Differential Equations for Engineers
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Lecture No 31
Properties of Bessel Functions (continued)

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So we were looking at the orthogonal relation between the Bessel functions. We have seen, we have shown that they are actually orthogonal to each other when they are different. So we look at the other, other part, that when you take the same function in that relation that, when you take lambda i equal to lambda j. That is same as

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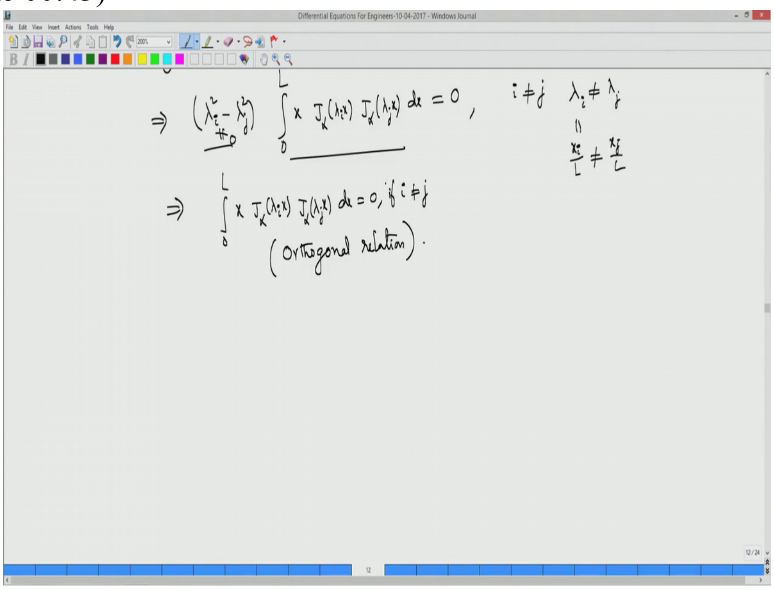

i equal to j, the case when i equal to j.

What if i equal to j this case what you have is zero to l x times J alpha of lambda i x d x so i is, both are same, this is actually equal to, so we will see that this will be equal to some number that is let me write this a square by 2 sorry l square by 2 j square n plus 1 of lambda i l. Or this is same as l square by 2 j n minus 1 square of lambda i l. This is not n, this is alpha. So this is alpha, this is what you will get, Ok.

So this is the, either this or this.

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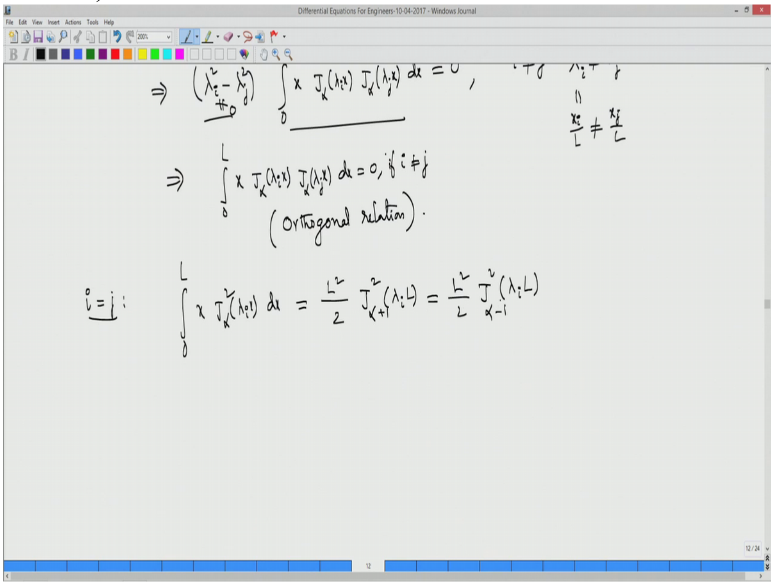

So we will show how it is done, Ok, this part, so this part what we do is you take the equations. So we will only have one, one function so that is J alpha of lambda i x satisfies Bessel equation that is x square that is J alpha double dash of lambda i x plus x J alpha dash of lambda i x, Ok.

So let us call this may be some function of x, u x,

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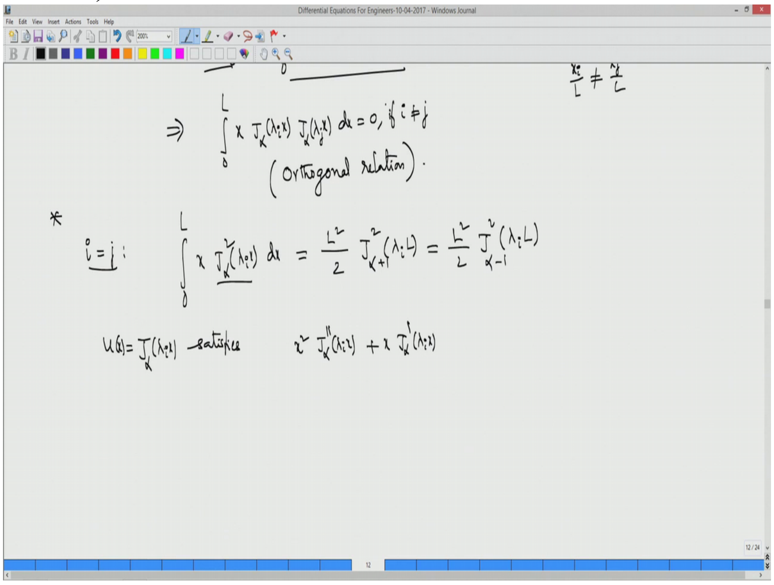

Ok so then you have u double dash, that will be easier for you. So x u dash plus x square, so you have here, this is lambda i square x square minus alpha square into u, u of x equal to zero. So u is J alpha of lambda i x, Ok. What I do is for this equation, I multiply,

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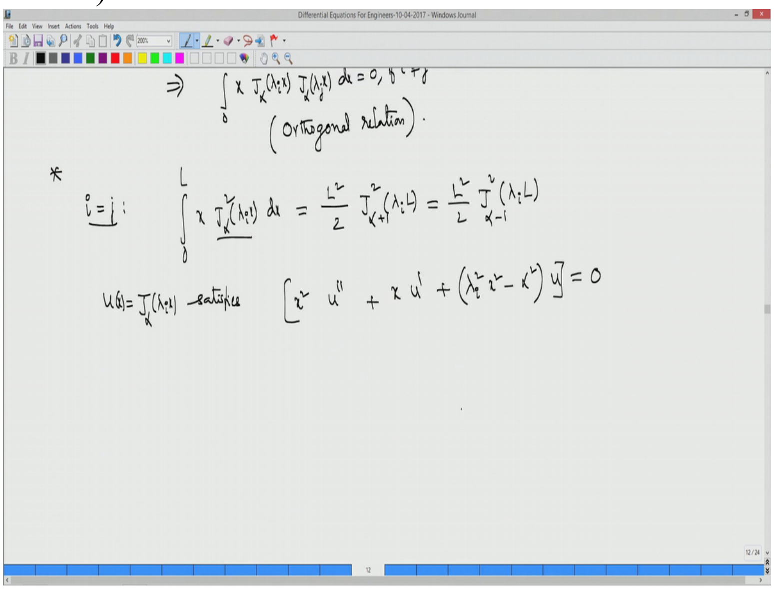

I multiply simply 2 u dash something, Ok, so, if I, this is zero, so both sides I multiply 2 u dash, multiply 2 u dash.

