

**Differential Equations for Engineers**  
**Professor Doctor Srinivasa Rao Manam**  
**Department of Mathematics**  
**Indian Institute of Technology Madras**  
**Lecture No 31**  
**Properties of Bessel Functions (continued)**

(Refer Slide Time 00:19)



So we were looking at the orthogonal relation between the Bessel functions. We have seen, we have shown that they are actually orthogonal to each other when they are different. So we look at the other, other part, that when you take the same function in that relation that, when you take lambda i equal to lambda j. That is same as

(Refer Slide Time 00:43)

$$\Rightarrow \underbrace{(\lambda_i^2 - \lambda_j^2)}_{\neq 0} \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_j x) dx = 0, \quad \begin{array}{l} i \neq j \quad \lambda_i \neq \lambda_j \\ \parallel \\ x_i \neq x_j \\ \parallel \\ L \end{array}$$

$$\Rightarrow \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_j x) dx = 0, \quad \text{if } i \neq j$$

(Orthogonal relation).

i equal to j, the case when i equal to j.

What if  $i$  equal to  $j$  this case what you have is  $1 \times 2$  times  $J_\alpha$  of  $\lambda_i x$   $d x$  so  $i$  is, both are same, this is actually equal to, so we will see that this will be equal to some number that is let me write this a square by 2 sorry  $1$  square by  $2 j$  square  $n$  plus  $1$  of  $\lambda_i$   $1$ . Or this is same as  $1$  square by  $2 j$   $n$  minus  $1$  square of  $\lambda_i$   $1$ . This is not  $n$ , this is  $\alpha$ . So this is  $\alpha$ , this is what you will get, Ok.

So this is the, either this or this.

(Refer Slide Time 01:33)

$$\Rightarrow \left( \lambda_i \neq \lambda_j \right) \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_j x) dx = 0, \quad \lambda_i \neq \lambda_j$$

$$\Rightarrow \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_j x) dx = 0, \quad \text{if } i \neq j$$
 (Orthogonal relation).

$$\underline{i = j}: \int_0^L x J_\alpha(\lambda_i x)^2 dx = \frac{L^2}{2} J_\alpha'(\lambda_i L)^2 = \frac{L^2}{2} J_\alpha^2(\lambda_i L)$$

So we will show how it is done, Ok, this part, so this part what we do is you take the equations. So we will only have one, one function so that is  $J_\alpha$  of  $\lambda_i x$  satisfies Bessel equation that is  $x^2$  that is  $J_\alpha$  double dash of  $\lambda_i x$  plus  $x J_\alpha$  dash of  $\lambda_i x$ , Ok.

So let us call this may be some function of  $x$ ,  $u x$ ,

(Refer Slide Time 02:06)

$\frac{x^2}{L} \neq \frac{x^2}{L}$

$$\Rightarrow \int_0^L x J_x(\lambda_i x) J_x(\lambda_j x) dx = 0, \text{ if } i \neq j$$

(Orthogonal relation).

\*  $i = j$ :

$$\int_0^L x J_x(\lambda_i x) dx = \frac{L^2}{2} J_{x+1}(\lambda_i L) = \frac{L^2}{2} J_{x-1}(\lambda_i L)$$

$u(x) = J_x(\lambda_i x)$  satisfies  $x^2 u'' + x u' + (\lambda_i^2 x^2 - x^2) u = 0$

Ok so then you have u double dash, that will be easier for you. So x u dash plus x square, so you have here, this is lambda i square x square minus alpha square into u, u of x equal to zero. So u is J alpha of lambda i x, Ok. What I do is for this equation, I multiply,

(Refer Slide Time 02:35)

$$\Rightarrow \int_0^L x J_x(\lambda_i x) J_x(\lambda_j x) dx = 0, \text{ if } i \neq j$$

(Orthogonal relation).

\*  $i = j$ :

$$\int_0^L x J_x(\lambda_i x) dx = \frac{L^2}{2} J_{x+1}(\lambda_i L) = \frac{L^2}{2} J_{x-1}(\lambda_i L)$$

$u(x) = J_x(\lambda_i x)$  satisfies  $[x^2 u'' + x u' + (\lambda_i^2 x^2 - x^2) u] = 0$

I multiply simply 2 u dash something, Ok, so, if I, this is zero, so both sides I multiply 2 u dash, multiply 2 u dash.

That is 2 times J alpha dash of lambda i x. So if I do both sides to get,

(Refer Slide Time 03:04)

$$\Rightarrow \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_j x) dx = 0, \quad i \neq j$$
 (Orthogonal relation).

$$* \quad i = j: \quad \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_i x) dx = \frac{L^2}{2} J_{\alpha+1}^2(\lambda_i L) = \frac{L^2}{2} J_{\alpha-1}^2(\lambda_i L)$$

$$u(x) = J_\alpha(\lambda_i x) \text{ satisfies } [x^2 u'' + x u' + (\lambda_i^2 x^2 - \alpha^2) u] = 0$$

Multiply  $2 u' = 2 J_\alpha'(\lambda_i x)$  both sides to get

what you get if you do this, you have  $2 u' x^2$ ,  $2 x^2 u''$  rather, if you do it  $2 x^2 u''$ ,  $2 x u'$ ,  $2 u u'' + 2 x u' + (\lambda_i^2 x^2 - \alpha^2) u = 0$ , Ok.

(Refer Slide Time 03:29)

$$\Rightarrow \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_j x) dx = 0, \quad i \neq j$$
 (Orthogonal relation).

$$* \quad i = j: \quad \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_i x) dx = \frac{L^2}{2} J_{\alpha+1}^2(\lambda_i L) = \frac{L^2}{2} J_{\alpha-1}^2(\lambda_i L)$$

$$u(x) = J_\alpha(\lambda_i x) \text{ satisfies } [x^2 u'' + x u' + (\lambda_i^2 x^2 - \alpha^2) u] = 0$$

Multiply  $2 u' = 2 J_\alpha'(\lambda_i x)$  both sides to get

$$2 x^2 u' u'' + 2 x u'^2 + (\lambda_i^2 x^2 - \alpha^2) 2 u u' = 0$$

So what the next step is, you can see that  $2 u u''$ ,  $2 u u'$ , so you have this one, so you just have to write this in a simplified manner.

That is, so you can see that  $x^2 u''$ , what is the derivative of this? Derivative of this if you consider, this is equal to  $x^2 u'$ , there is  $2 x$ , if you do this, this is  $2 x u' x^2 + x^2 u''$ , so you have  $2 u' x^2$ , so you can see that this, this together,

(Refer Slide Time 04:17)

$u(x) = J_x(\lambda) \text{ satisfies } [x^2 u'' + x u' + (\lambda^2 x^2 - x^2) u] = 0$   
 Multiply by  $u' = 2 J_x(\lambda)$  both sides to get  
 $2 x^2 u' u'' + 2 x u'^2 + (\lambda^2 x^2 - x^2) 2 u u' = 0$   
 $\frac{d}{dx} (x^2 u'^2) = 2 x u'^2 + 2 x^2 u' u''$

so these two parts you can write like as a derivative, Ok.

So what happens the other part is  $\frac{d}{dx}$  of  $\frac{d}{dx}$  of  $\lambda^2 x^2 u^2$ . What is this one, this one  $\lambda^2 x^2 u^2$ , if I differentiate  $u^2$ . Now one more term is there, that is  $\lambda^2 x^2$ , you differentiate  $x^2$  so you get  $2 x u^2$ . This is what you need, and then so, so I can replace this part, this part with this.

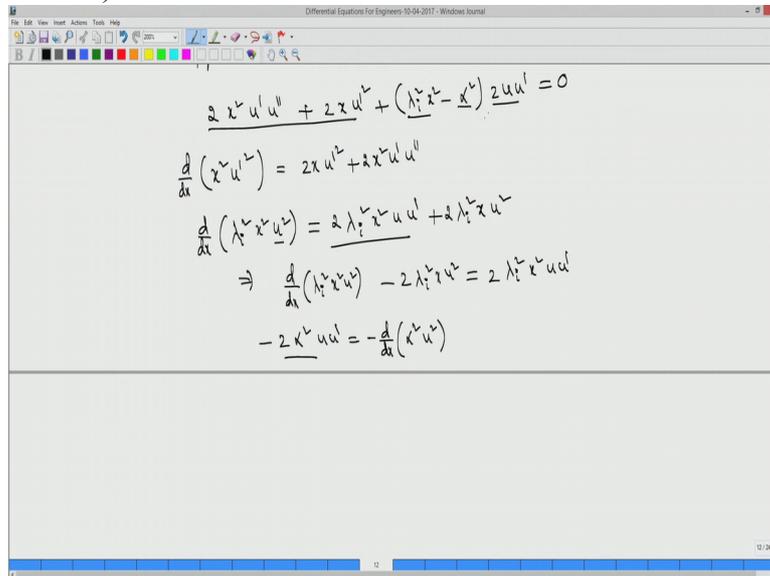
So this implies, so we can rewrite  $\frac{d}{dx}$  of  $\lambda^2 x^2 u^2$  minus  $2 \lambda^2 x^2 u u'$ , I can replace with  $2 \lambda^2 x^2 u u'$ ,

(Refer Slide Time 05:33)

Multiply by  $u' = 2 J_x(\lambda)$  both sides to get  
 $2 x^2 u' u'' + 2 x u'^2 + (\lambda^2 x^2 - x^2) 2 u u' = 0$   
 $\frac{d}{dx} (x^2 u'^2) = 2 x u'^2 + 2 x^2 u' u''$   
 $\frac{d}{dx} (\lambda^2 x^2 u^2) = 2 \lambda^2 x^2 u u' + 2 \lambda^2 x u^2$   
 $\Rightarrow \frac{d}{dx} (\lambda^2 x^2 u^2) - 2 \lambda^2 x u^2 = 2 \lambda^2 x^2 u u'$

Ok. So what is left is minus 2 alpha square u u dash. This I can write like minus d d x of alpha square u square, Ok. So replace each of these, with

(Refer Slide Time 05:52)



$$2x^\lambda u^\lambda u'' + 2x u^\lambda + (\lambda^2 x^\lambda - x^\lambda) 2uu' = 0$$

$$\frac{d}{dx} (x^\lambda u^\lambda) = 2x u^\lambda + \lambda x^\lambda u^\lambda$$

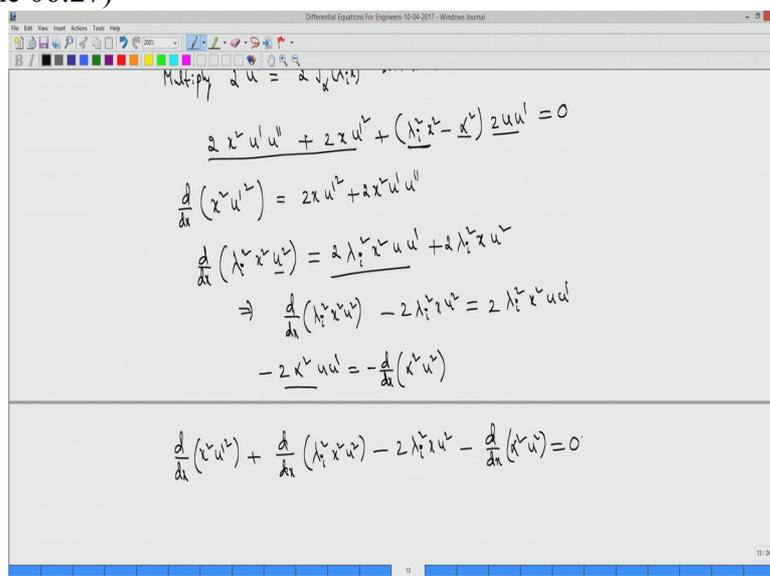
$$\frac{d}{dx} (\lambda^2 x^\lambda u^\lambda) = 2\lambda^2 x^\lambda u u' + \lambda^2 x^\lambda u^\lambda$$

$$\Rightarrow \frac{d}{dx} (\lambda^2 x^\lambda u^\lambda) - 2\lambda^2 x^\lambda u^\lambda = 2\lambda^2 x^\lambda u u'$$

$$-2x^\lambda u u' = -\frac{d}{dx} (x^\lambda u^\lambda)$$

alpha square, this is with this, as a derivatives. If I do this, you can write the first two terms, I can write like d d x of square u dash square Ok plus d d x of lambda i square x square u square minus 2 lambda i square x u square that is what is this term, first term in the bracket. And what is left is finally minus d d x of alpha square u square equal to zero.

