get c 1 so what is this one gives me minus Mu c 1 Sin Mu 0 so that is 0, okay Sin Mu x plus Mu c 2 Cos Mu x at x equal to 0 equal to 0 so this whole thing at x equal to 0. So this means you have this is 0 so that means c 1 is arbitrary so what you get is c 2 Mu is anyway positive Cos Mu into 0 that is 1 so c 2 equal to 0. So what happens now your general solution becomes c 1 Cos Mu x.

Now you apply other boundary condition (X L) X at L equal to 0 will give me c 1 Cos Mu L equal to 0 so c 1 is 0 or Cos Mu L is 0, Cos Mu L can be 0 for certain Mu positive values, okay. So so for those Mu values c 1 can be arbitrary. So that means so you have a nonzero solution for such Mu values by taking c 1 as nonzero. So if you so we want a nonzero solution so that means you make a Cos Mu L equal to 0 implies I know that it has a solutions m L equal to 2n plus 1 by 2 Pi, okay and running from n is from 0, 1, 2, 3 onwards. If you put n equal 0 that is Pi by 2 Cos Pi by 2 is 0.

So this is how you get so this implies I have all these Mu which is 2n plus 1 Pi by 2L, n is running from 1, 2, 3 and so on. So for these Mu values Mu is depending on n so we call this Mu n, so you have lambda n these are eigenvalues lambda n's are minus Mu square that is 2n plus 1 whole square Pi square by 4L square. So these are your eigenvalues, n is from 0 onwards 0, 1, 2, 3 and so on.

And what are the eigenfunctions eigenfunctions what are the eigenfunctions basically you have a c 1 is arbitrary here in the solution so Cos Mu so that is you call this eigenfunctions Phi n of x we call them Cos Mu n, so what is Mu n 2n plus 1 Pi by 2L x again n is running from 0, 1, 2, 3 and so on. So these are your eigenvalues and eigenfunctions. So corresponding to this n equal to 1, 2, 3 take this as a lambda and try to now you solve completely this problem. So now construct we choose those eigenvalues in the place of lambda and try to solve this ordinary differential equation for t.

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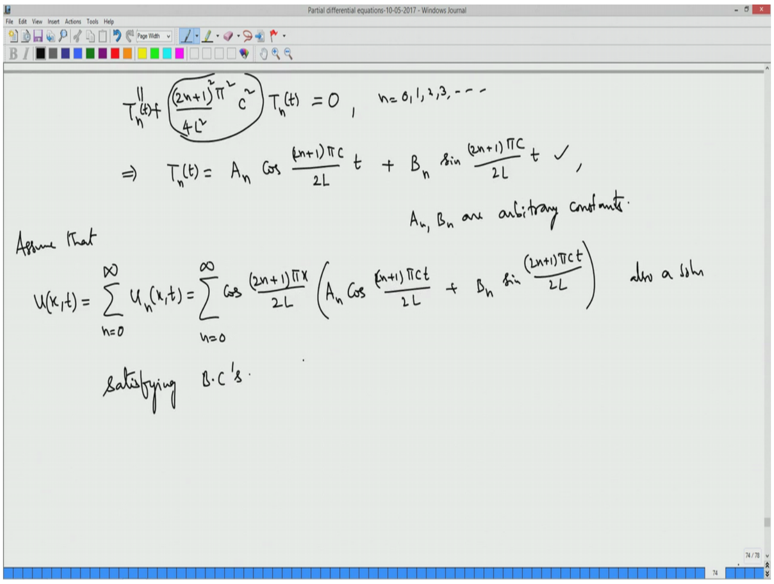

So you have T double dash minus lambda is minus minus plus 2n plus 1 by 4L square into Pi square so you have this is your lambda so because T depending on n for each lambda lambda n's so it is depending on so calling it the ODE for each n I have a differential equation like this T n double dash plus lambda c square T n of t equal to 0, okay. So T n double dash of t minus lambda c square T of t minus lambda, lambda is minus so minus minus plus this is what you get when again n is from 0, 1, 2, 3 onwards.