That is 2 times J alpha dash of lambda i x. So if I do both sides to get,

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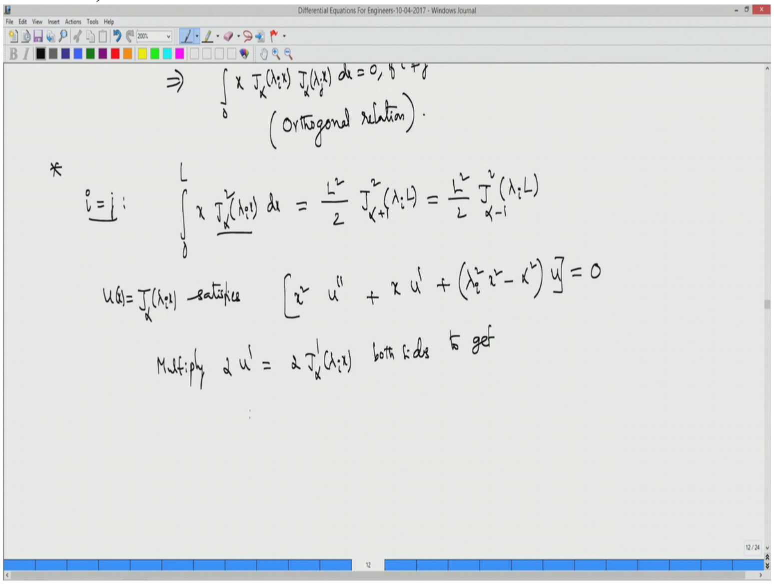

what you get if you do this, you have 2 u dash x square, 2 x square rather, if you do it 2 x square, u dash, u double dash plus 2 x u dash square plus lambda i square x square minus alpha square times 2 u u dash equal to zero, Ok.

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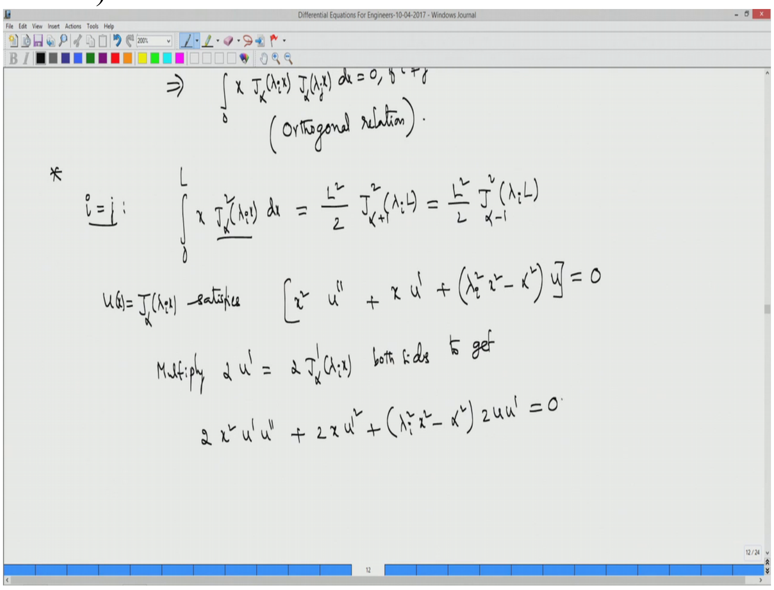

So what the next step is, you can see that 2 u u dash, 2 u dash, so you have this one, so you just have to write this in a simplified manner.

That is, so you can see that x square u dash square, what is the derivative of this? Derivative of this if you consider, this is equal to x square, there is 2 x, if you do this, this is 2 x u dash square plus x square, so you have 2 u dash u double dash, so you can see that this, this together,

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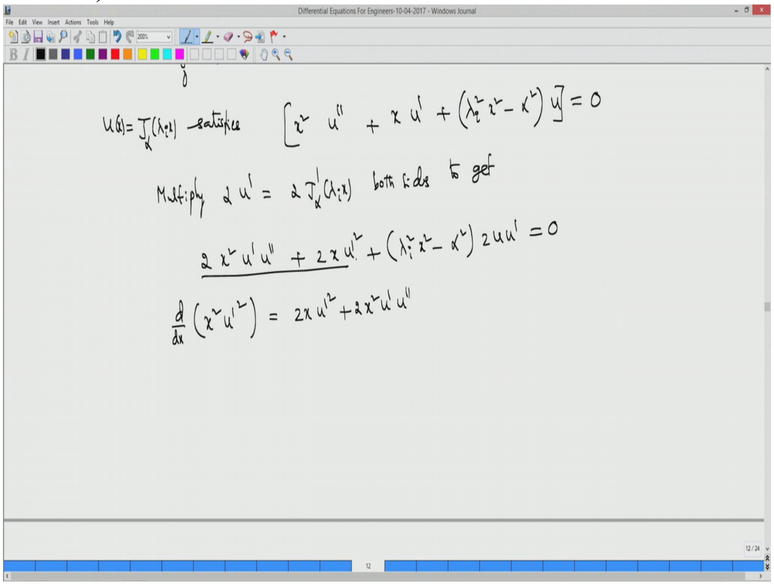

so these two parts you can write like as a derivative, Ok.

So what happens the other part is d d x of, d d x of lambda i square x square u square. What is this one, this one lambda i square x square 2 u u dash, if I differentiate u square. Now one more term is there, that is lambda i square, you differentiate x square so you get 2 x u square. This is what you need, and then so, so I can replace this part, this part with this.

So this implies, so we can rewrite d d x of lambda i square x square u square minus 2 lambda i square x u square, I can replace with 2 lambda i square x square u u dash,

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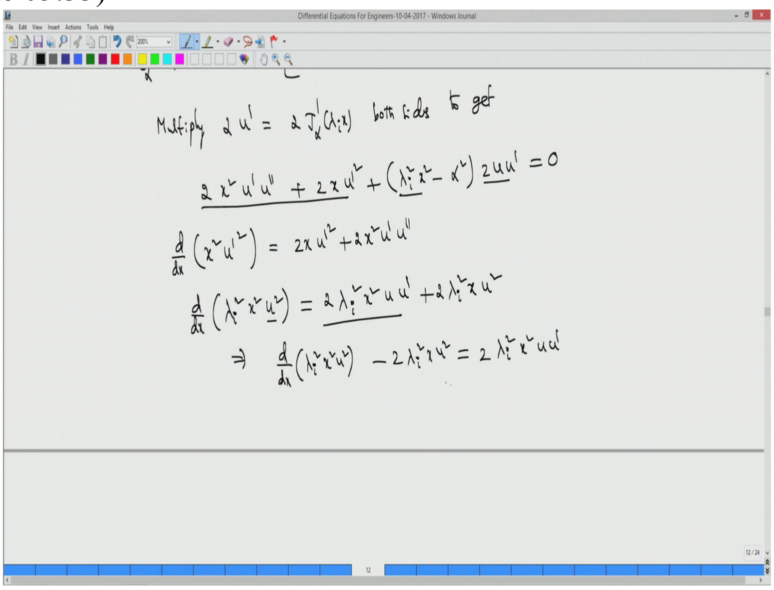

Ok. So what is left is minus 2 alpha square u u dash. This I can write like minus d d x of alpha square u square, Ok. So replace each of these, with

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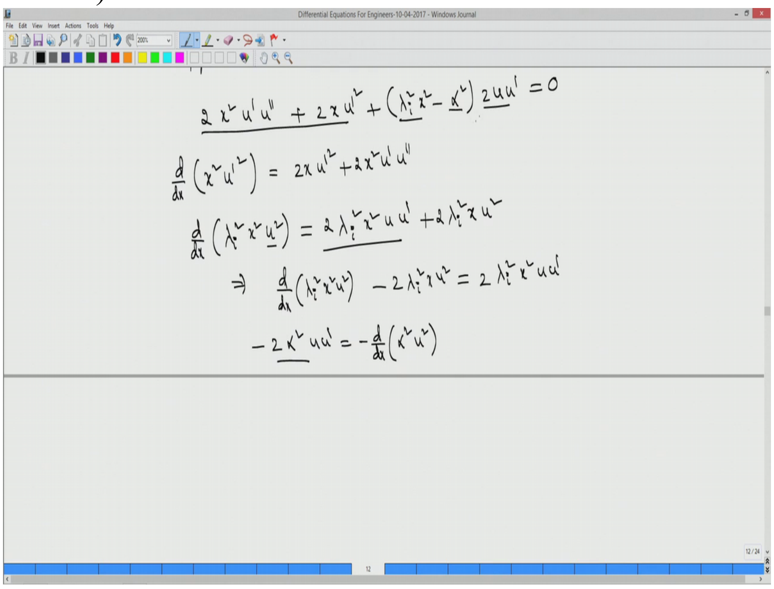

alpha square, this is with this, as a derivatives. If I do this, you can write the first two terms, I can write like d d x of square u dash square Ok plus d d x of lambda i square x square u square minus 2 lambda i square x u square that is what is this term, first term in the bracket. And what is left is finally minus d d x of alpha square u square equal to zero.