(Refer Slide Time 06:27)



Multiply  $d u = 2 \lambda u^\lambda$

$$2x^\lambda u^\lambda u'' + 2x u^\lambda + (\lambda^2 x^\lambda - x^\lambda) 2uu' = 0$$

$$\frac{d}{dx} (x^\lambda u^\lambda) = 2x u^\lambda + \lambda x^\lambda u^\lambda$$

$$\frac{d}{dx} (\lambda^2 x^\lambda u^\lambda) = 2\lambda^2 x^\lambda u u' + \lambda^2 x^\lambda u^\lambda$$

$$\Rightarrow \frac{d}{dx} (\lambda^2 x^\lambda u^\lambda) - 2\lambda^2 x^\lambda u^\lambda = 2\lambda^2 x^\lambda u u'$$

$$-2x^\lambda u u' = -\frac{d}{dx} (x^\lambda u^\lambda)$$

$$\frac{d}{dx} (x^\lambda u^\lambda) + \frac{d}{dx} (\lambda^2 x^\lambda u^\lambda) - 2\lambda^2 x^\lambda u^\lambda - \frac{d}{dx} (x^\lambda u^\lambda) = 0$$

So derivatives you put it together, so what you have is, this you take it the other side, so you have lambda i square x u square equal to d d x of x square u dash square plus lambda i square

$x^2 u'' - \alpha^2 u = 0$ . Now you differentiate both sides from zero to  $l$  with respect to  $x$ .

(Refer Slide Time 07:02)

$$-2xu' = -\frac{d}{dx}(x^2u)$$

$$\frac{d}{dx}(x^2u') + \frac{d}{dx}(\lambda^2 x^2 u) - 2\lambda^2 x u' - \frac{d}{dx}(x^2 u) = 0$$

$$\int_0^l \lambda^2 x u' dx = \int_0^l (x^2 u'' + \lambda^2 x^2 u' - x^2 u') dx$$

with respect to  $x$ .

This is equal to  $x^2 u'' + \lambda^2 x^2 u' - \alpha^2 x^2 u$ . We put the values from zero to  $l$ .

(Refer Slide Time 07:17)

$$-2xu' = -\frac{d}{dx}(x^2u)$$

$$\frac{d}{dx}(x^2u') + \frac{d}{dx}(\lambda^2 x^2 u) - 2\lambda^2 x u' - \frac{d}{dx}(x^2 u) = 0$$

$$\int_0^l \lambda^2 x u' dx = \int_0^l (x^2 u'' + \lambda^2 x^2 u' - x^2 u') dx$$

$$= [x^2 u' + \lambda^2 x^2 u - x^2 u]_0^l$$

Now you see that  $u$  at  $l$  is zero, Ok so this nothing but it will give you  $l^2 u'$  at  $l$  minus  $0$ , Ok, minus so when you put zero, this is zero, this is zero and you have minus minus plus,  $\alpha^2 u$  at zero square.

Do this properly so you have lambda i square 2 x square so 2 x square lambda i square u dash I replace with this, minus 2 lambda i square x u square, Ok. So you have x square this, alpha square u square that is finally alpha square u square, Ok. So this is what you get.

So

(Refer Slide Time 08:09)

$$\frac{d}{dx} (x^2 u') + \frac{d}{dx} (\lambda_i x^2 u') - 2 \lambda_i x u' - \alpha (x^2 u) = 0$$

$$\int_0^L \lambda_i x^2 u' dx = \int_0^L (x^2 u' + \lambda_i x^2 u' - \alpha x^2 u) dx$$

$$= [x^2 u + \lambda_i x^2 u - \alpha x^2 u]_0^L$$

$$= L^2 (u'(L)) + \alpha (u(0)) - L^2$$

when you replace x by l, u is zero at l, u is zero at l, so this will contribute to l square, this and you put minus, minus of this and you put x equal to zero, this is zero, this is zero, this is going to be plus alpha square this one. So this is what you have. So this divided by 2, so this implies, so what you get is zero to l lambda i square, lambda i square comes out, so this is a constant so you have x, u is J alpha of lambda i x square, Ok d x is equal to 1 by, so that is l square by 2 lambda i square, this is J alpha dash of lambda i x whole square plus alpha square by 2 lambda i square into J alpha of lambda i at zero, Ok, this square.

But J alpha

(Refer Slide Time 09:21)

The image shows a software window titled "Differential Equations For Engineers-10-04-2017 - Windows Journal". The window contains the following handwritten mathematical derivation:

$$\int_0^L \lambda_i^2 x u'' dx = \int_0^L \frac{d}{dx} (x^2 u' + \lambda_i^2 x^2 u - x^2 u') dx$$

$$= \left[ x^2 u' + \lambda_i^2 x^2 u - x^2 u' \right]_0^L$$

$$= L^2 (u'(L)) + x^2 (u(0))$$

$$\Rightarrow \int_0^L x \ddot{J}_x(\lambda_i) dx = \frac{L^2}{2 \lambda_i^2} (\ddot{J}_x(\lambda_i)) + \frac{L^2}{2 \lambda_i^2} (\ddot{J}_x(\lambda_i))^2$$

at zero, u at zero is also zero, Ok. So J alpha at zero is zero. Since J alpha of x if you actually see as a sum which has x power 2 m plus alpha, alpha is positive, Ok. So you can see that this is zero, J alpha, if you put x equal to zero, this is zero, Ok, so this will be actually zero,

(Refer Slide Time 09:47)

The image shows a software window titled "Differential Equations For Engineers-10-04-2017 - Windows Journal". The window contains the following handwritten mathematical derivation:

$$\int_0^L \lambda_i^2 x u'' dx = \int_0^L \frac{d}{dx} (x^2 u' + \lambda_i^2 x^2 u - x^2 u') dx$$

$$= \left[ x^2 u' + \lambda_i^2 x^2 u - x^2 u' \right]_0^L$$

$$= L^2 (u'(L)) + x^2 (u(0))$$

$$\Rightarrow \int_0^L x \ddot{J}_x(\lambda_i) dx = \frac{L^2}{2 \lambda_i^2} (\ddot{J}_x(\lambda_i)) + \frac{L^2}{2 \lambda_i^2} (\ddot{J}_x(\lambda_i))^2$$

To the right of the main equation, there is a note:  $\therefore \ddot{J}_x(x) = \sum_{l=0}^{\infty} x^{2m+\alpha}$  and  $\lim_{x \rightarrow 0} = 0$ .

so this is what you get. So this is equal to 1 square by 2 lambda i square J alpha dash of lambda i x whole square, lambda i l, right,

(Refer Slide Time 10:04)

$$= \mathcal{L}\{u'(x)\} + x^{\alpha} u(0)$$

$$\Rightarrow \int_0^L x J_{\alpha}'(\lambda_1 x) dx = \frac{L}{2\lambda_1^2} (J_{\alpha}'(\lambda_1 L))^2 - \frac{L}{2\lambda_1^2} (J_{\alpha}'(\lambda_1 0))^2$$

$$= \frac{L}{2\lambda_1^2} (J_{\alpha}'(\lambda_1 L))^2$$

$x > 0$   
 $\therefore J_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha+k}}{k!} = 0$

so it is lambda i l.

This is fixed thing so lambda i l whole square, Ok. This is what you get. Now how do you find, this make it equal to this type, alpha plus 1, alpha minus 1.

(Refer Slide Time 10:23)

$$\Rightarrow \int_0^L x J_{\alpha}(\lambda_1 x) J_{\alpha}(\lambda_2 x) dx = 0, \lambda_1^2 + \lambda_2^2 = 0$$

(Orthogonal relation).

$$* \quad i=j: \quad \int_0^L x J_{\alpha}^2(\lambda_1 x) dx = \frac{L}{2} J_{\alpha+1}^2(\lambda_1 L) = \frac{L}{2} J_{\alpha-1}^2(\lambda_1 L)$$

$$u(x) = J_{\alpha}(\lambda_1 x) \text{ satisfies } [x^2 u'' + x u' + (\lambda_1^2 x^2 - \alpha^2) u] = 0$$

Multiply both sides by  $2 J_{\alpha}'(\lambda_1 x)$  both sides to get

$$2 x^2 u' u'' + 2 x u'^2 + (\lambda_1^2 x^2 - \alpha^2) 2 u u' = 0$$

$$\frac{d}{dx} (x^2 u'^2) = 2 x u'^2 + 2 x^2 u' u''$$

$$\frac{d}{dx} (\lambda_1^2 x^2 u^2) = 2 \lambda_1^2 x^2 u u' + 2 \lambda_1^2 x u^2$$

This is where we use the properties, the properties of those, properties of these, these properties if we use,

(Refer Slide Time 10:31)

$$J_k(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+k+1)} \left(\frac{x}{2}\right)^{2m+k}$$

1. 
$$\frac{d}{dx} \left( x^k J_k(x) \right) = x^k J_{k-1}(x) \checkmark$$

2. 
$$\frac{d}{dx} \left( x^{-k} J_k(x) \right) = -x^{-k} J_{k+1}(x) \checkmark$$

Proof for 1: 
$$\frac{d}{dx} \left( x^k J_k(x) \right) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+k+1)} x^{2m+2k}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2k)}{m! \Gamma(m+k+1)} x^{2m+2k-1}$$

$$= x^k \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+k+1)} \frac{2m+2k}{x} x^{2m+k}$$

$$= x^k J_{k-1}(x)$$

Proof for 2: 
$$\frac{d}{dx} \left( x^{-k} J_k(x) \right) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+k+1)} x^{2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m 2m}{m! \Gamma(m+k+1)} x^{2m-1}$$

$$= -x^{-k} J_{k+1}(x)$$

properties of the Bessel functions, you can actually get those things. You simply, differentiate, differentiate 1.

If you differentiate this, what you are getting end up is x power alpha so you just expand this

(Refer Slide Time 10:53)

1. 
$$\frac{d}{dx} \left( x^k J_k(x) \right) = x^k J_{k-1}(x) \checkmark \Rightarrow$$

2. 
$$\frac{d}{dx} \left( x^{-k} J_k(x) \right) = -x^{-k} J_{k+1}(x) \checkmark$$

Proof for 1: 
$$\frac{d}{dx} \left( x^k J_k(x) \right) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+k+1)} x^{2m+2k}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2k)}{m! \Gamma(m+k+1)} x^{2m+2k-1}$$

$$= x^k \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+k+1)} \frac{2m+2k}{x} x^{2m+k}$$

$$= x^k J_{k-1}(x)$$

Proof for 2: 
$$\frac{d}{dx} \left( x^{-k} J_k(x) \right) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+k+1)} x^{2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m 2m}{m! \Gamma(m+k+1)} x^{2m-1}$$

$$= -x^{-k} J_{k+1}(x)$$

derivative Ok, which is equal to this,

(Refer Slide Time 10:56)

1.  $\frac{d}{dx} \left( x^\alpha J_\alpha(x) \right) = x^\alpha J_{\alpha-1}(x) \checkmark \Rightarrow$

2.  $\frac{d}{dx} \left( x^{-\alpha} J_\alpha(x) \right) = -x^\alpha J_{\alpha+1}(x) \checkmark$

Proof 1:  $\frac{d}{dx} \left( x^\alpha J_\alpha(x) \right) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+\alpha}}{2^{2m+\alpha} m! \Gamma(m+\alpha+1)} = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+\alpha) x^{2m+\alpha-1}}{m! \Gamma(m+\alpha) \Gamma(m+\alpha+1)} = x^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+\alpha-1}}{m! \Gamma(m+\alpha+1) 2^{2m+\alpha-1}} = x^\alpha J_{\alpha-1}(x)$

Proof 2:  $\frac{d}{dx} \left( x^{-\alpha} J_\alpha(x) \right) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\alpha} m! \Gamma(m+\alpha+1)} = \sum_{m=1}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m+\alpha} m! \Gamma(m+\alpha+1)} = -x^\alpha J_{\alpha+1}(x)$

either of these you can use,

(Refer Slide Time 10:59)

1.  $\frac{d}{dx} \left( x^\alpha J_\alpha(x) \right) = x^\alpha J_{\alpha-1}(x) \checkmark \Rightarrow$

2.  $\frac{d}{dx} \left( x^{-\alpha} J_\alpha(x) \right) = -x^\alpha J_{\alpha+1}(x) \checkmark$

Proof 1:  $\frac{d}{dx} \left( x^\alpha J_\alpha(x) \right) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+\alpha}}{2^{2m+\alpha} m! \Gamma(m+\alpha+1)} = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+\alpha) x^{2m+\alpha-1}}{m! \Gamma(m+\alpha) \Gamma(m+\alpha+1)} = x^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+\alpha-1}}{m! \Gamma(m+\alpha+1) 2^{2m+\alpha-1}} = x^\alpha J_{\alpha-1}(x)$

Proof 2:  $\frac{d}{dx} \left( x^{-\alpha} J_\alpha(x) \right) = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\alpha} m! \Gamma(m+\alpha+1)} = \sum_{m=1}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m+\alpha} m! \Gamma(m+\alpha+1)} = -x^\alpha J_{\alpha+1}(x)$

one of these, one of these you can use to show that, to get this one, Ok. So I will just do 1. 1, I will just use, 1 to find this value, J alpha dash, so

(Refer Slide Time 11:12)

$$\Rightarrow \int_0^\infty x^\alpha J_\alpha(\lambda x) dx = \frac{\Gamma(\alpha+1)}{2\lambda^{\alpha+1}} J_\alpha(\lambda x) + \frac{\Gamma(\alpha+1)}{2\lambda^{\alpha+1}} J_\alpha(\lambda x)$$

$$= \frac{\Gamma(\alpha+1)}{2\lambda^{\alpha+1}} J_\alpha(\lambda x)$$

let us take this first, what is that, so recurrence relation is  $\frac{d}{dx} x^\alpha J_\alpha(x)$  equal to  $x^{\alpha-1} J_{\alpha-1}(x)$ , so let's choose this one,

(Refer Slide Time 11:31)

$$\Rightarrow \int_0^\infty x^\alpha J_\alpha(\lambda x) dx = \frac{\Gamma(\alpha+1)}{2\lambda^{\alpha+1}} J_\alpha(\lambda x) + \frac{\Gamma(\alpha+1)}{2\lambda^{\alpha+1}} J_\alpha(\lambda x)$$

$$= \frac{\Gamma(\alpha+1)}{2\lambda^{\alpha+1}} J_\alpha(\lambda x)$$

$$\frac{d}{dx} (x^\alpha J_\alpha(x)) = x^{\alpha-1} J_{\alpha-1}(x)$$

Ok.