So this will give me clearly T n of t as you call this for each n two arbitrary constant I am calling it A n Cos 2n plus 1. So this is Cos 2n plus 1 Pi so lambda 1 is square so lambda n is basically have a square. So you have a square here so missed here so Pi c by 4L that is 2L into t so this is Cos of this number. So root of this number this whole thing I am writing it here so Cos at number into t plus other arbitrary constant B n with Sin Sin 2n plus 1 Pi c by 2L t. So these are your solutions where A n, B n are arbitrary constants, okay.

So what happens to your solutions u n of x for each n you have u n of x, t that is x n of x that is what is your x n of x that is your calling it eigenfunctions. So if you call it x n of n so this is Cos 2n plus 1, so Cos 2n plus 1 by into Pi x by 2L into T n x T n of t is this one. So if you write this A n Cos 2n plus 1 Pi c t by 2L plus B n Sin 2n plus 1 Pi c t by 2L. So this is what you have u n.

So this if you make a sum of all solutions n is now running from 0 to infinity, n is now running from 0 to infinity there is we call it u of x t be the solution, okay each term is a solution of wave equation satisfying the boundary conditions so assume that assume that assume that this sum this infinite sum make sense and assume that it is a solution by super position of all the each of the solutions, okay assume that is a solution is also a solution satisfying boundary conditions.

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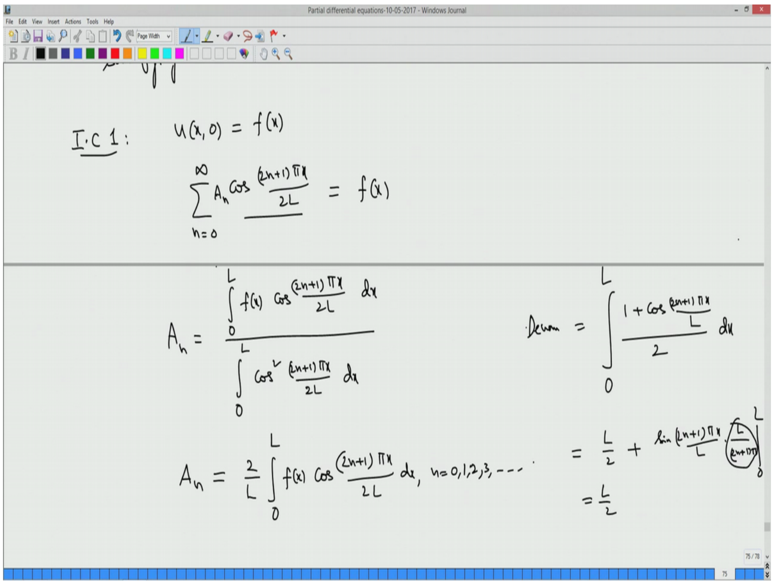

Now what is if it is a solution only thing is A n, B n's are not known arbitrary constants still to be determined. So make use of initial data initial condition 1 that is u at x, 0 equal to f x. So if you put u left hand side u at x, 0 is that is Cos so Cos if you put n is from 0 to infinity u at x so Cos as it is 2n plus 1 Pi x by 2L and if we put t equal to 0 so B n will be will not be there. So you have A n Cos so you have that is 1 so that is what it is equal to f x and we put t equal to 0 only A n. So this whole thing is A n, okay.

So how do from this you can get your A n's A n's for each n simply multiply this x n's okay make it dot product so multiply and integrate from 0 to L that is what is your that is where is you are using this dot product from the Sturm-Liouville problem. So because these are all the functions these are all real lambda is real and eigenfunctions are also real so because Cosine all these are real so bar does not make does not matter.

So you have A n's 0 to L right hand side of 0 to L A n you will get A n as integral 0 to L that is right hand side f x x n of x that is Cos 2n plus 1 Pi x by 2L dx this one and the left hand side when you multiply again with Cos same Cos 2n plus 1 Pi x by 2L only A n's will remain with the integral. So that integral if you divide it so we get 0 to L Cos square 2n plus 1 Pi x by 2L so that is what is so this you can evaluate the denominator so the integral in the denominator you can actually evaluate and you can write it.

So if you evaluate this 1 plus Cos 2 so that is 2n plus 1 Pi x by L right divided by 2, so this is from 0 to L this is what you will get, okay. So if you do this you will see that if you integrate from 0 to L so 0 to L that is what you will get so this is the integrant so 2 Cos square half of this that is what is this divided by 2, okay this is from 0 to L. So you get Cos 2n plus 1 Pi L so that is actually it will become Cos 2n plus 1 Pi so that is 1 by 2 so you have that is what is you have to integrate this this is what is the 0 to L.