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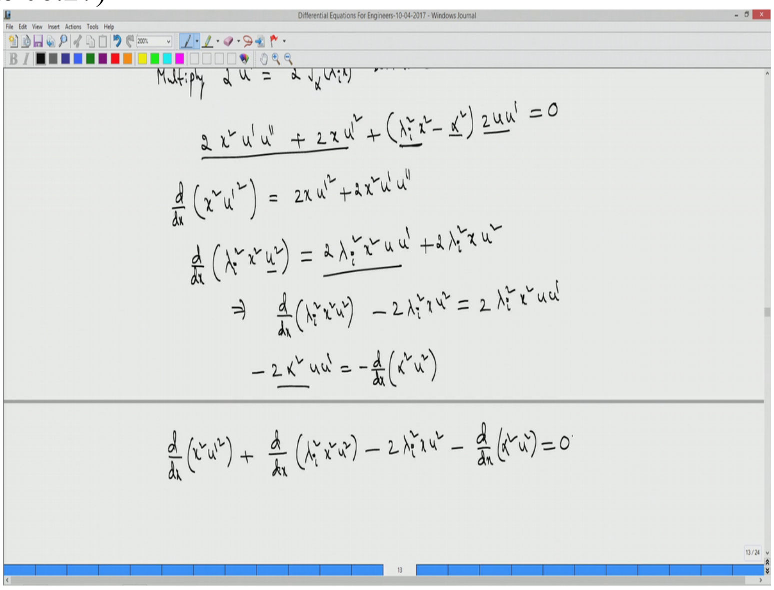

So derivatives you put it together, so what you have is, this you take it the other side, so you have lambda i square x u square equal to d d x of x square u dash square plus lambda i square x square u square minus alpha square u square. Now you differentiate both sides from zero to l d x, zero to

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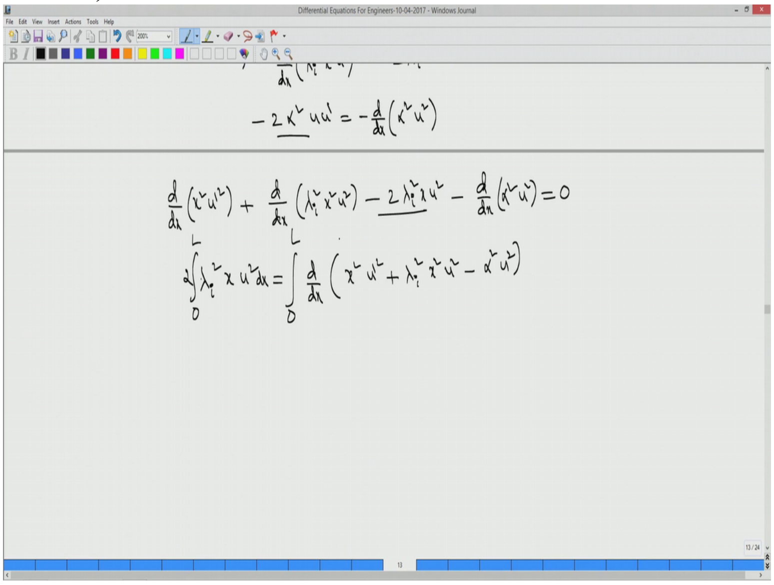

l d x.

This is equal to x square u dash square plus lambda i square x square u square minus alpha square, u square we put the values from zero to l.

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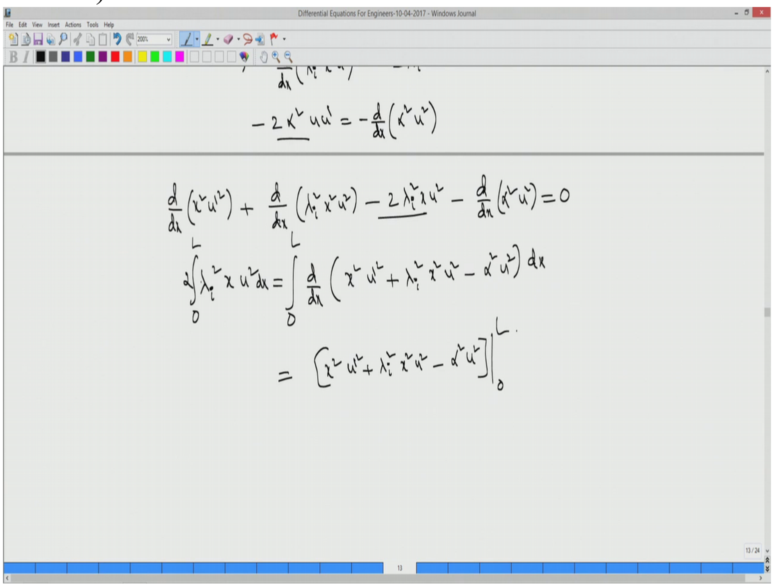

Now you see that u at l is zero, Ok so this nothing but it will give you l square u dash at l square, Ok, minus so when you put zero, this is zero, this is zero and you have minus minus plus, alpha square u at zero square.

Do this properly so you have lambda i square 2 x square so 2 x square lambda i square u u dash I replace with this, minus 2 lambda i square x u square, Ok. So you have x square this, alpha square u square that is finally alpha square u square, Ok. So this is what you get.

So

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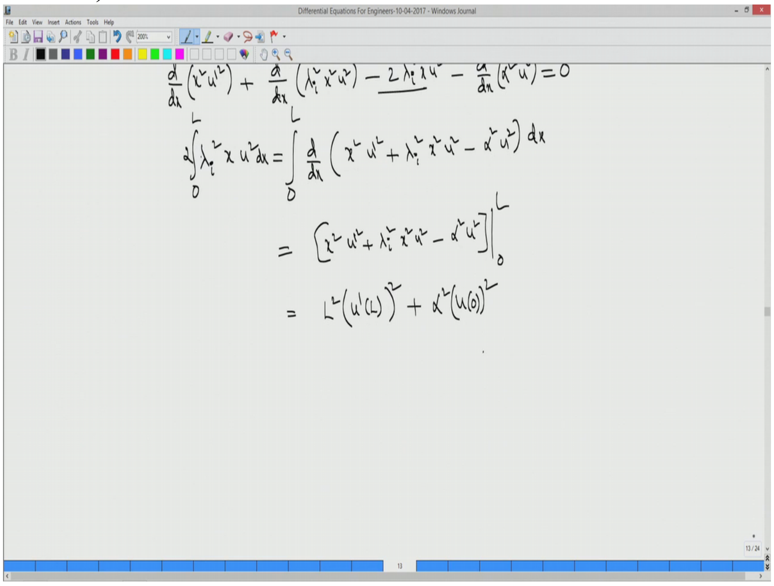

when you replace x by l, u is zero at l, u is zero at l, so this will contribute to l square, this and you put minus, minus of this and you put x equal to zero, this is zero, this is zero, this is going to be plus alpha square this one. So this is what you have. So this divided by 2, so this implies, so what you get is zero to l lambda i square, lambda i square comes out, so this is a constant so you have x, u is J alpha of lambda i x square, Ok d x is equal to 1 by, so that is l square by 2 lambda i square, this is J alpha dash of lambda i x whole square plus alpha square by 2 lambda i square into J alpha of lambda i at zero, Ok, this square.