If you expand this  $x^\alpha J_\alpha(x)$  into  $x^{\alpha-1} J_\alpha(x)$ , that is the derivative, the left hand part I have written. You have  $x^{\alpha-1} J_\alpha(x)$ . Now you cancel both sides  $x^{\alpha-1}$ ,

(Refer Slide Time 11:54)

$$= \frac{\Gamma'}{\Gamma} \left( \frac{\Gamma'}{\Gamma} \right)$$

$$\frac{d}{dx} \left( x^\alpha \Gamma_\alpha(x) \right) = x^\alpha \Gamma_{\alpha-1}(x)$$

$$\Rightarrow x^\alpha \Gamma_\alpha'(x) + \alpha x^{\alpha-1} \Gamma_\alpha(x) = x^\alpha \Gamma_{\alpha-1}(x)$$

so to get  $\Gamma_\alpha'(x) + \alpha \Gamma_\alpha(x) = \Gamma_{\alpha-1}(x)$ , so this is what you get.

(Refer Slide Time 12:11)

$$= \frac{\Gamma'}{\Gamma} \left( \frac{\Gamma'}{\Gamma} \right)$$

$$\frac{d}{dx} \left( x^\alpha \Gamma_\alpha(x) \right) = x^\alpha \Gamma_{\alpha-1}(x)$$

$$\Rightarrow x^\alpha \Gamma_\alpha'(x) + \alpha x^{\alpha-1} \Gamma_\alpha(x) = x^\alpha \Gamma_{\alpha-1}(x)$$

$$\Rightarrow \Gamma_\alpha'(x) = -\frac{\alpha}{x} \Gamma_\alpha(x) + \Gamma_{\alpha-1}(x)$$

Ok.

So now use that, this one if you use,  $\Gamma_\alpha'(x)$  and put  $x = \lambda + 1$  which is equal to  $\lambda + 1 - \alpha$   $\Gamma_\alpha(\lambda + 1)$ , and I know that this is zero, plus  $\Gamma_{\alpha-1}(\lambda + 1)$ , Ok. So you want this square and this whole thing is zero so this square. Ok

(Refer Slide Time 12:43)

$$= \frac{L^2}{2\lambda_i^2} \left( J_k(\lambda_i) \right)^2$$

$$\frac{d}{dx} \left( x^k J_k(x) \right) = x^k J_{k-1}(x)$$

$$\Rightarrow x^k J_k'(x) + k x^{k-1} J_k(x) = x^k J_{k-1}(x)$$

$$\Rightarrow J_k'(x) = -\frac{k}{x} J_k(x) + J_{k-1}(x)$$

$$\left( J_k'(\lambda_i) \right)^2 = -\frac{k}{\lambda_i} \frac{J_k(\lambda_i)}{\lambda_i} + \left( J_{k-1}(\lambda_i) \right)^2$$

So this is exactly what you have. So you need this is equal to lambda square by 2 lambda i square J alpha minus 1 of lambda i l whole square.

(Refer Slide Time 12:56)

$$\Rightarrow \int_0^L x J_k'(x) dx = \frac{L^2}{2\lambda_i^2} \left( J_k'(\lambda_i) \right)^2 + \frac{L^2}{2\lambda_i^2} \left( J_k(\lambda_i) \right)^2$$

$$= \frac{L^2}{2\lambda_i^2} \left( J_k'(\lambda_i) \right)^2 = \frac{L^2}{2\lambda_i^2} \left( J_{k-1}(\lambda_i) \right)^2$$

$$\frac{d}{dx} \left( x^k J_k(x) \right) = x^k J_{k-1}(x)$$

$$\Rightarrow x^k J_k'(x) + k x^{k-1} J_k(x) = x^k J_{k-1}(x)$$

$$\Rightarrow J_k'(x) = -\frac{k}{x} J_k(x) + J_{k-1}(x)$$

$$\left( J_k'(\lambda_i) \right)^2 = -\frac{k}{\lambda_i} \frac{J_k(\lambda_i)}{\lambda_i} + \left( J_{k-1}(\lambda_i) \right)^2$$

So maybe I missed this part so what you have is, that is what you have. 2 lambda i square that is also there. lambda square by 2 lambda i square so you have actually lambda i square is also here, lambda i square that is the result, Ok. I didn't make any mistake here so the mistake comes here, so actually, you have, when you differentiate this,

(Refer Slide Time 13:30)

Multiply by  $x u' = 2 J_x^\lambda(x)$  both sides to get

$$2 x^\lambda u' u'' + 2 x u'^\lambda + (\lambda x^\lambda - x^\lambda) 2 u u' = 0$$

$$\frac{d}{dx} (x^\lambda u'^\lambda) = 2 x u'^\lambda + 2 x^\lambda u' u''$$

$$\frac{d}{dx} (\lambda x^\lambda u') = 2 \lambda x^\lambda u' + 2 \lambda x^\lambda u''$$

$$\Rightarrow \frac{d}{dx} (\lambda x^\lambda u') - 2 \lambda x^\lambda u'' = 2 \lambda x^\lambda u' u'$$

$$-2 x^\lambda u u' = -\frac{d}{dx} (x^\lambda u'^\lambda)$$

$$\frac{d}{dx} (x^\lambda u'^\lambda) + \frac{d}{dx} (\lambda x^\lambda u') - 2 \lambda x^\lambda u'' - \frac{d}{dx} (x^\lambda u') = 0$$

when you differentiate this, you have 2 lambda a square x square u double dash 3 d x of u dash, i dash, 2 x square u u dash, it's fine no, yeah.

So when you are doing this u dash, so u dash, u is actually, u dash means d d x, d d x of u dash, u dash, u is, u dash of x is, u dash l is d d x of J alpha of lambda i x. This is J alpha dash of lambda i x into lambda i, Ok so when you are differen, when you are squaring it, so you have, this is this square, this square, this whole square equal to this square into lambda i square. So you have, lambda i square comes here.

(Refer Slide Time 14:47)

$\int_0^L \lambda_i x u' dx = \int_0^L \frac{d}{dx} (x^\lambda u'^\lambda + \lambda_i x^\lambda u' - x u) dx$

$$= [x^\lambda u'^\lambda + \lambda_i x^\lambda u' - x u]_0^L \quad (u'(L) \frac{d}{dx} (J_x^\lambda(x_i)) = (J_x^\lambda(x_i))^\lambda \lambda_i)$$

$$= L^\lambda (u'(L))^\lambda + \lambda_i (u(0))^\lambda$$

$$\Rightarrow \int_0^L x J_x^\lambda(x) dx = \frac{L^\lambda \lambda_i}{2 \lambda_i} (J_x^\lambda(x_i))^\lambda + \frac{\lambda_i}{2 \lambda_i} (J_x^\lambda(x_0))^\lambda$$

$$= \frac{L^\lambda}{2 \lambda_i} (J_x^\lambda(x_i))^\lambda = \frac{L^\lambda}{2 \lambda_i} (J_x^{\lambda-1}(x_i))^\lambda$$

$$\frac{d}{dx} (x^\lambda J_x^\lambda(x)) = x^\lambda J_x^{\lambda-1}(x)$$

Ok, so that goes.

So this is, this is what you have. So otherwise it is fine, so we don't have this lambda i square, so that's how you get this lambda without this lambda i square. Similarly if you use other recurrence

(Refer Slide Time 15:10)

Handwritten notes on a whiteboard:

$$\Rightarrow \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_j x) dx = 0, \text{ if } i \neq j$$

(Orthogonal relation).

$$* \text{ if } i = j: \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_i x) dx = \frac{L^2}{2} J_{\alpha+1}^2(\lambda_i L) = \frac{L^2}{2} J_{\alpha-1}^2(\lambda_i L)$$

$u(x) = J_\alpha(\lambda_i x)$  satisfies  $[x^2 u'' + x u' + (\lambda_i^2 x^2 - \alpha^2) u] = 0$

Multiply  $2 u' = 2 J_\alpha'(\lambda_i x)$  both sides to get

$$2 x^2 u' u'' + 2 x u' u' + (\lambda_i^2 x^2 - \alpha^2) 2 u u' = 0$$

$$\frac{d}{dx} (x^2 u'^2) = 2 x u'^2 + 2 x^2 u' u''$$

relation. If other property, if you use the other property, so if I use this one, if I use this property, you get this result, Ok.

If I use the other result that is, d d x of x power minus alpha J alpha of x which is given as x power minus alpha times J alpha plus 1 of x. If I use this one, I get

(Refer Slide Time 15:38)

Handwritten notes on a whiteboard:

$$\Rightarrow \int_0^L x J_\alpha(\lambda_i x) J_\alpha(\lambda_i x) dx = \frac{L^2}{2 \lambda_i^2} (J_\alpha'(\lambda_i L))^2 + \frac{L^2}{2 \lambda_i^2} (J_\alpha(\lambda_i L))^2$$

$$= \frac{L^2}{2 \lambda_i^2} (J_\alpha'(\lambda_i L))^2 = \frac{L^2}{2} (J_{\alpha-1}(\lambda_i L))^2$$

$$\frac{d}{dx} (x^\alpha J_\alpha(x)) = x^\alpha J_{\alpha-1}(x) \quad \frac{d}{dx} (x^{-\alpha} J_\alpha(x)) = -x^{-\alpha} J_{\alpha+1}(x)$$

$$\Rightarrow x^\alpha J_\alpha'(x) + \alpha x^{\alpha-1} J_\alpha(x) = x^\alpha J_{\alpha-1}(x)$$

$$\Rightarrow J_\alpha'(x) = -\frac{\alpha}{x} J_\alpha(x) + J_{\alpha-1}(x)$$

$$(J_\alpha'(\lambda_i L))^2 = -\frac{\alpha}{\lambda_i L} J_\alpha(\lambda_i L) + (J_{\alpha-1}(\lambda_i L))^2$$

other, other result, which is 1 square by j, 1 square by 2 J alpha plus 1 at lambda i 1 whole square, Ok. Both the things so

(Refer Slide Time 15:51)

$$\Rightarrow \int_0^L x J'_x(\lambda x) dx = \frac{L^{-x}}{2 \lambda x^2} (J'_x(\lambda x))' + \frac{x^{-x}}{2 \lambda x^2} (J'_{x-1}(\lambda x))'$$

$$= \frac{L^{-x}}{2 \lambda x^2} (J'_x(\lambda x))' = \frac{L^{-x}}{2} (J'_{x-1}(\lambda x))' = \frac{L^{-x}}{2} (J'_{x+1}(\lambda x))'$$

$$\frac{d}{dx} (x^{-x} J'_x(x)) = x^{-x} J'_{x-1}(x) \quad \frac{d}{dx} (x^{-x} J'_x(x)) = x^{-x} J'_{x+1}(x)$$

$$\Rightarrow x^{-x} J'_x(x) + x^{-x-1} J'_x(x) = x^{-x} J'_{x-1}(x)$$

$$\Rightarrow J'_x(x) = -\frac{x}{\lambda x^2} J'_x(\lambda x) + J'_{x-1}(x)$$

$$(J'_x(\lambda x))' = -\frac{x}{\lambda x^2} J'_x(\lambda x) + (J'_{x-1}(\lambda x))'$$

if you use this one, I get this one. Because I use this one, I got this result. OK that is what is