So you can write this as this this is your dx that is the integral denominator denominator is this. So you have this is L by 2 plus this will be Sin 2n plus 1 Pi x by L into L by 2n plus 1 Pi this if you do it from 0 to L this is a constant comes out L if you substitute that is 0 0. So you simply have L by 2. So when you can write this as A n as 2 by L integral 0 to L f x Cos 2n plus 1 Pi x by 2L dx, n is running from 0, 1, 2, 3 and so on, so this is how you get your a n's by applying the initial condition 1.

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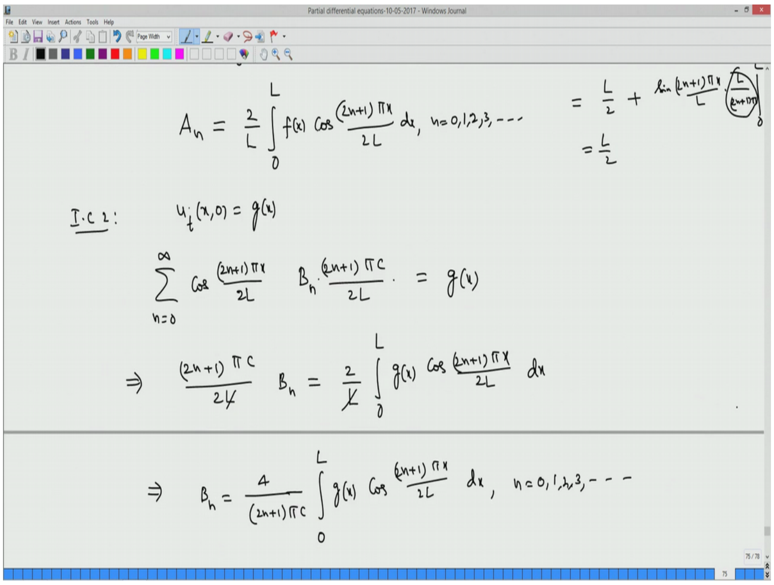

So in this solution the super position solution I know what is my A n, so if I find B n by applying the initial condition 2 and through. So I C 2 initial condition 2 is u t at x, 0 is g x, g is given this is the data given. So you have n is from 0 to infinity. Now you have A Cos 2n plus 1 Pi x by 2L this is A n if you now if you differentiate this with respect to t then you will get and put t equal to 0 Cos becomes Sin and you put t equal to 0 that is 0 so this will not be contributing.

So what you will have is here 2n plus 1 Pi c by 2L into cosine of that. So if you differentiate this with respect to t 2n plus 1 Pi c by 2L Cos at t equal to 0 that is 1 so you have into B n so this into B n is so this is your number so this is your number, this is your function so this is equal to g of x. Now again you can see that from this if you apply the dot product both sides you can see that 2n plus 1 Pi c by 2L into B n you can get it as again 2 by L integral 0 to L g x by the same way like earlier so you have g x Cos 2n plus 1 Pi x by 2L dx, okay.

So this gives me B n B n as so L, L goes 4 by 2n plus 1 Pi c into 0 to L g x Cos 2n plus 1 Pi x by 2L dx, so again n is from 0, 1, 2, 3 onwards. So now I know what is my A n's and B n's in this general solution so this is your general solution. So this is the solution that satisfy all the condition.

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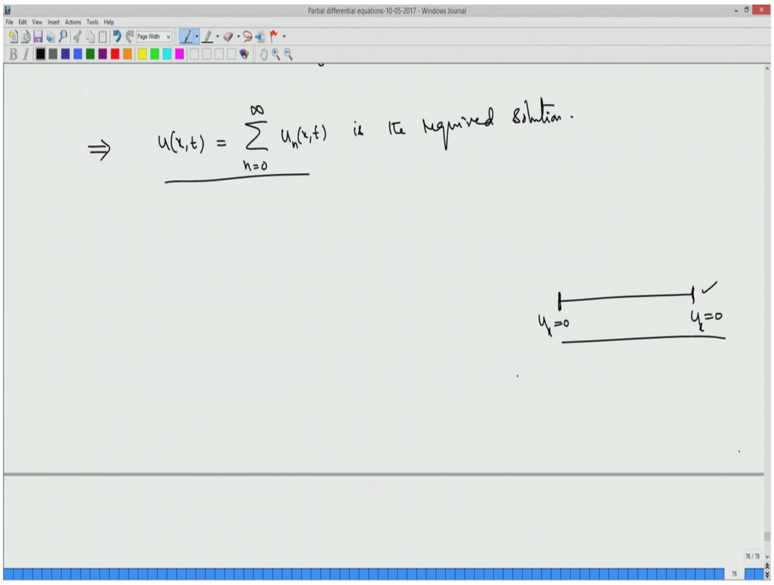