But J alpha

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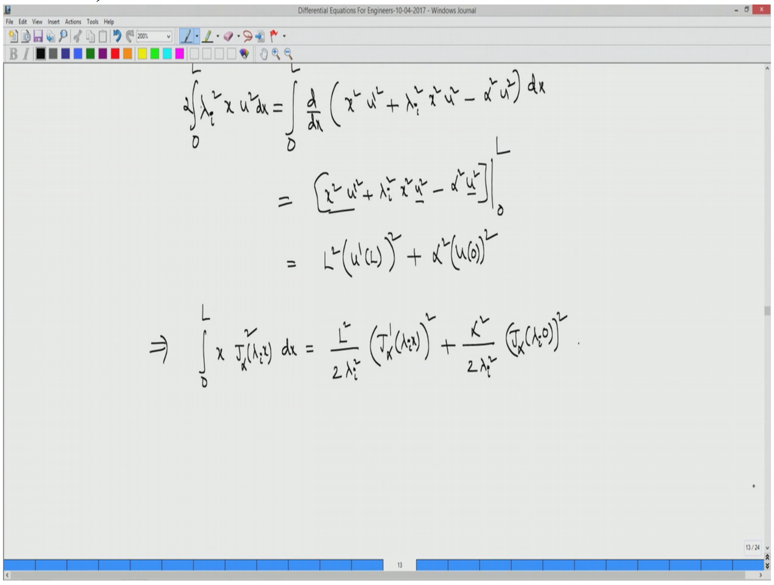

at zero, u at zero is also zero, Ok. So J alpha at zero is zero. Since J alpha of x if you actually see as a sum which has x power 2 m plus alpha, alpha is positive, Ok. So you can see that this is zero, J alpha, if you put x equal to zero, this is zero, Ok, so this will be actually zero,

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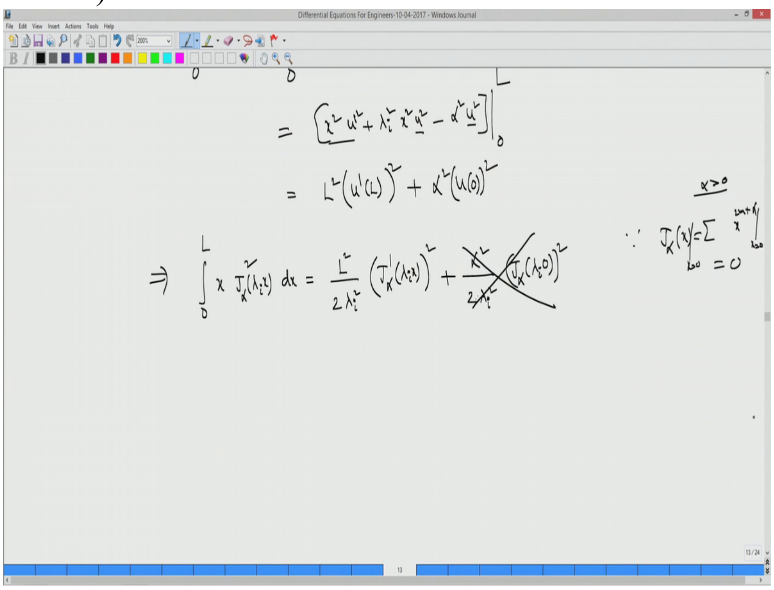

so this is what you get. So this is equal to l square by 2 lambda i square J alpha dash of lambda i x whole square, lambda i l, right,

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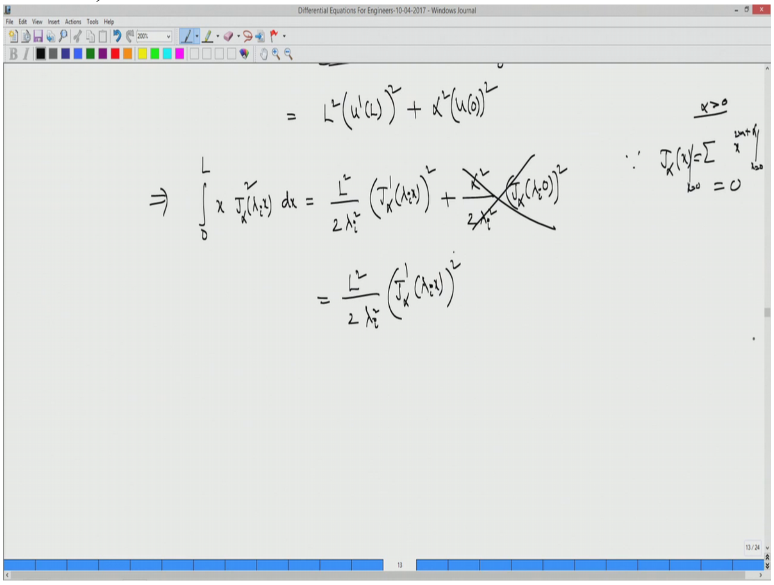

so it is lambda i l.

This is fixed thing so lambda i l whole square, Ok. This is what you get. Now how do you find, this make it equal to this type, alpha plus 1, alpha minus 1.

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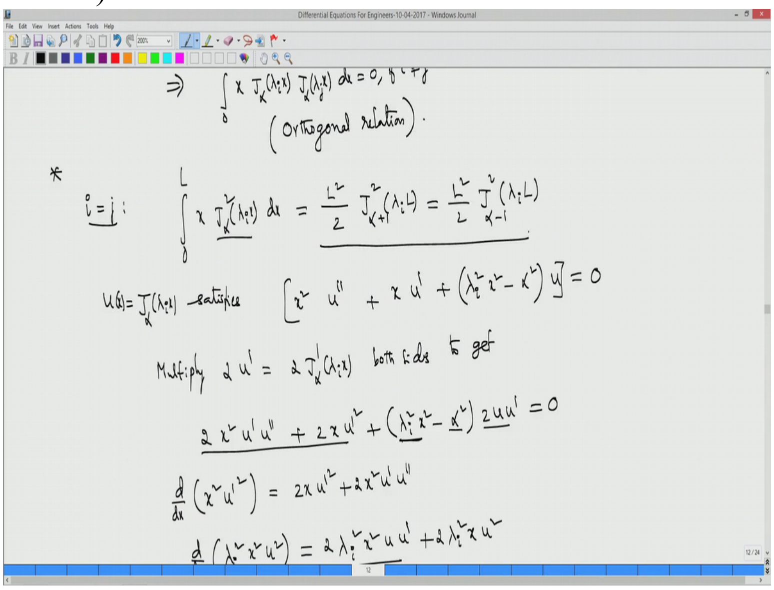

This is where we use the properties, the properties of those, properties of these, these properties if we use,

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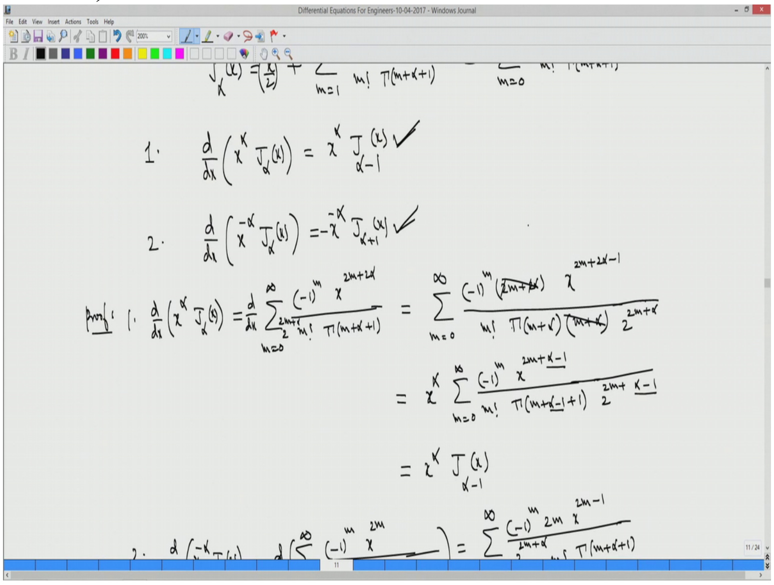

properties of the Bessel functions, you can actually get those things. You simply, differentiate, differentiate 1.