(Refer Slide Time 15:56)

$$= \frac{L^{-x}}{2 \lambda x^2} (J'_x(\lambda x))' = \frac{L^{-x}}{2} (J'_{x-1}(\lambda x))' = \frac{L^{-x}}{2} (J'_{x+1}(\lambda x))'$$

$$\frac{d}{dx} (x^{-x} J'_x(x)) = x^{-x} J'_{x-1}(x) \quad \frac{d}{dx} (x^{-x} J'_x(x)) = x^{-x} J'_{x+1}(x)$$

$$\Rightarrow x^{-x} J'_x(x) + x^{-x-1} J'_x(x) = x^{-x} J'_{x-1}(x)$$

$$\Rightarrow J'_x(x) = -\frac{x}{\lambda x^2} J'_x(\lambda x) + J'_{x-1}(x)$$

$$(J'_x(\lambda x))' = -\frac{x}{\lambda x^2} J'_x(\lambda x) + (J'_{x-1}(\lambda x))'$$

the, this integral value. This is the, this integral value, Ok. So that's what we find. So

(Refer Slide Time 16:04)

(Orthogonal relation)

$$* \quad i = j: \quad \int_0^L x J_{\alpha}^2(\lambda_i x) dx = \frac{L^2}{2} J_{\alpha+1}^2(\lambda_i L) = \frac{L^2}{2} J_{\alpha-1}^2(\lambda_i L)$$

$$u(x) = J_{\alpha}(\lambda_i x) \text{ satisfies } [x^2 u'' + x u' + (\lambda_i^2 x^2 - \alpha^2) u] = 0$$

Multiply  $2 u' = 2 J_{\alpha}'(\lambda_i x)$  both sides to get

$$2 x^2 u' u'' + 2 x u'^2 + (\lambda_i^2 x^2 - \alpha^2) 2 u u' = 0$$

$$\frac{d}{dx} (x^2 u'^2) = 2 x u'^2 + 2 x^2 u' u''$$

$$\frac{d}{dx} (\lambda_i^2 x^2 u^2) = 2 \lambda_i^2 x^2 u u' + 2 \lambda_i^2 x u^2$$

$$\Rightarrow \frac{d}{dx} (x^2 u'^2) - 2 \lambda_i^2 x u^2 = 2 \lambda_i^2 x^2 u u'$$

this is what we derived.

So if I use the other property you can get this result. That you can do yourself. So, so these are the (()), some important properties of Bessel's, Bessel functions. We have seen some special case when  $2\alpha$  equal to, the difference between the indicial roots,  $2\alpha$  is 1, 3, 5, Ok. So the general solutions, solutions are, two linear independent solutions are  $J_{\alpha} x$ ,  $J_{\alpha - \frac{1}{2}} x$ , Ok.

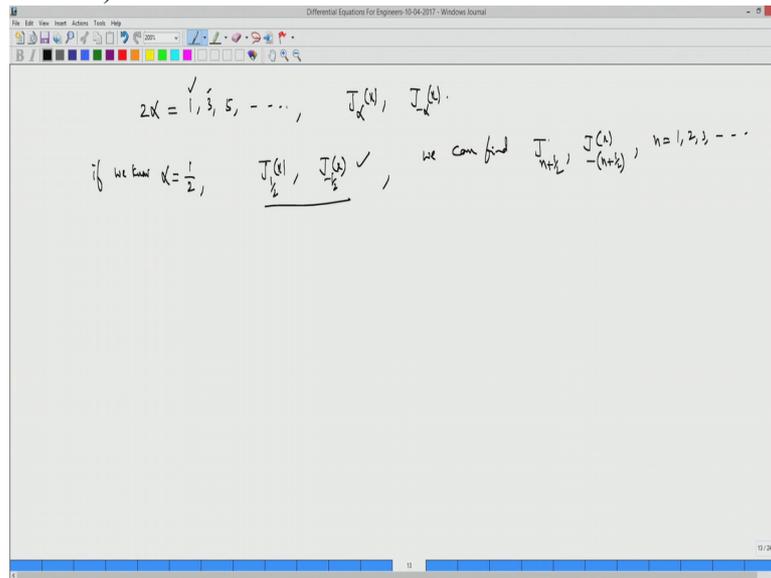
In this special case, for all these cases, you need, if you know one,  $J_{\alpha - \frac{1}{2}}$

(Refer Slide Time 16:50)

$$2\alpha = 1, 3, 5, \dots, \quad J_{\alpha}(x), \quad J_{\alpha - \frac{1}{2}}(x)$$

when alpha equal to half, Ok when alpha equal to half, you get J half x and J minus half x. So if I know this, I can get, from the recurrence, from those two properties I can get all other values, 3, 5. So I can get, if I know this, if we know for alpha equal to this one, we can find, we can find all other things Ok, we can find all other things like J n plus half and J, minus n, rather minus n, minus n plus half, Ok and n is from 1, 2, 3 and so on

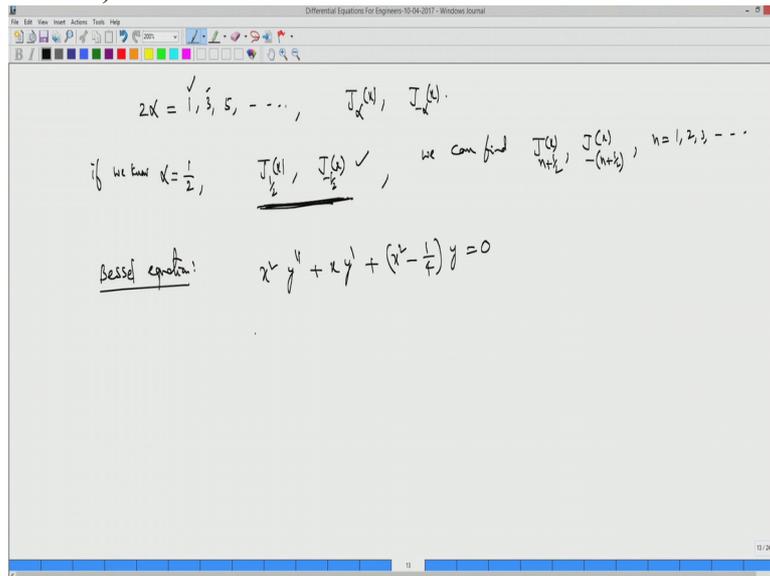
(Refer Slide Time 17:41)



of x, Ok by recurrence relations.

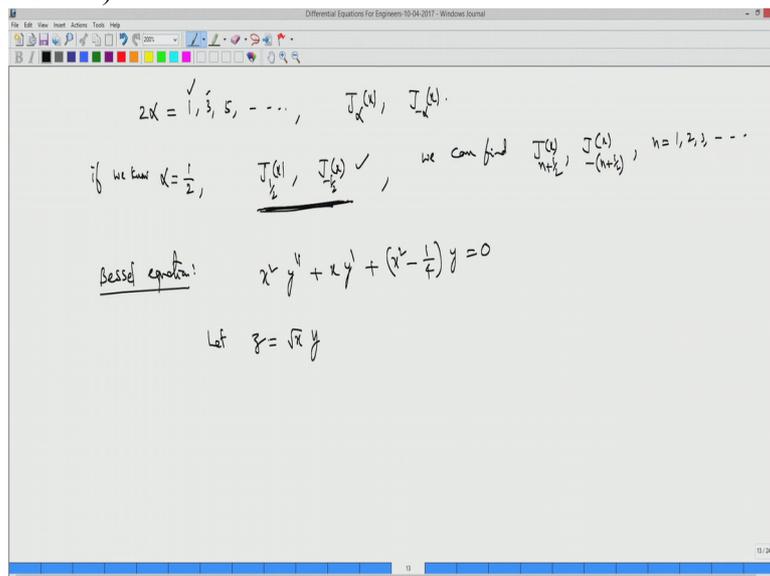
So that is called recurrence relations. Just by properties 1 and 2 if you use you can get them and then you can use. So how do I get this? We can nicely find the formula for J half and J minus half, that we do by simply consider the equation, Bessel equation in this case, Bessel equation is  $x^2 y'' + x y' + (x^2 - \alpha^2) y = 0$ . So this is the equation. So

(Refer Slide Time 18:26)



if I choose some z equal to root x into y. Let be the transformation. Ok. So let dependent variable

(Refer Slide Time 18:37)



I am changing to z, z of x, Ok. If I use that, z of x, so if I use this transformation, what you have is d y by d x equal to, d y by d x is, d y by d x, d y by, d y by d x, this is d z by d x. I want d z by d x, Ok. So d z by d x, you can write d z by d x, so how do I transform this?

So you simply replace d d x of z is root x y, so this you have root x d y by d x plus 1 by 2 root x into y, this is what you replace, d z by d x, Ok. Then what

(Refer Slide Time 19:49)

$2x = 1, 3, 5, \dots, J_x(x), J_{-x}(x).$   
 if we know  $x = \frac{1}{2}$ ,  $J_{\frac{1}{2}}(x), J_{-\frac{1}{2}}(x)$  ✓, we can find  $J_{n+\frac{1}{2}}(x), J_{-(n+\frac{1}{2})}(x), n = 1, 2, 3, \dots$   
Bessel equation:  $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$   
 Let  $z = \sqrt{x} y$   $\frac{dr}{dx} = \frac{d}{dx}(\sqrt{x} y) = \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y$

you need is d square z by d x square. This is equal to d d x of, d z by d x is, we already found so that is d y by d x plus 1 by 2 root x y.

(Refer Slide Time 20:07)

$2x = 1, 3, 5, \dots, J_x(x), J_{-x}(x).$   
 if we know  $x = \frac{1}{2}$ ,  $J_{\frac{1}{2}}(x), J_{-\frac{1}{2}}(x)$  ✓, we can find  $J_{n+\frac{1}{2}}(x), J_{-(n+\frac{1}{2})}(x), n = 1, 2, 3, \dots$   
Bessel equation:  $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$   
 Let  $z = \sqrt{x} y$   $\frac{dr}{dx} = \frac{d}{dx}(\sqrt{x} y) = \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y$   
 $\frac{dz}{dx} = \frac{d}{dx} \left( \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y \right)$

This is equal to 1 by 2 root x d y by d x plus root x d square y by d x square plus d y by d x, 2 root x d y by d x, Ok plus 1 by 2 into x power

(Refer Slide Time 20:35)

$2n = 1, 3, 5, \dots, J_{n+1/2}, J_{n-1/2}$   
 if we take  $n = \frac{1}{2}$ ,  $J_{\frac{1}{2}}, J_{-\frac{1}{2}}$ , we can find  $J_{\frac{3}{2}}, J_{-\frac{3}{2}}$ ,  $n = 1, 2, 3, \dots$   
Bessel equation:  $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$   
 Let  $z = \sqrt{x} y$   $\frac{dz}{dx} = \frac{d}{dx}(\sqrt{x} y) = \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y$   
 $\frac{d^2z}{dx^2} = \frac{d}{dx}(\sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y)$   
 $= \frac{1}{2\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{d^2y}{dx^2} + \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y$

minus half that is, what is the derivative of x power minus half, that is minus half x power minus 3 by 2.

So this is going to be minus, so 1 by 2 into 1 by 2 square, that's going to be 4, x power 3 by 2.

This into y, this is what you have, so substitute

(Refer Slide Time 21:03)

if we take  $n = \frac{1}{2}$ ,  $J_{\frac{1}{2}}, J_{-\frac{1}{2}}$ , we can find  $J_{\frac{3}{2}}, J_{-\frac{3}{2}}$ ,  $n = 1, 2, 3, \dots$   
Bessel equation:  $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$   
 Let  $z = \sqrt{x} y$   $\frac{dz}{dx} = \frac{d}{dx}(\sqrt{x} y) = \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y$   $\frac{d^2z}{dx^2} = \frac{d}{dx}(\sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y)$   
 $= \frac{1}{2\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{d^2y}{dx^2} + \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y$

for d y by d x. From this you can calculate d y by d x. So this is 1 by root x d y by d x plus root x d square y by d x square minus 1 by 4 x cube plus 3 by 2 y.