So implies u of x, t take a super position of all the solutions of u n of x, t with this A n's and B n's is the required solution. So these are the vibrations you can see for all times for a finite length of string string of finite length that is one end is fixed, other end is lefted free. So that is how this is how it vibrates so you have this solution, okay. So you can also do the other problems when the string is both ends are lefted free so you can allow both ends are 0, u x is 0 here and u x is also 0 here.

So you can work out bring extract this Sturm Liouville problem with different boundary conditions so you find the eigenvalues and eigenfunctions, then apply the initial conditions to get the solution in a separable form. So nicely you get vibration of string whatever may be the boundary conditions, okay so leave it as an exercise for you to work out remaining problems, okay.

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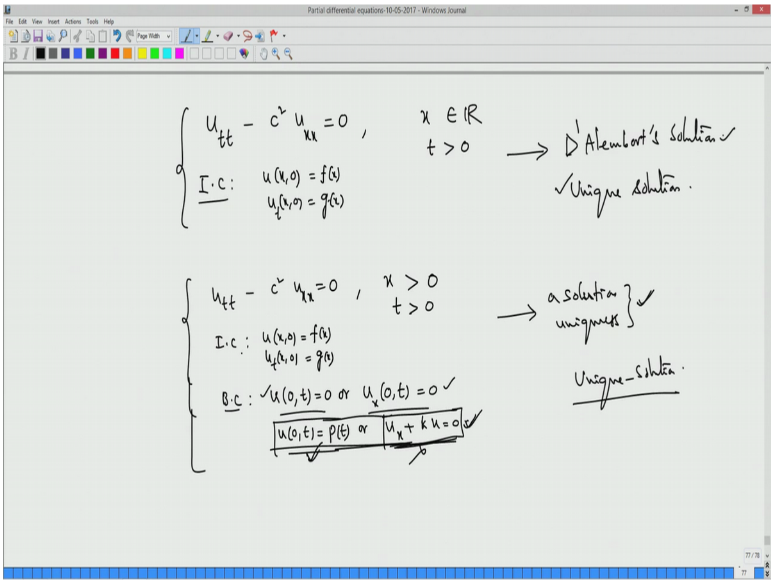

So far what we have done so let us recap what we have done so u tt minus c square this is your wave equation so what you have is x belongs so first of all when you have x in the full real line t positive you only have the initial data that is u at x equal to 0 equal to f x given at u t at x equal to 0 is g x because there is no boundary this has a solution. So this is the D'Alembert's solution we found we had a solution D'Alembert's solution and you have a solution for this problem and we have also seen that this is the unique solution by the energy argument by considering the energy you have the unique solution uniqueness is also proved, okay so you have a unique solution you have shown that D'Alembert's solution is the unique solution for this problem this initial problem for the wave equation on a full domain you have a infinite string infinite string at both ends when you give the displacement and the velocity initially you know that vibration of the string for all time you know as a D'Alembert's solution that is actually that is D solution is because of uniqueness.

And we also have seen how to reduce same problems in a same problem but when you consider a semi-infinite domain. So x is positive and t is positive and initial conditions are same that is (u x u 0) u at x, 0 is f x u t at x, 0 equal to g x because you now you have the boundary you have a boundary condition so you can give the boundary condition as either fixed it or allow it to be free. So u at 0, t is 0 or u x at 0, t equal to 0.

So in each case we simply extends the domain into extend all the functions involved even or odd functions and then extend the domain as a full real line and make use of this D'Alembert's solution and we find the solution so you have a solution, okay we have a solution. And we also have seen that uniqueness uniqueness is shown, okay uniqueness is shown by the same energy argument earlier. So you have again unique solution, so in this case, okay.