If you differentiate this, what you are getting end up is x power alpha so you just expand this

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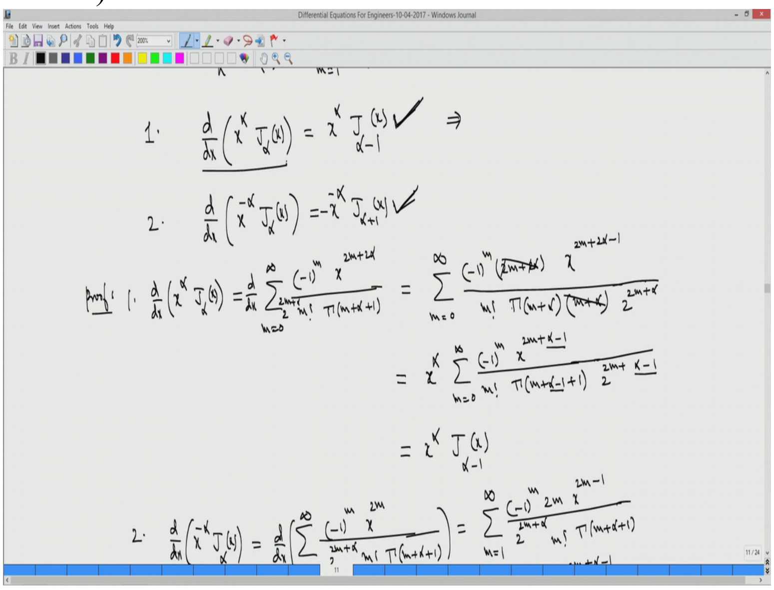

derivative Ok, which is equal to this,

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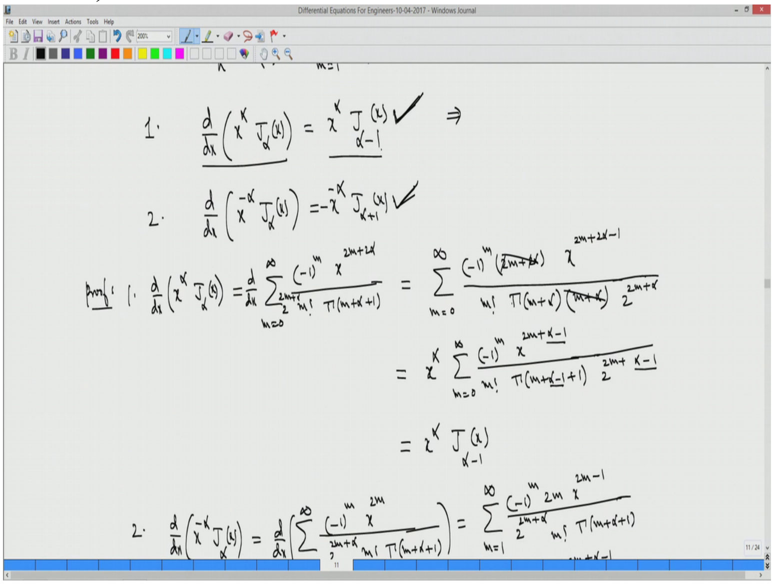

either of these you can use,

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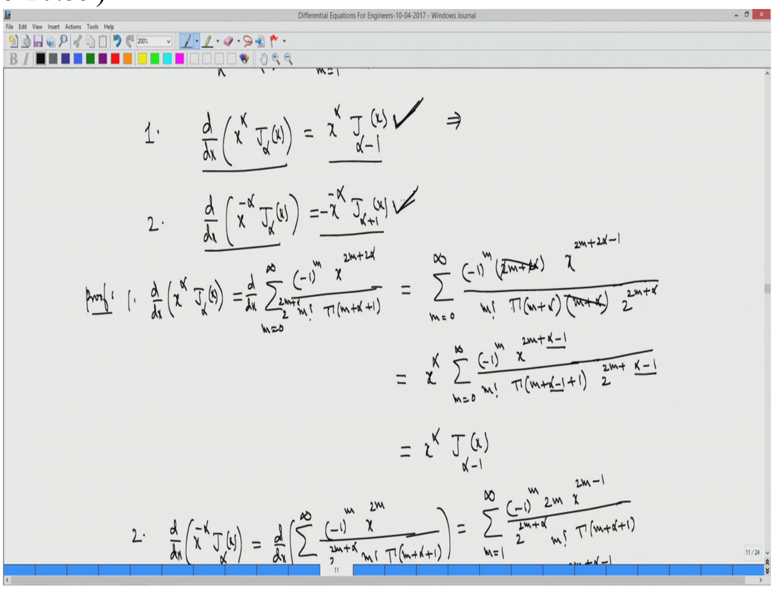

one of these, one of these you can use to show that, to get this one, Ok. So I will just do 1. 1, I will just use, 1 to find this value, J alpha dash, so

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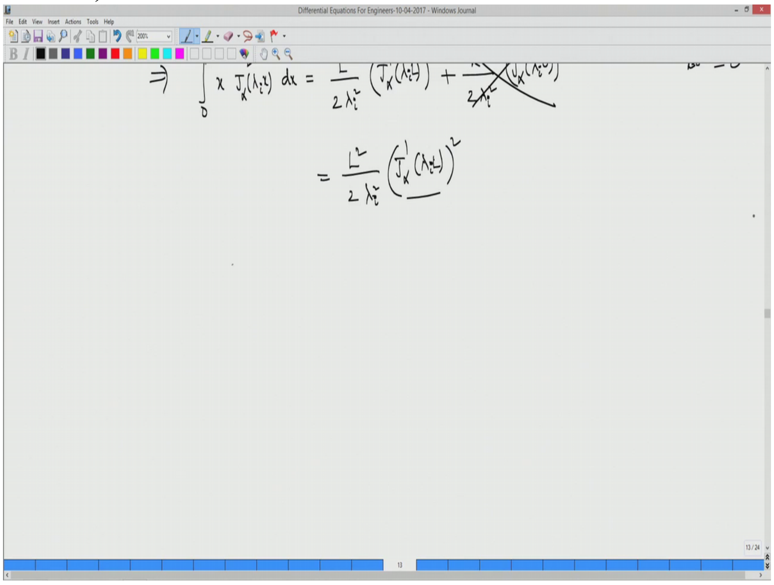

let us take this first, what is that, so recurrence relation is d d x of x power alpha J alpha of x equal to x power alpha J alpha minus 1 of x, so let's choose this one,

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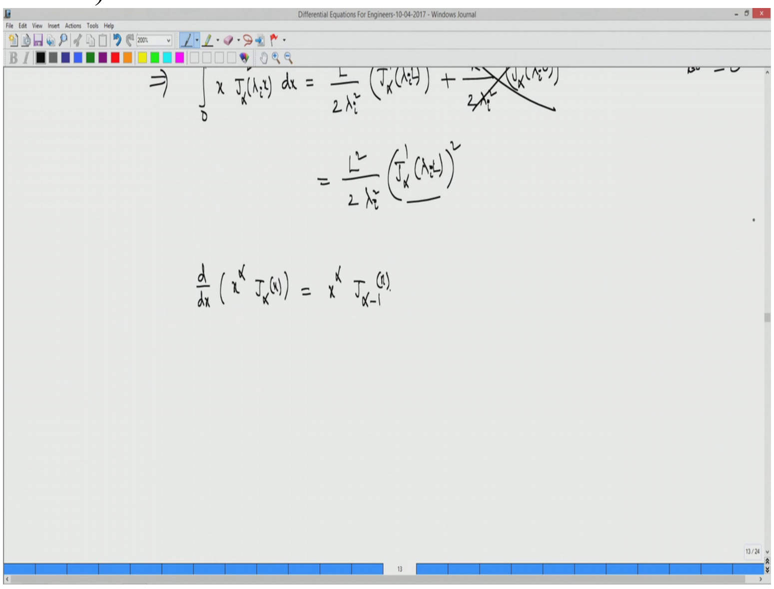

Ok.