(Refer Slide Time 21:25)

Bessel equation:  $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$

Let  $z = \sqrt{x} y$       $\frac{dz}{dx} = \frac{d}{dx}(\sqrt{x} y) = \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y$       $\frac{d^2z}{dx^2} = -\frac{1}{4} x^{-3/2} y$

$$\frac{d^2z}{dx^2} = \frac{d}{dx} \left( \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y \right)$$

$$= \frac{1}{2\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{d^2y}{dx^2} + \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y$$

$$= \frac{1}{\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{d^2y}{dx^2} - \frac{1}{4x^{3/2}} y$$

Rather, so instead of these calculations, we can consider,

(Refer Slide Time 21:33)

Bessel equation:  $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$

Let  $z = \sqrt{x} y$       $\frac{dz}{dx} = \frac{d}{dx}(\sqrt{x} y) = \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y$       $\frac{d^2z}{dx^2} = -\frac{1}{4} x^{-3/2} y$

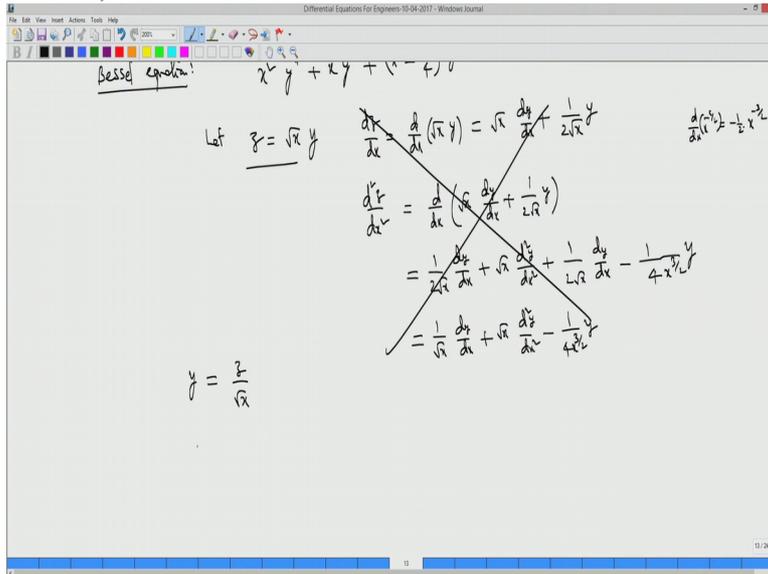
~~$$\frac{d^2z}{dx^2} = \frac{d}{dx} \left( \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y \right)$$

$$= \frac{1}{2\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{d^2y}{dx^2} + \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y$$

$$= \frac{1}{\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{d^2y}{dx^2} - \frac{1}{4x^{3/2}} y$$~~

you can consider, if you choose this transformation, I can get  $dy$  as  $z$  by root  $x$ ,

(Refer Slide Time 21:39)

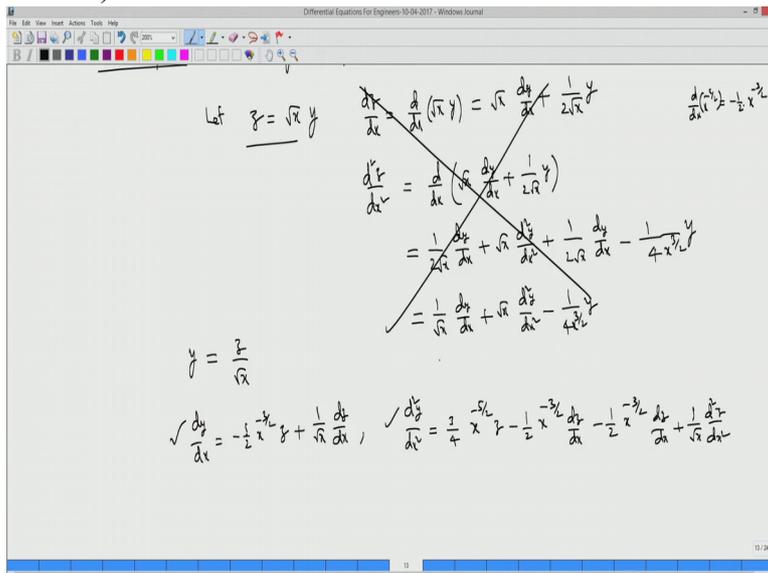


so that you can calculate  $dy$  by  $dx$  as directly.

So you have  $z$  minus half  $x$  power minus 3 by 2  $z$  plus 1 by root  $z$ , root  $x$   $dz$  by  $dx$ . Similarly we can replace, we can get this  $d^2y$  by  $dx^2$  as 3 by 4  $x$  power minus 5 by 2  $z$  plus minus, minus 1 by 2  $x$  power minus 3 by 2  $dz$  by  $dx$  plus or rather minus half  $x$  power minus 3 by 2  $z$  by  $dx$  plus 1 by root  $x$ ,  $d^2z$  by  $dx^2$ .

So once you calculate this  $dy$  by  $dx$  and  $d^2y$  by  $dx^2$ , substitute

(Refer Slide Time 22:36)



into the equation. So if you substitute into the equation, you get  $x^2 y''$  that is  $x^2$  into this part. So you get 3 by 4,  $x^2$  will give you 1 by root  $x$ , 1 by root  $x$   $z$ ,

by root x Ok, minus this, this together will give me minus 1 that is 1, minus 1 x power minus 3 by 2 plus 2 that is root x, root x times d z by d x plus x power 3 by 2 times d square z by d x square.

So this is what is x square y double dash plus x times, x times d y by d x you can write this part, for this you multiply x, so you have minus 1 by 2 root x z plus root x d z by d x and then plus what you have is x square minus 4, x square minus 1 by 4, y is z by root x equal to zero.

(Refer Slide Time 23:55)

The image shows a presentation slide with handwritten mathematical work. The title of the slide is "Differential Equations For Engineers-10-04-2017 - Windows Journal". The work includes the following steps:

$$\frac{dy}{dx} = \frac{d}{dx} \left( \sqrt{x} \frac{dz}{dx} + \frac{1}{2\sqrt{x}} y \right)$$

$$= \frac{1}{2\sqrt{x}} \frac{dz}{dx} + \sqrt{x} \frac{d^2z}{dx^2} + \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y$$

$$= \frac{1}{\sqrt{x}} \frac{dz}{dx} + \sqrt{x} \frac{d^2z}{dx^2} - \frac{1}{4x^{3/2}} y$$

$$y = \frac{z}{\sqrt{x}}$$

$$\sqrt{x} \frac{dy}{dx} = -\frac{1}{2} x^{-3/2} z + \frac{1}{\sqrt{x}} \frac{dz}{dx}, \quad \sqrt{x} \frac{d^2z}{dx^2} = \frac{1}{4} x^{-5/2} z - \frac{1}{2} x^{-3/2} \frac{dz}{dx} - \frac{1}{2} x^{-3/2} \frac{dz}{dx} + \frac{1}{\sqrt{x}} \frac{d^2z}{dx^2}$$

$$\frac{1}{4} \frac{z}{\sqrt{x}} - \sqrt{x} \frac{dz}{dx} + \sqrt{x} \frac{d^2z}{dx^2} - \frac{1}{2\sqrt{x}} z + \sqrt{x} \frac{dz}{dx} + \left( x^{-3/2} - \frac{1}{2} \right) \frac{z}{\sqrt{x}} = 0$$

So you see that nicely this transforms into simpler equation.

As you see, multiply so you can see that x root x d z by d x cancels. And this part, minus 1 by 4 z by root x Ok and 3 by 4,

(Refer Slide Time 24:15)

$$\begin{aligned}
 &= \frac{1}{4x^2} \frac{dy}{dx} + \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y \\
 &= \frac{1}{\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y \\
 y &= \frac{z}{\sqrt{x}} \\
 \sqrt{x} \frac{dy}{dx} &= -\frac{1}{2} x^{-3/2} z + \frac{1}{\sqrt{x}} \frac{dz}{dx}, \quad \sqrt{x} \frac{dy}{dx} = \frac{1}{4} x^{-5/2} z - \frac{1}{2} x^{-3/2} \frac{dz}{dx} - \frac{1}{2} x^{-3/2} \frac{dz}{dx} + \frac{1}{\sqrt{x}} \frac{dz}{dx} \\
 \frac{1}{4} \frac{z}{\sqrt{x}} - \sqrt{x} \frac{dz}{dx} + x^{3/2} \frac{dz}{dx} - \frac{1}{2\sqrt{x}} z + \sqrt{x} \frac{dz}{dx} + \left(x - \frac{1}{2}\right) \frac{z}{\sqrt{x}} &= 0 \\
 \Rightarrow
 \end{aligned}$$

3 by 4 z by root x and minus 1 by 2 z by root x, this becomes zero, so 3 by 4,

(Refer Slide Time 24:25)

$$\begin{aligned}
 &= \frac{1}{4x^2} \frac{dy}{dx} + \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y \\
 &= \frac{1}{\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y \\
 y &= \frac{z}{\sqrt{x}} \\
 \sqrt{x} \frac{dy}{dx} &= -\frac{1}{2} x^{-3/2} z + \frac{1}{\sqrt{x}} \frac{dz}{dx}, \quad \sqrt{x} \frac{dy}{dx} = \frac{1}{4} x^{-5/2} z - \frac{1}{2} x^{-3/2} \frac{dz}{dx} - \frac{1}{2} x^{-3/2} \frac{dz}{dx} + \frac{1}{\sqrt{x}} \frac{dz}{dx} \\
 \frac{1}{4} \frac{z}{\sqrt{x}} - \sqrt{x} \frac{dz}{dx} + x^{3/2} \frac{dz}{dx} - \frac{1}{2\sqrt{x}} z + \sqrt{x} \frac{dz}{dx} + \left(x - \frac{1}{2}\right) \frac{z}{\sqrt{x}} &= 0 \\
 \Rightarrow
 \end{aligned}$$

it becomes minus half minus 1 by 4 z by root x. This part will be zero. So what

(Refer Slide Time 24:30)

$$y = \frac{z}{\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{1}{2}x^{-3/2}z + \frac{1}{\sqrt{x}}\frac{dz}{dx}, \quad \frac{d^2y}{dx^2} = \frac{3}{4}x^{-5/2}z - \frac{1}{2}x^{-3/2}\frac{dz}{dx} - \frac{1}{2}x^{-3/2}\frac{dz}{dx} + \frac{1}{\sqrt{x}}\frac{d^2z}{dx^2}$$

$$\frac{3}{4}\frac{z}{x^{5/2}} - \sqrt{x}\frac{dz}{dx} + x^{3/2}\frac{d^2z}{dx^2} - \frac{1}{2\sqrt{x}}z + \sqrt{x}\frac{dz}{dx} + \left(x^{3/2} - \frac{1}{2}\right)\frac{z}{x^{5/2}} = 0$$

$$\Rightarrow x^{3/2}\frac{d^2z}{dx^2} + \frac{3}{2}x^{1/2}\frac{dz}{dx} = 0$$

you are left with is, simply  $x^3$  by  $2 dx^2 z$  by  $dx^2$  plus  $x^3$  by  $2$  because  $x^2$  divided by  $\sqrt{x}$  into  $z$  equal to zero.