And if you change this to different things for example u at 0, t as some general function P of t still you can actually show the uniqueness solution you can work out by the general way. So without extending even or odd function we can also just by looking at the general solution of the wave equation and then make use of the boundary condition for to find the arbitrary functions one in the negative side.

So by that method you can actually seen the solution and again energy argument you can show works uniqueness is there so you have the unique solution even in this case, only problem is when you have this absorbing boundary condition u x plus u constant times u so if you attach with some string then you may not have you have not shown the uniqueness unique solution but it is little tricky so it is not straight forward but you have the solution is uniqueness unique solution is there one can prove uniqueness solution by different argument but we are not giving so this you need not worry.

So there also we can work out similar solutions this this we have not given these two solutions but we can work out by the general argument, you have a solution and that is actually unique but I have not shown here and one can show the uniqueness uniqueness argument by the energy argument you can show the uniqueness even in this case for the for the same problem with these boundary condition, okay. So this is what you have seen.

So now let us look at the initial value problem here, so this initial value problem as a non-homogeneous function so if you consider this is a forcing is 0 there is no forcing outside force external force is 0 here. So you have a wave equation there is no external force initially these are the this is what you have. so if you give some forcing outside from the forcing so you can write f of x t for all times you can give some function let us say some h of x, t if you give as a forcing term, okay.

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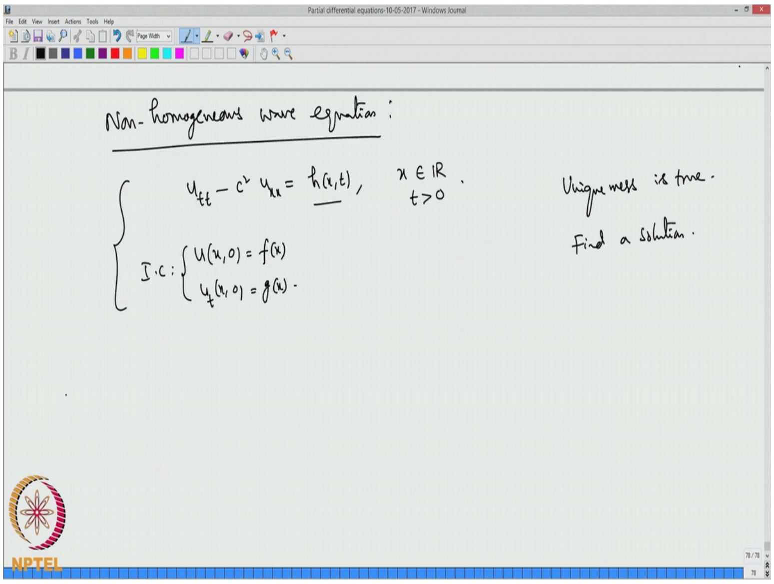

So let us say non-homogeneous equation so non-homogeneous equation so if we will write this non-homogeneous equation okay. So let me write non-homogeneous wave equation so let us take this full domain so u tt minus c square u xx equal to some h of x, t given this is a forcing external force for x belongs to full R and t positive so you have full infinite string and you have the initial data u at again x, 0 is f x u t at x, 0 equal to g x so this is your initial data.

So we want to know how do we find the solution here so again by the same energy argument one can show that by the same argument if you assume u 1, u 2 are two solutions for this problem, okay for this problem if you consider two solutions u 1 and u 1 of x, t and u 2 of x, t be two solutions, you take the difference that you call it w, w is the solution of the wave equation with no forcing because if u 1 satisfies this you take the difference it will be 0.

So h right hand side will be 0, simply satisfies the homogeneous wave equation and the initial data will be 0, 0. So by using the energy same energy argument whatever have given earlier proof that works even here so that you have the uniqueness of the solution uniqueness is guaranteed uniqueness uniqueness is true, only thing you have to find the solution one needs to find a solution so that you have a unique solution even for this non-homogeneous wave equation, okay.

So that is what this is what we will show in the next video, I will just give you we will give you the method to find solution we will try to break this problem into two parts one is you already know that it is a D'Alembert's solution homogeneous part and non-homogeneous solution that particular thing you can get the solution by simple integration taking it. So we will see this in the next video, thank you very much.