If you expand this x power alpha, J alpha dash of x plus alpha into x power alpha minus 1 J alpha of x, that is the derivative, the left hand part I have written. You have x power alpha J alpha minus 1 of x. Now you cancel both sides x power alpha,

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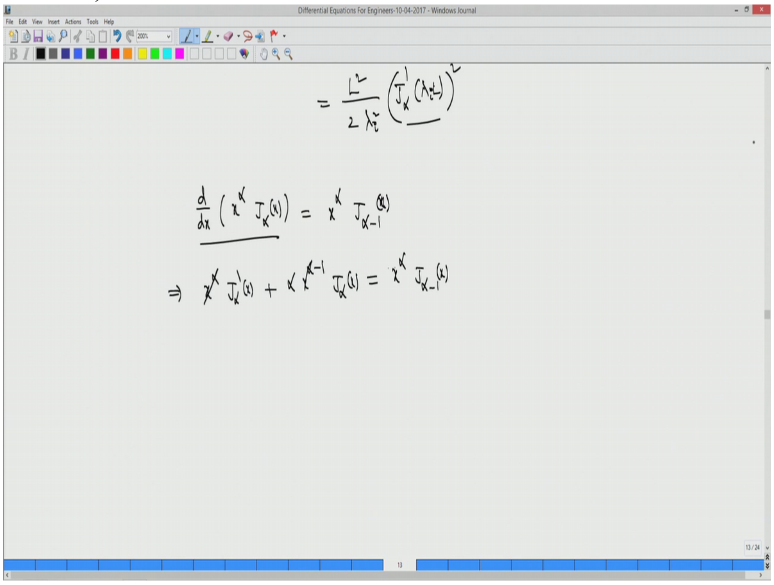

so to get J alpha dash of x plus alpha by x J alpha of x plus J alpha minus 1 of x, so this is what you get.

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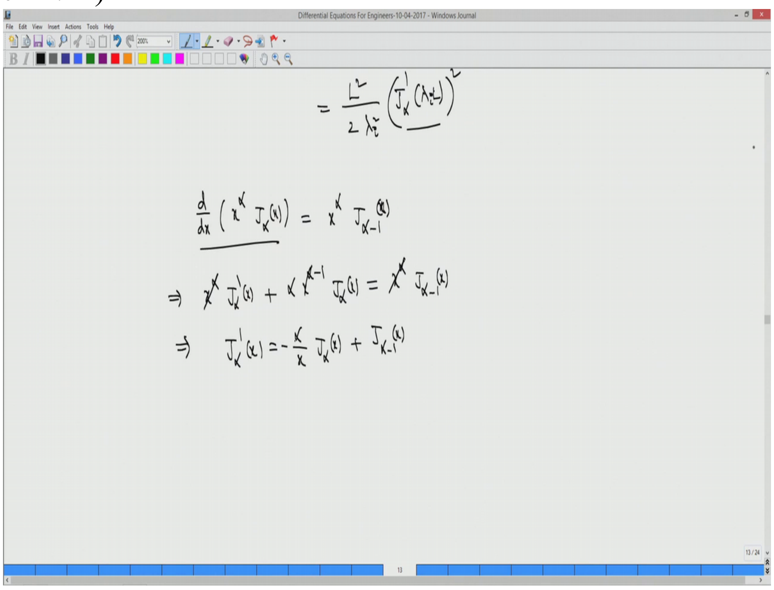

Ok.

So now use that, this one if you use, J alpha dash and put x equal to lambda i l which is equal to minus alpha by lambda i l J alpha of lambda i l, and I know that this is zero, plus J alpha minus 1 of lambda i l, Ok. So you want this square and this whole thing is zero so this square. Ok

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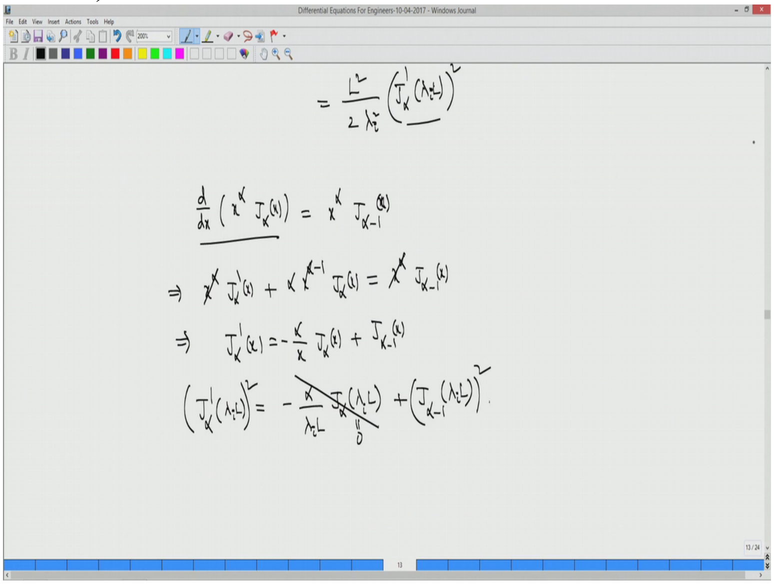

So this is exactly what you have. So you need this is equal to lambda square by 2 lambda i square J alpha minus 1 of lambda i l whole square.

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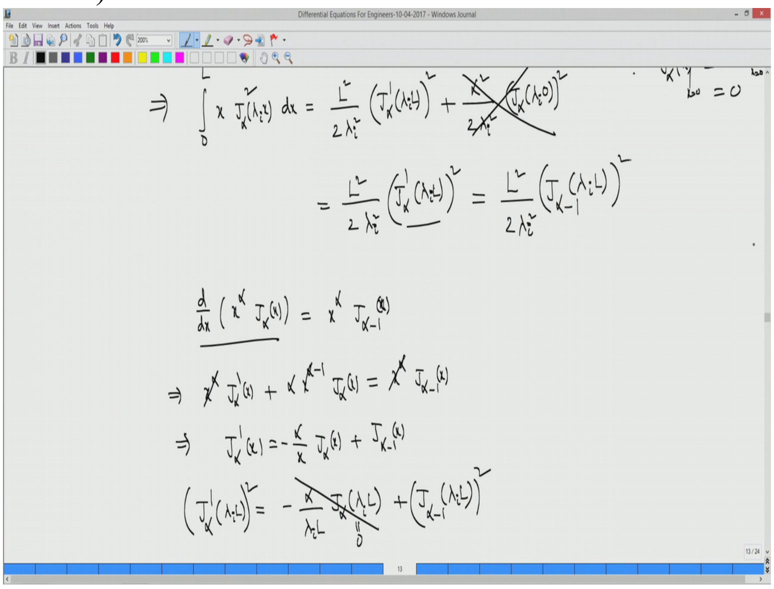

So maybe I missed this part so what you have is, that is what you have. 2 lambda i square that is also there. lambda square by 2 lambda i square so you have actually lambda i square is also here, lambda i square that is the result, Ok. I didn't make any mistake here so the mistake comes here, so actually, you have, when you differentiate this,

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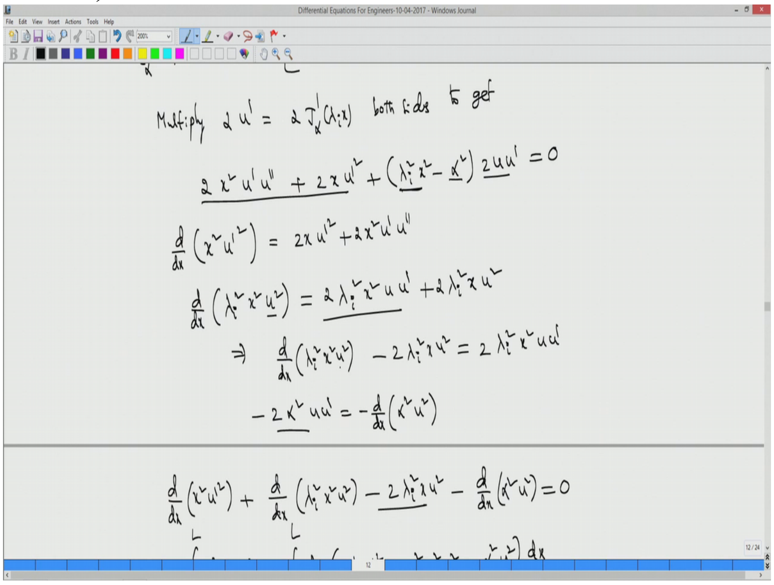

when you differentiate this, you have 2 lambda a square x square u double dash 3 d x of u dash, i dash, 2 x square u u dash, it's fine no, yeah.