So this will give me, now

(Refer Slide Time 24:45)

$$y = \frac{z}{\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{1}{2}x^{-3/2}z + \frac{1}{\sqrt{x}}\frac{dz}{dx}, \quad \frac{d^2y}{dx^2} = \frac{3}{4}x^{-5/2}z - \frac{1}{2}x^{-3/2}\frac{dz}{dx} - \frac{1}{2}x^{-3/2}\frac{dz}{dx} + \frac{1}{\sqrt{x}}\frac{d^2z}{dx^2}$$

$$\frac{3}{4}\frac{z}{x^{5/2}} - \sqrt{x}\frac{dz}{dx} + x^{3/2}\frac{d^2z}{dx^2} - \frac{1}{2\sqrt{x}}z + \sqrt{x}\frac{dz}{dx} + \left(x^{3/2} - \frac{1}{2}\right)\frac{z}{x^{5/2}} = 0$$

$$\Rightarrow x^{3/2}\frac{d^2z}{dx^2} + \frac{3}{2}x^{1/2}\frac{dz}{dx} = 0$$

$x^3$  both sides it will cancel, you get  $z'' + z = 0$ . So if you use this transformation,

(Refer Slide Time 24:53)

$$\begin{aligned}
 &= \frac{1}{\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y \\
 &= \frac{1}{\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y \\
 \checkmark \quad y &= \frac{z}{\sqrt{x}} \\
 \checkmark \quad \frac{dy}{dx} &= -\frac{1}{2} \frac{z}{x} + \frac{1}{\sqrt{x}} \frac{dz}{dx}, \quad \checkmark \quad \frac{dy}{dx} = \frac{1}{4} x^{-3/2} z - \frac{1}{2} x^{-3/2} \frac{dz}{dx} - \frac{1}{2} x^{-3/2} \frac{dz}{dx} + \frac{1}{\sqrt{x}} \frac{dz}{dx} \\
 \frac{z}{4\sqrt{x}} - \sqrt{x} \frac{dz}{dx} + \sqrt{x} \frac{dz}{dx} - \frac{1}{2\sqrt{x}} z + \sqrt{x} \frac{dz}{dx} + \left( \frac{1}{2} - \frac{1}{2} \right) \frac{z}{\sqrt{x}} &= 0 \\
 \Rightarrow \sqrt{x} \frac{dz}{dx} + \sqrt{x} z &= 0 \Rightarrow z'' + z = 0
 \end{aligned}$$

the equation given, Bessel equation when alpha equals to half becomes simpler. So this we know z x as the general solution of this equation is c 1 cos x plus c 2 sin x. So once I know

(Refer Slide Time 25:09)

$$\begin{aligned}
 \checkmark \quad \frac{dy}{dx} &= -\frac{1}{2} \frac{z}{x} + \frac{1}{\sqrt{x}} \frac{dz}{dx}, \quad \checkmark \quad \frac{dy}{dx} = \frac{1}{4} x^{-3/2} z - \frac{1}{2} x^{-3/2} \frac{dz}{dx} - \frac{1}{2} x^{-3/2} \frac{dz}{dx} + \frac{1}{\sqrt{x}} \frac{dz}{dx} \\
 \frac{z}{4\sqrt{x}} - \sqrt{x} \frac{dz}{dx} + \sqrt{x} \frac{dz}{dx} - \frac{1}{2\sqrt{x}} z + \sqrt{x} \frac{dz}{dx} + \left( \frac{1}{2} - \frac{1}{2} \right) \frac{z}{\sqrt{x}} &= 0 \\
 \Rightarrow \sqrt{x} \frac{dz}{dx} + \sqrt{x} z &= 0 \Rightarrow z'' + z = 0 \\
 \Rightarrow z(x) &= C_1 \cos x + C_2 \sin x
 \end{aligned}$$

z what is my y x, simply c 1 cos x by root x plus c 2 sin x by root x

(Refer Slide Time 25:20)

The image shows a whiteboard with handwritten mathematical work. At the top, there is a title bar for a software application named "Differential Equations For Engineers-10-04-2017 - Windows Journal". Below the title bar is a toolbar with various drawing tools. The main content of the whiteboard consists of several lines of handwritten equations:

$$\frac{z}{4} - \sqrt{x} \frac{dz}{dx} + \frac{3}{2} \frac{dz}{dx} - \frac{1}{2\sqrt{x}} z + \sqrt{x} \frac{dz}{dx} + \left(\frac{x^2}{4}\right) \frac{z}{\sqrt{x}} = 0$$

$$\Rightarrow \cancel{\frac{z}{4}} - \cancel{\sqrt{x} \frac{dz}{dx}} + \frac{3}{2} \frac{dz}{dx} - \frac{1}{2\sqrt{x}} z + \cancel{\sqrt{x} \frac{dz}{dx}} + \left(\frac{x^2}{4}\right) \frac{z}{\sqrt{x}} = 0$$

$$\Rightarrow \frac{3}{2} \frac{dz}{dx} + \frac{z}{\sqrt{x}} = 0 \Rightarrow z'' + z = 0$$

$$\Rightarrow z(x) = C_1 \cos x + C_2 \sin x$$

$$\Rightarrow y(x) = C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}}$$

because  $z = x$  by root  $x$ ,  $z = \sqrt{x}$  by root  $x$  is my general solution  $y$  for the Bessel equation.

So this is the, is the general solution, fine. So this is the general solution of

(Refer Slide Time 25:35)

This image is very similar to the previous one, showing the same handwritten derivation on a whiteboard. The steps are identical, but the final line of the derivation is slightly different:

$$\Rightarrow y(x) = C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}} \text{ is the general solution.}$$

the, what you found so far is the general solution

(Refer Slide Time 25:39)

if we take  $k = \frac{1}{2}$ ,  $J_{\frac{1}{2}}, J_{-\frac{1}{2}}$  ✓, we can find  $J_{n+\frac{1}{2}}, J_{-(n+\frac{1}{2})}$ ,  $n=1,2,3,\dots$

Bessel equation:  $x^2 y'' + x y' + (x^2 - \frac{1}{4}) y = 0$  ✓

Let  $z = \sqrt{x} y$

$$\frac{d}{dx} (\sqrt{x} y) = \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y$$

$$\frac{d^2 z}{dx^2} = \frac{d}{dx} \left( \sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y \right)$$

$$= \frac{1}{2\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{d^2 y}{dx^2} + \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{1}{4x^{3/2}} y$$

$$= \frac{1}{\sqrt{x}} \frac{dy}{dx} + \sqrt{x} \frac{d^2 y}{dx^2} - \frac{1}{4x^{3/2}} y$$

✓  $y = \frac{z}{\sqrt{x}}$  ✓

of the Bessel equation. But we already know that  $J_{\frac{1}{2}}$  and  $J_{-\frac{1}{2}}$  are the two linearly independent solutions, Ok.  $J_{\frac{1}{2}}$  and  $J_{-\frac{1}{2}}$  are two linear independent solutions.

(Refer Slide Time 25:51)

⇒  $x^2 \frac{d^2 z}{dx^2} + x^2 z = 0 \Rightarrow z'' + z = 0$

⇒  $z(x) = C_1 \cos x + C_2 \sin x$

⇒  $y(x) = C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}}$  is the general solution.

$J_{\frac{1}{2}}, J_{-\frac{1}{2}}$  - 2LTI

Here now I found that  $\cos x$  by  $x$   $\sin x$  by  $x$   $\sqrt{x}$  are two linearly independent solutions.

Because

(Refer Slide Time 25:59)

$\Rightarrow \cancel{x} \frac{d^2 y}{dx^2} + \cancel{x} y = 0 \Rightarrow \underline{y'' + y = 0}$   
 $\Rightarrow y(x) = C_1 \cos x + C_2 \sin x$   
 $\Rightarrow y(x) = C_1 \frac{\cos x}{\sqrt{2}} + C_2 \frac{\sin x}{\sqrt{2}}$  is the general solution.

$\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} - 2L \cdot \mathbb{R}$   
 $\frac{\cos x}{\sqrt{2}}, \frac{\sin x}{\sqrt{2}} - 2L \cdot \mathbb{R}$

these 2 are linearly independent, so this is the general solution. So this is how you get. So I found these 2 are linearly independent. So that means this must be one of them

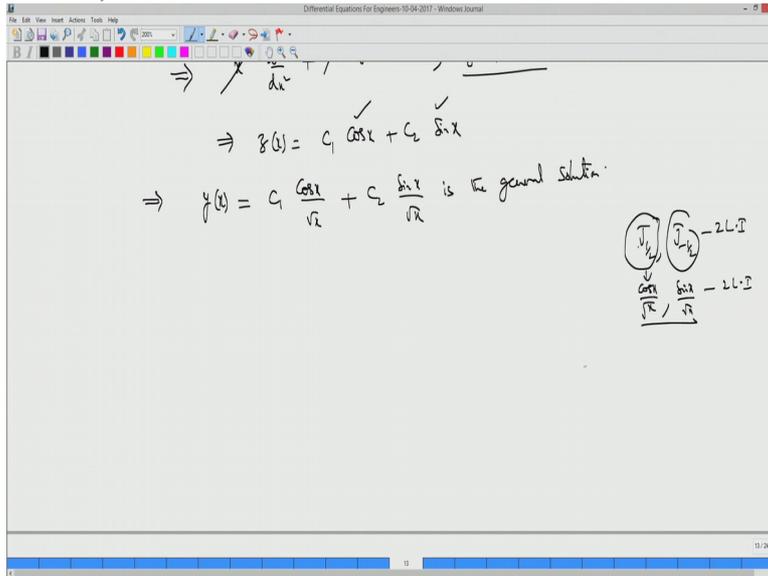
(Refer Slide Time 26:10)

$\Rightarrow \cancel{x} \frac{d^2 y}{dx^2} + \cancel{x} y = 0 \Rightarrow \underline{y'' + y = 0}$   
 $\Rightarrow y(x) = C_1 \cos x + C_2 \sin x$   
 $\Rightarrow y(x) = C_1 \frac{\cos x}{\sqrt{2}} + C_2 \frac{\sin x}{\sqrt{2}}$  is the general solution.

$\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} - 2L \cdot \mathbb{R}$   
 $\frac{\cos x}{\sqrt{2}}, \frac{\sin x}{\sqrt{2}} - 2L \cdot \mathbb{R}$

with some constant multiple, Ok. So this, I should be able to write this in terms of this and this in terms of this, Ok.

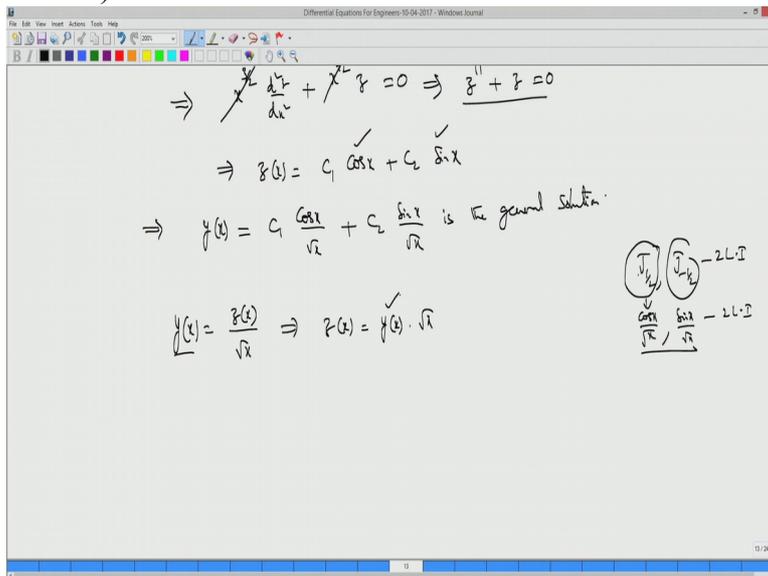
(Refer Slide Time 26:18)



So let us do one in terms of the other, Ok. What we do is, so we know that  $x$  half, so that is what is, right, so you know that  $y$  is, so you know that  $y$  is  $z$  by root  $x$ , right so  $y$ , so we see, let us see  $y$  of  $x$  is, the transformation is  $z$  of  $x$  by root  $x$ , Ok. This is the transformation. That means my  $y$   $x$ , if I, I know that  $y$   $x$ ,  $y$  is satisfying the Bessel equation so that is  $J$  alpha,  $J$  half of  $x$  is one solution, Ok.

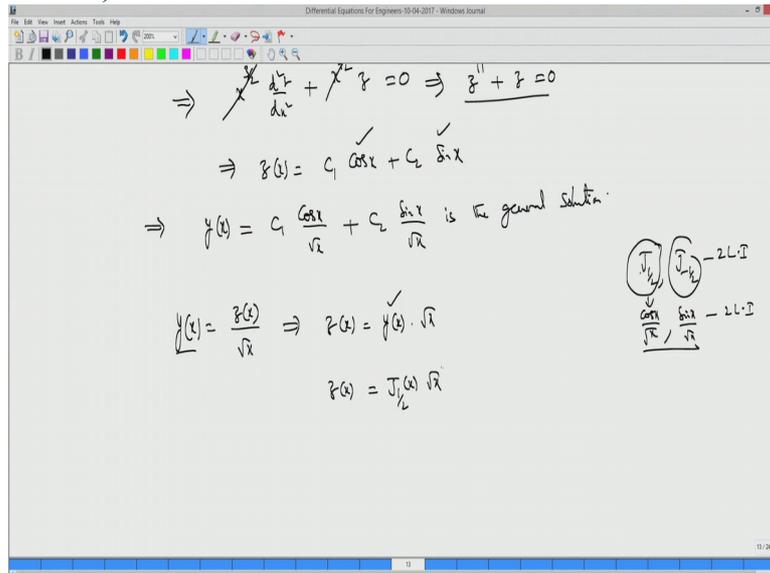
So this is the solution for  $y$  which is equal to  $z$ . Some  $z$  will be, so if I choose my  $y$   $x$ , sorry, I will expand properly, so you have, this implies  $z$   $x$  equal to root  $y$   $x$  into root  $x$ , Ok. So if I choose  $y$

(Refer Slide Time 27:39)



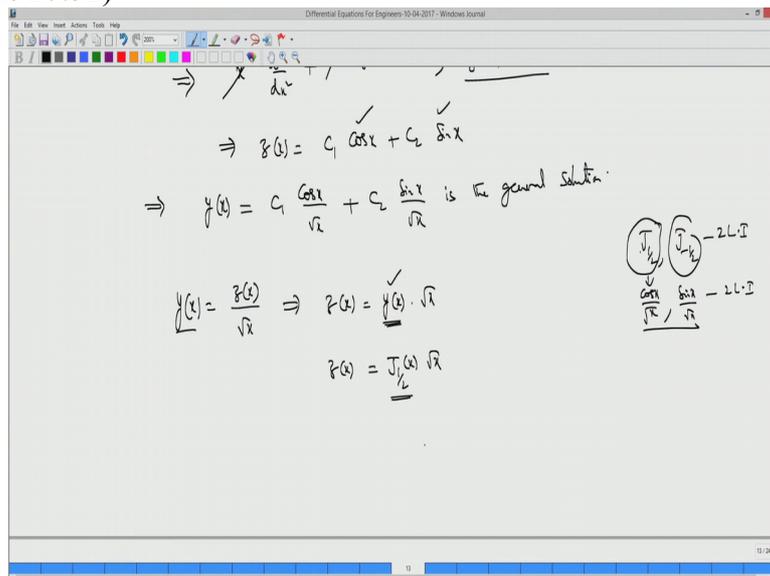
as  $J$  half of  $x$  into root  $x$ , this is also the solution, Ok. That means  $z$   $x$

(Refer Slide Time 27:46)



will be like this. If I choose in the, for  $y$  if I choose as, I know that  $J$  half is the solution,

(Refer Slide Time 27:52)



so once I choose this, root  $x$  into  $J$  half is the solution of this transformed equation, Ok. This satisfies  $z'' + z = 0$ .

So that means

(Refer Slide Time 28:07)

$\Rightarrow \frac{d^2 z}{dx^2} + z = 0 \Rightarrow z'' + z = 0$   
 $\Rightarrow z(x) = C_1 \cos x + C_2 \sin x$   
 $\Rightarrow y(x) = C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}}$  is the general solution.  
 $y(x) = \frac{z(x)}{\sqrt{x}} \Rightarrow z(x) = y(x) \cdot \sqrt{x}$   
 $z(x) = J_{\frac{1}{2}}(x) \sqrt{x}$  satisfies  $z'' + z = 0$   
 $\frac{J_{\frac{1}{2}}(x)}{\sqrt{x}} = \frac{\sin x}{\sqrt{x}} - 2L \cdot I$   
 $\frac{J_{\frac{3}{2}}(x)}{\sqrt{x}} = \frac{\cos x}{\sqrt{x}} - 2L \cdot I$

this is the solution of this equation, one solution. So that implies J half of x into root x is equal to some linear combination of some d 1 times cos x by root x, because these are the solutions, not root x, what are the solutions of z double dash plus z x, cos x plus d 2 sin x. Put x equal to zero,

(Refer Slide Time 28:35)

$y(x) = \frac{z(x)}{\sqrt{x}} \Rightarrow z(x) = y(x) \cdot \sqrt{x}$   
 $z(x) = J_{\frac{1}{2}}(x) \sqrt{x}$  satisfies  $z'' + z = 0$   
 $\Rightarrow \sqrt{x} J_{\frac{1}{2}}(x) = d_1 \cos x + d_2 \sin x$   
 $\frac{J_{\frac{1}{2}}(x)}{\sqrt{x}} = \frac{\sin x}{\sqrt{x}} - 2L \cdot I$   
 $\frac{J_{\frac{3}{2}}(x)}{\sqrt{x}} = \frac{\cos x}{\sqrt{x}} - 2L \cdot I$

that will give me d 1 equal to zero, Ok. So that means this is not true.

(Refer Slide Time 28:43)

$y(x) = \frac{P(x)}{\sqrt{x}} \Rightarrow P(x) = \sqrt{x} \cdot J_{\frac{1}{2}}(x)$   
 $P(x) = J_{\frac{1}{2}}(x) \sqrt{x}$  satisfies  $P'' + P = 0$   
 $\Rightarrow \sqrt{x} J_{\frac{1}{2}}(x) = d_1 \cos x + d_2 \sin x$   
 $\Rightarrow d_1 = 0$  if  $x=0$ .

$\frac{J_{\frac{1}{2}}(x)}{\sqrt{x}} = \frac{\sin x}{\sqrt{x}} - 2L \cdot I$

So this implies root x J half of x is actually equal to d 2 sin x.

(Refer Slide Time 28:51)

$y(x) = \frac{P(x)}{\sqrt{x}} \Rightarrow P(x) = \sqrt{x} \cdot J_{\frac{1}{2}}(x)$   
 $P(x) = J_{\frac{1}{2}}(x) \sqrt{x}$  satisfies  $P'' + P = 0$   
 $\Rightarrow \sqrt{x} J_{\frac{1}{2}}(x) = d_1 \cos x + d_2 \sin x$   
 $\Rightarrow d_1 = 0$  if  $x=0$ .

$\frac{J_{\frac{1}{2}}(x)}{\sqrt{x}} = \frac{\sin x}{\sqrt{x}} - 2L \cdot I$

$\Rightarrow \sqrt{x} J_{\frac{1}{2}}(x) = d_2 \sin x$

So how do we find my d 2? So that I can get my J half in nice expression. So this implies you write simply d 2 as root x J half of x divided by sin x, Ok. If d 2 is constant, now if I can take this limit x, this should be true, this should be a constant. Even at x equal to zero or any other value it should give the same value, Ok. So at x equal to zero if I take this is well-defined at x equal to zero, well-defined,

(Refer Slide Time 29:26)

$$p(x) = J_{\frac{1}{2}}(x) \sqrt{x} \text{ satisfies } p'' + p = 0$$

$$\Rightarrow \sqrt{x} J_{\frac{1}{2}}(x) = d_1 \cos x + d_2 \sin x \Rightarrow \sqrt{x} J_{\frac{1}{2}}(x) = d_2 \sin x$$

$$\Rightarrow d_1 = 0 \text{ if } x=0.$$

$$\Rightarrow d_2 = \lim_{x \rightarrow 0} \frac{\sqrt{x} J_{\frac{1}{2}}(x)}{\sin x}$$

well-defined at  $x$  equal to zero. So I can consider this limit. This is equal to, limit  $x$  close to zero, this I write  $\sin x$  by  $x$ , so that if I write  $\sin x$  by  $x$ , I should have, I should have  $J$  half of  $x$  by root  $x$ , Ok.

(Refer Slide Time 29:50)

$$p(x) = J_{\frac{1}{2}}(x) \sqrt{x} \text{ satisfies } p'' + p = 0$$

$$\Rightarrow \sqrt{x} J_{\frac{1}{2}}(x) = d_1 \cos x + d_2 \sin x \Rightarrow \sqrt{x} J_{\frac{1}{2}}(x) = d_2 \sin x$$

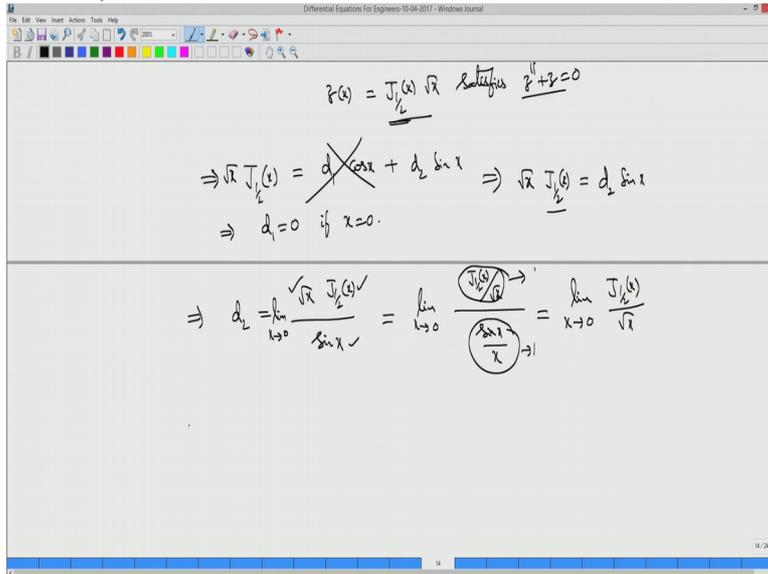
$$\Rightarrow d_1 = 0 \text{ if } x=0.$$

$$\Rightarrow d_2 = \lim_{x \rightarrow 0} \frac{\sqrt{x} J_{\frac{1}{2}}(x)}{\sin x} = \lim_{x \rightarrow 0} \frac{J_{\frac{1}{2}}(x)/\sqrt{x}}{x/x}$$

These two are same.

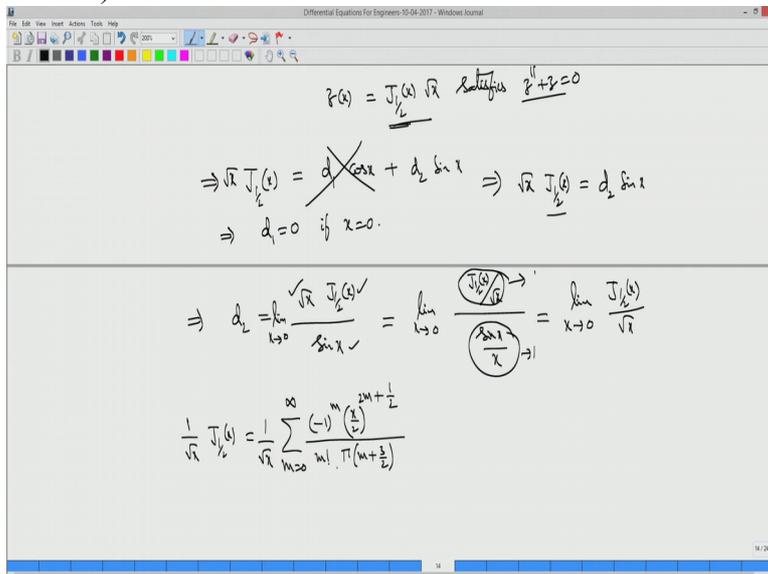
So this is nothing but, now you see that bottom, this limit goes to 1, Ok. This goes to 1 and this goes to simply 1, what is the value? So this you have to calculate, Ok. So this is actually equal to limit  $J$  half of  $x$  by root  $x$ ,  $x$  close to zero. So we will find what that  $J$  alpha,  $J$  half of that, Ok.

(Refer Slide Time 30:20)



So 1 by root x, J half of x is actually equal to, what you get, you will get sigma m is from zero to infinity minus 1 power m, x by 2 power 2 m plus alpha that is half, Ok divided by m factorial into gamma of m plus 3 by 2, m plus half plus 1, this is what is the J half of x, I am dividing with 1 by root x. This is my J half.

(Refer Slide Time 31:05)



If I divide with this thing, this you will get simply m is from zero to infinity, minus 1 power m x by 2, x power 2 m plus, x power 2 m divided by, x x goes Ok, root x and I have 2 power and a root, Ok, 2 power 2 m plus half into m factorial, gamma m plus 3 by 2.

So this is what, for which I am looking for the limit, Ok. So this is, if you take the limit x close to zero, this limit x close to zero,

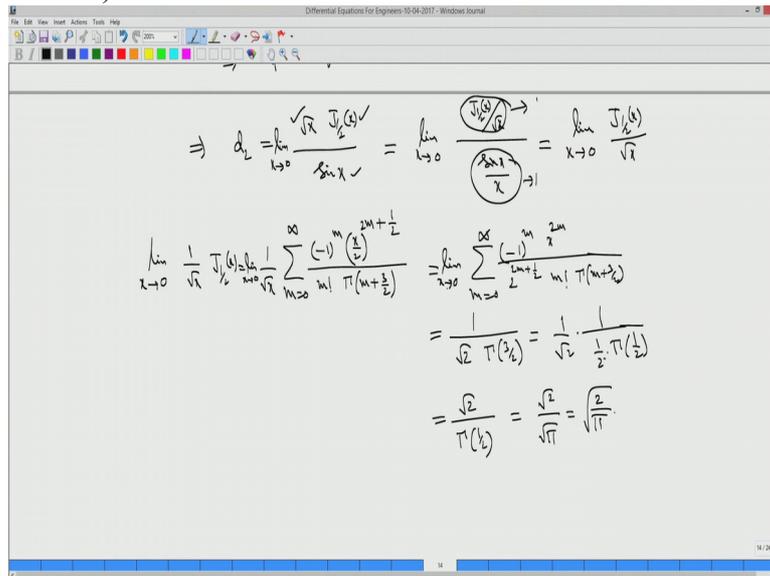
(Refer Slide Time 31:49)

so what happens, so if I, so these are the terms starting from constant plus x square, x power 4. All the x square, x power 4 coefficients everything will be zero as x close to zero. Only that constant term that is corresponds m equal to zero, you have simply 1 divided by root 2 m factorial, zero factorial is 1, gamma of 3 by 2. This is 1 by root 2, 1 by 2 into gamma half, Ok.