So when you are doing this u dash, so u dash, u is actually, u dash means d d x, d d x of u dash, u dash, u is, u dash of x is, u dash l is d d x of J alpha of lambda i x. This is J alpha dash of lambda i x into lambda i, Ok so when you are differen, when you are squaring it, so you have, this is this square, this square, this whole square equal to this square into lambda i square. So you have, lambda i square comes here.

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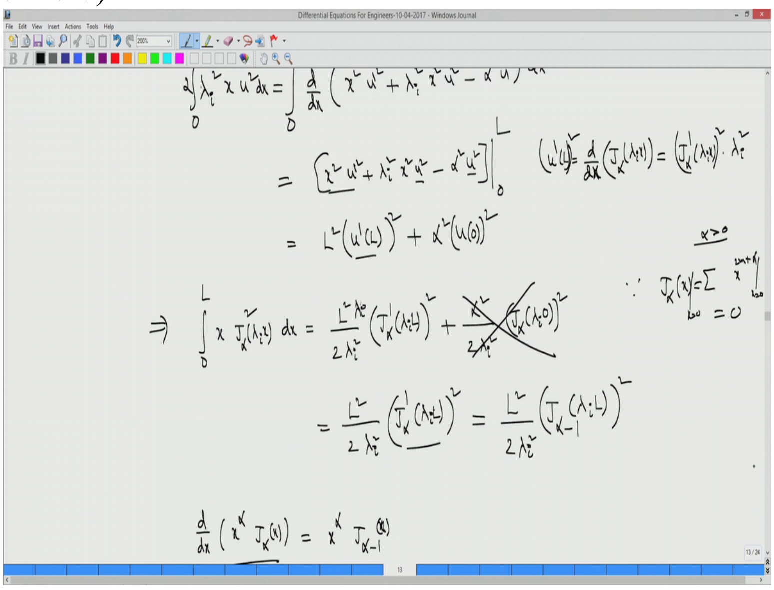

Ok, so that goes.

So this is, this is what you have. So otherwise it is fine, so we don't have this lambda i square, so that's how you get this lambda without this lambda i square. Similarly if you use other recurrence

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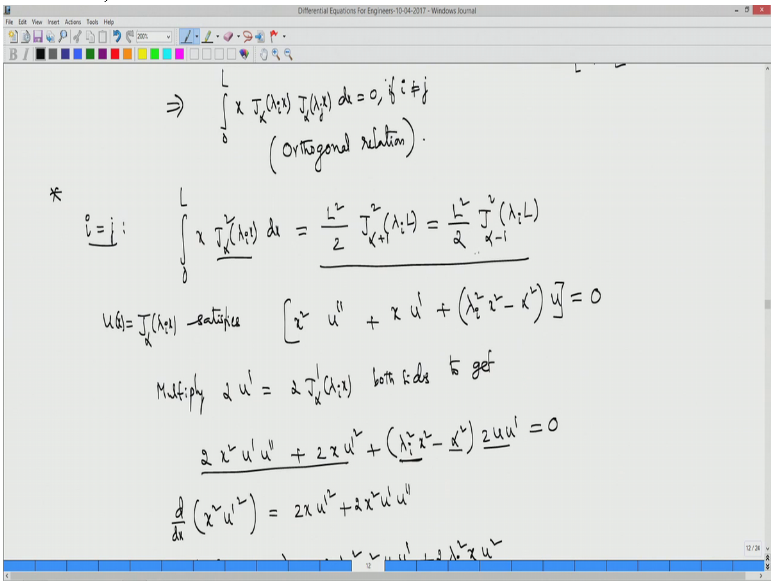

relation. If other property, if you use the other property, so if I use this one, if I use this property, you get this result, Ok.

If I use the other result that is, d d x of x power minus alpha J alpha of x which is given as x power minus alpha times J alpha plus 1 of x. If I use this one, I get

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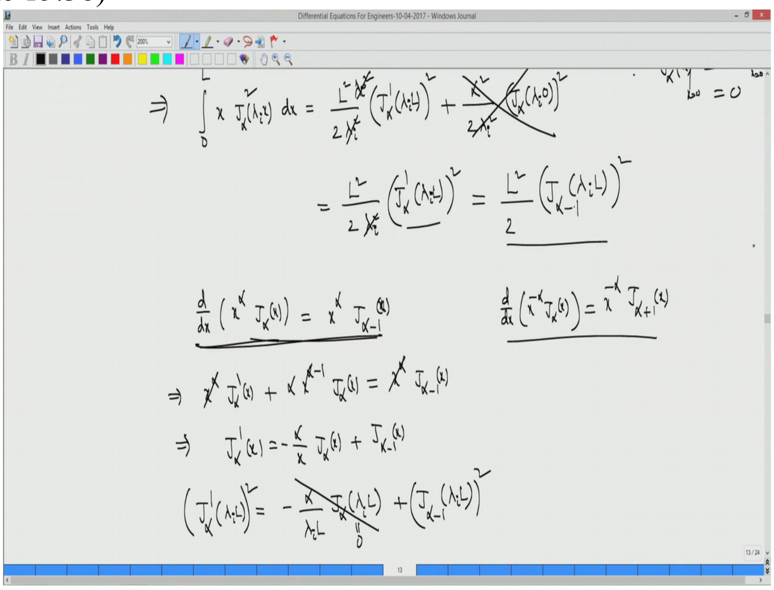

other, other result, which is l square by j, l square by 2 J alpha plus 1 at lambda i l whole square, Ok. Both the things so

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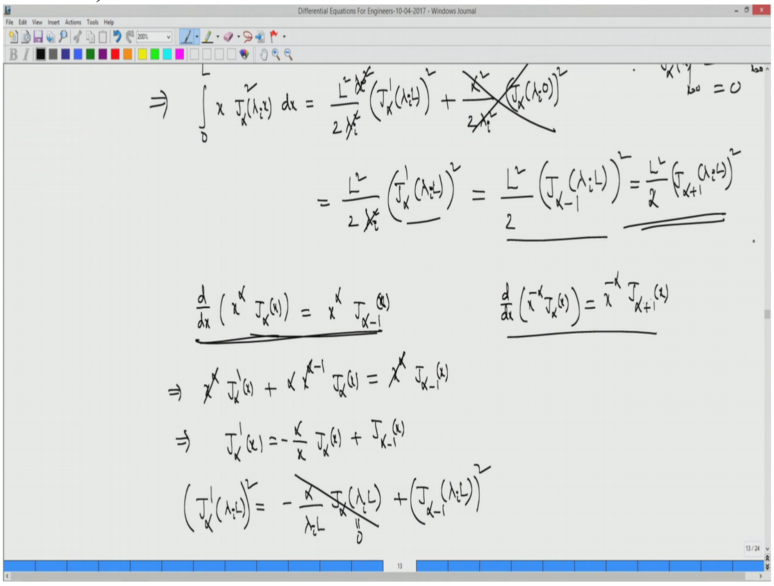

if you use this one, I get this one. Because I use this one, I got this result. OK that is what is