(Refer Slide Time 32:21)

So this is nothing but, 2, 2 so this becomes 1 by root 2, that becomes root 2 plus numerator into gamma half. So gamma half is root 2 by pi. So root, gamma half is root pi, Ok, so you get root 2 by pi.

(Refer Slide Time 32:40)



$$\Rightarrow d_1 = \lim_{x \rightarrow 0} \frac{\sqrt{x} J_{\frac{1}{2}}(x)}{\sin x} = \lim_{x \rightarrow 0} \frac{J_{\frac{1}{2}}(x)}{\frac{\sin x}{\sqrt{x}}} = \lim_{x \rightarrow 0} \frac{J_{\frac{1}{2}}(x)}{\sqrt{x}}$$

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} J_{\frac{1}{2}}(x) = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m! \Gamma\left(m+\frac{3}{2}\right)}$$

$$= \lim_{x \rightarrow 0} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\frac{1}{2}} m! \Gamma\left(m+\frac{3}{2}\right)}$$

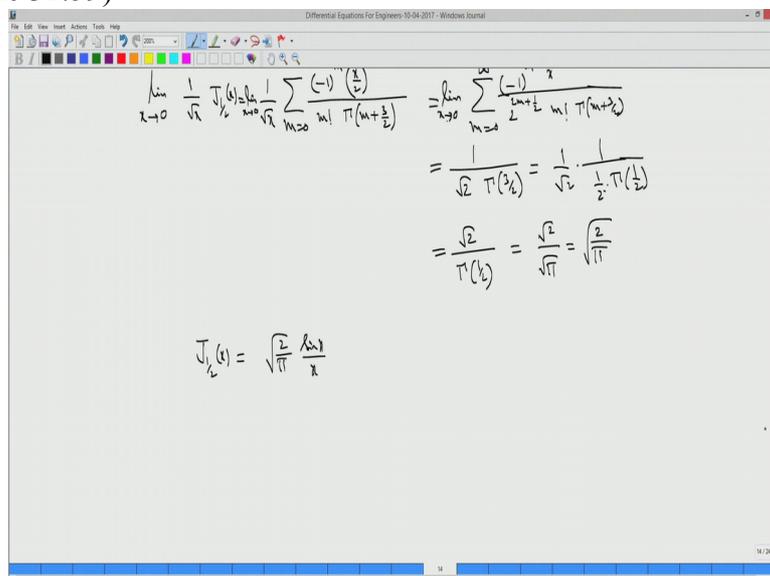
$$= \frac{1}{\sqrt{2} \Gamma\left(\frac{3}{2}\right)} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\frac{1}{2} \Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)} = \frac{\sqrt{2}}{\sqrt{\pi}} = \sqrt{\frac{2}{\pi}}$$

So once you get root 2 by, go and put it, so this is what you get as root 2 by pi.

So that implies, so from here, you can get J half of x, J half of x is nothing but root 2 by pi sin x by x,

(Refer Slide Time 32:59)



$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} J_{\frac{1}{2}}(x) = \lim_{x \rightarrow 0} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{2^{2m+\frac{1}{2}} m! \Gamma\left(m+\frac{3}{2}\right)}$$

$$= \frac{1}{\sqrt{2} \Gamma\left(\frac{3}{2}\right)} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\frac{1}{2} \Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)} = \frac{\sqrt{2}}{\sqrt{\pi}} = \sqrt{\frac{2}{\pi}}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}$$

Ok. So if you want gamma of, gamma half, how to find gamma half, gamma z plus 1 is integral zero to infinity, that's what we defined yesterday, right, so we have, gamma is e power minus x, x power z minus 1, z, x power z d x. So if you want gamma half, you take gamma z is z minus 1 d x. So you take

(Refer Slide Time 33:39)

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} J_{\frac{1}{2}}(x) = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m! \Gamma(m+\frac{1}{2})} = \lim_{x \rightarrow 0} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\frac{1}{2}} m! \Gamma(m+\frac{1}{2})}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\Gamma(\frac{1}{2})} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\frac{1}{2} \Gamma(\frac{1}{2})}$$

$$= \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} = \frac{\sqrt{2}}{\sqrt{\pi}} = \frac{\sqrt{2}}{\sqrt{\pi}}$$

$$J_{\frac{1}{2}}(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin x}{x}$$

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx$$

z equal to half, so gamma half, if you calculate, so zero to infinity e power minus x, x power, z is half so you have minus half d x.

(Refer Slide Time 33:50)

$$\lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} J_{\frac{1}{2}}(x) = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m! \Gamma(m+\frac{1}{2})} = \lim_{x \rightarrow 0} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\frac{1}{2}} m! \Gamma(m+\frac{1}{2})}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\Gamma(\frac{1}{2})} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\frac{1}{2} \Gamma(\frac{1}{2})}$$

$$= \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} = \frac{\sqrt{2}}{\sqrt{\pi}} = \frac{\sqrt{2}}{\sqrt{\pi}}$$

$$J_{\frac{1}{2}}(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin x}{x}$$

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx$$

$$\Gamma\left(\frac{z}{2}\right) = \int_0^{\infty} e^{-x} x^{\frac{z}{2}-1} dx$$

So you just do the integration by parts and you can find that its value is 1 by root pi you will get, Ok so you can do that as an exercise. So this is

(Refer Slide Time 34:04)

$$= \frac{1}{\sqrt{2} \Gamma(\frac{3}{2})} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\frac{1}{2} \Gamma(\frac{1}{2})}$$

$$= \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} = \frac{\sqrt{2}}{\sqrt{\pi}} = \sqrt{\frac{2}{\pi}}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}$$

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$$

$$\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-x} x^{-\frac{1}{2}} dx = \sqrt{\pi} \checkmark$$

gamma half is root pi, so you will get that. So you use that to get this one.

(Refer Slide Time 34:12)

$$= \frac{1}{\sqrt{2} \Gamma(\frac{3}{2})} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\frac{1}{2} \Gamma(\frac{1}{2})}$$

$$= \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} = \frac{\sqrt{2}}{\sqrt{\pi}} = \sqrt{\frac{2}{\pi}}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}$$

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$$

$$\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-x} x^{-\frac{1}{2}} dx = \sqrt{\pi} \checkmark$$

Now to find what is the, next question is J half of x. What is my J half of x? This also you can show that root 2 by pi cos x by x Ok, how do I show

(Refer Slide Time 34:26)

The image shows a software window titled "Differential Equations For Engineers-10-04-2017 - Windows Journal". The main content is handwritten mathematical work. On the left, the limit of the Bessel function of the first kind is derived as  $x \rightarrow 0$ . The expression is  $\lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} J_{\frac{1}{2}}(x) = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m! \Gamma(m+\frac{1}{2})}$ . This is simplified to  $\lim_{x \rightarrow 0} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\frac{1}{2}} m! \Gamma(m+\frac{1}{2})}$ , which then becomes  $\frac{1}{\sqrt{2} \Gamma(\frac{1}{2})} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\frac{1}{2} \Gamma(\frac{1}{2})}$ . Finally, it is shown that  $\frac{1}{\sqrt{2} \Gamma(\frac{1}{2})} = \frac{\sqrt{2}}{\sqrt{\pi}} = \sqrt{\frac{2}{\pi}}$ . Below this, the Bessel function is expressed as  $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}$  and  $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{x}$ . On the right side, the Gamma function is defined as  $\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx$  and  $\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-x} x^{-\frac{1}{2}} dx = \sqrt{\pi}$ .

that? That is not difficult to show.

So we use the property that, what is the property, that root x right, this should be root x, sin x by root x, so here also, cos x by root x, Ok,

(Refer Slide Time 34:59)

The image shows a software window titled "Differential Equations For Engineers-10-04-2017 - Windows Journal". The main content is handwritten mathematical work. On the left, the Bessel function is expressed as  $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{x}$  and  $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{x}$ . On the right side, the Gamma function is defined as  $\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx$  and  $\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-x} x^{-\frac{1}{2}} dx = \sqrt{\pi}$ .

this is what you get, this is what you want to show, Ok. To show, you can also show that this one, this is,



(Refer Slide Time 36:07)

Handwritten mathematical derivation on a whiteboard:

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{\sqrt{x}}$$

To show  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{\sqrt{x}}$

$$\frac{d}{dx} \left( x^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \right) = x^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \quad (\text{Property 1})$$

$$\text{L.H.S} \Rightarrow \frac{d}{dx} \left( \sqrt{\frac{2}{\pi}} \cos x \right) = x^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)$$

Gamma function definition:

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-x} x^{-\frac{1}{2}} dx = \sqrt{\pi}$$

So this gives me, what happens, so right, this property 1 if you use, this is what you will get but I know that  $d(x \sqrt{\frac{2}{\pi}} \cos x)$  into  $J_{\frac{1}{2}}$  is, I already know from here, so if you do that this is nothing but  $\sqrt{\frac{2}{\pi}} \cos x$  equal to  $x^{\frac{1}{2}} J_{\frac{1}{2}}$ .

(Refer Slide Time 36:36)

Handwritten mathematical derivation on a whiteboard:

To show  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{\sqrt{x}}$

$$\frac{d}{dx} \left( x^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \right) = x^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) \quad (\text{Property 1})$$

$$\text{L.H.S} \Rightarrow \frac{d}{dx} \left( \sqrt{\frac{2}{\pi}} \cos x \right) = x^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \cos x = \sqrt{x} \Gamma\left(\frac{1}{2}\right)$$

Gamma function definition:

$$\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-x} x^{-\frac{1}{2}} dx = \sqrt{\pi}$$

So implies  $J_{\frac{1}{2}}$  is  $\sqrt{\frac{2}{\pi}} \cos x$ , what you will get, Ok.

(Refer Slide Time 36:47)

$$\text{To show } J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{\sqrt{x}} \quad \checkmark$$

$$\frac{d}{dx} (x^{k/2} J_{\frac{k}{2}}(x)) = x^{k/2} J_{\frac{k}{2}-1}(x) \quad (\text{Property 1})$$

$$\text{L.H.S} = \frac{d}{dx} \left( \sqrt{\frac{2}{\pi}} \cos x \right) = x^{k/2} J_{\frac{k}{2}-1}(x)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \cos x = \sqrt{x} J_{\frac{1}{2}}(x)$$

$$\Rightarrow J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x \quad \checkmark$$

So here, we already know this one, Ok, this is what you

(Refer Slide Time 36:51)

$$\Rightarrow J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{\sqrt{x}}$$

$$\text{To show } J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{\sqrt{x}} \quad \checkmark$$

$$\frac{d}{dx} (x^{k/2} J_{\frac{k}{2}}(x)) = x^{k/2} J_{\frac{k}{2}-1}(x) \quad (\text{Property 1})$$

$$\text{L.H.S} = \frac{d}{dx} \left( \sqrt{\frac{2}{\pi}} \cos x \right) = x^{k/2} J_{\frac{k}{2}-1}(x)$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \cos x = \sqrt{x} J_{\frac{1}{2}}(x)$$

$$\Rightarrow T(x) = \sqrt{\frac{2}{\pi}} \cos x \quad \checkmark$$

$$T(\frac{1}{2}) = \int_0^{\infty} e^{-x} x^{2-1} dx$$

$$T(\frac{1}{2}) = \int_0^{\infty} e^{-x} x^{-k} dx$$

$$= \sqrt{\pi} \quad \checkmark$$

have shown. So you have nicely get J half and J minus half.

Ok, so

(Refer Slide Time 36:58)



we will, we have used the property 1 and property 2. You just get, you just expand property 1 and property 2 and add and subtract them to get 2 more relations. Using that you can actually get all  $J_{3/2}$ ,  $J_{5/2}$ , all  $J$ ,  $J_{3/2}$ ,  $J_{5/2}$ , or  $J_{-3/2}$ ,  $J_{-5/2}$  and so on, Ok. All these things you can get it as nice expressions in terms of  $\cos x$  and  $\sin x$  and  $\sqrt{x}$ , Ok, in terms of this you can express. We will see this in the next video.