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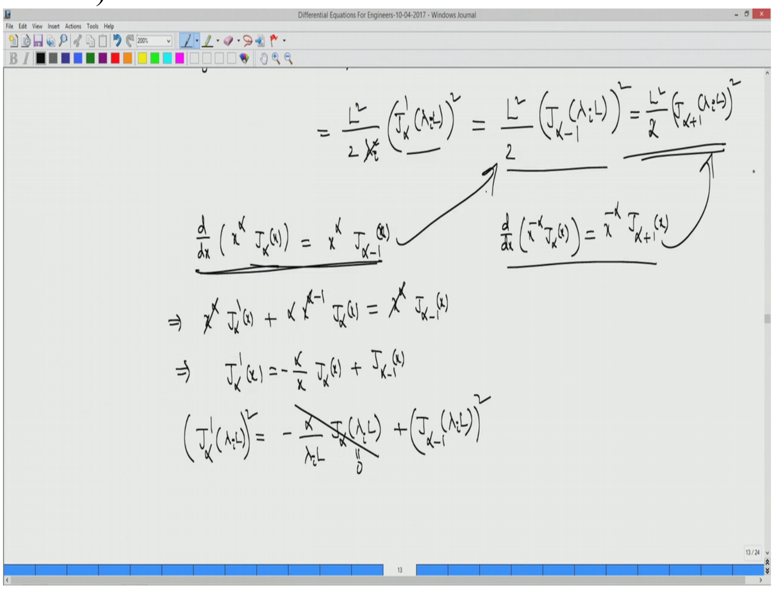

the, this integral value. This is the, this integral value, Ok. So that's what we find. So

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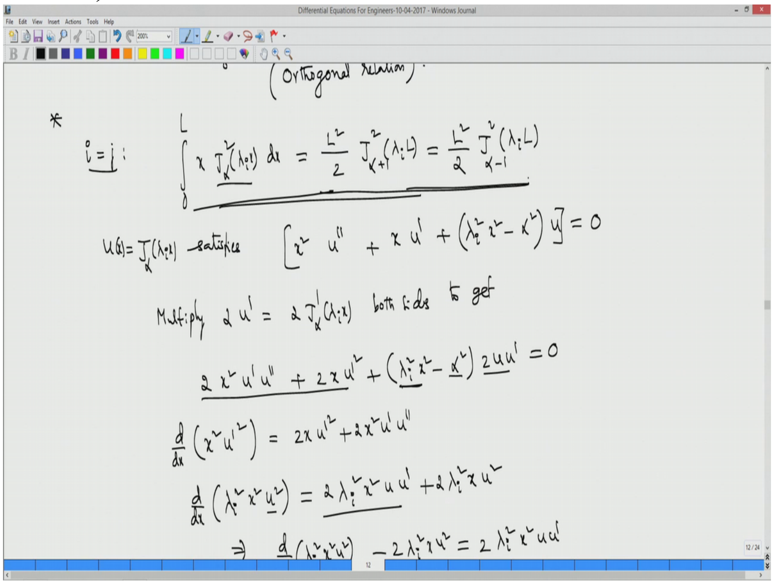

this is what we derived.

So if I use the other property you can get this result. That you can do yourself. So, so these are the (()), some important properties of Bessel's, Bessel functions. We have seen some special case when 2 alpha equal to, the difference between the indicial roots, 2 alpha is 1, 3, 5, Ok. So the general solutions, solutions are, two linear independent solutions are J alpha x, J minus alpha x, Ok.

In this special case, for all these cases, you need, if you know one, J minus half

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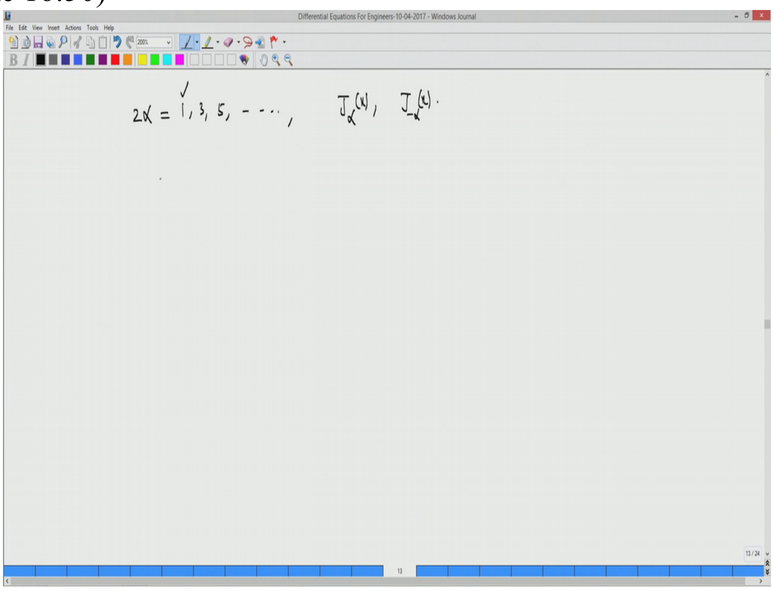

when alpha equal to half, Ok when alpha equal to half, you get J half x and J minus half x. So if I know this, I can get, from the recurrence, from those two properties I can get all other values, 3, 5. So I can get, if I know this, if we know for alpha equal to this one, we can find, we can find all other things Ok, we can find all other things like J n plus half and J, minus n, rather minus n, minus n plus half, Ok and n is from 1, 2, 3 and so on

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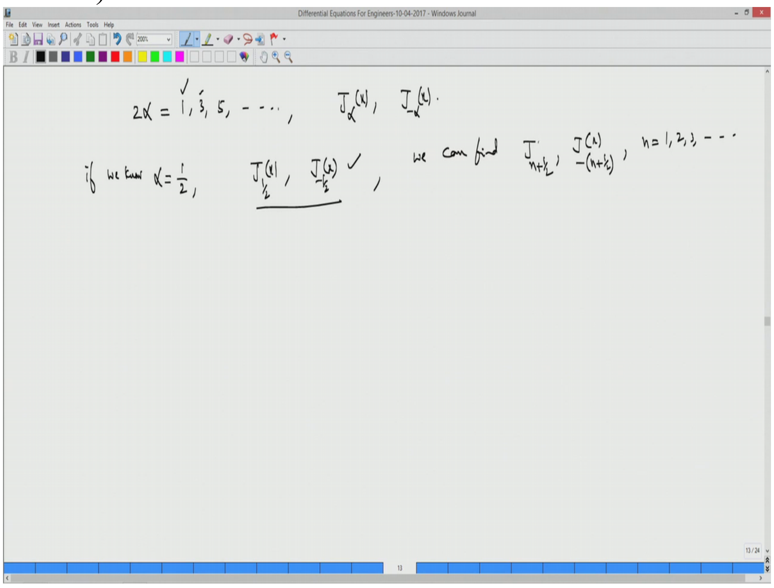

of x, Ok by recurrence relations.

So that is called recurrence relations. Just by properties 1 and 2 if you use you can get them and then you can use. So how do I get this? We can nicely find the formula for J half and J minus half, that we do by simply consider the equation, Bessel equation in this case, Bessel equation is x square y double dash plus x y dash plus x square minus alpha square is 1 by 4, y equal to zero. So this is the equation. So

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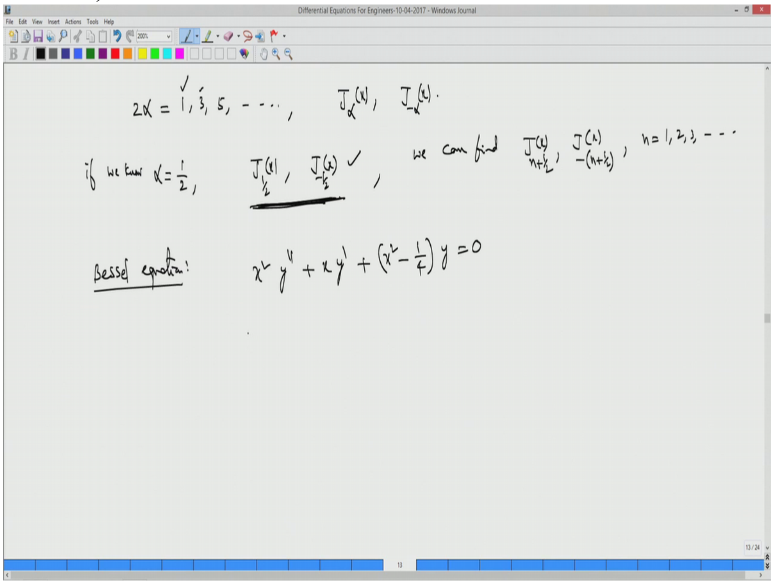

if I choose some z equal to root x into y. Let be the transformation. Ok. So let dependent variable

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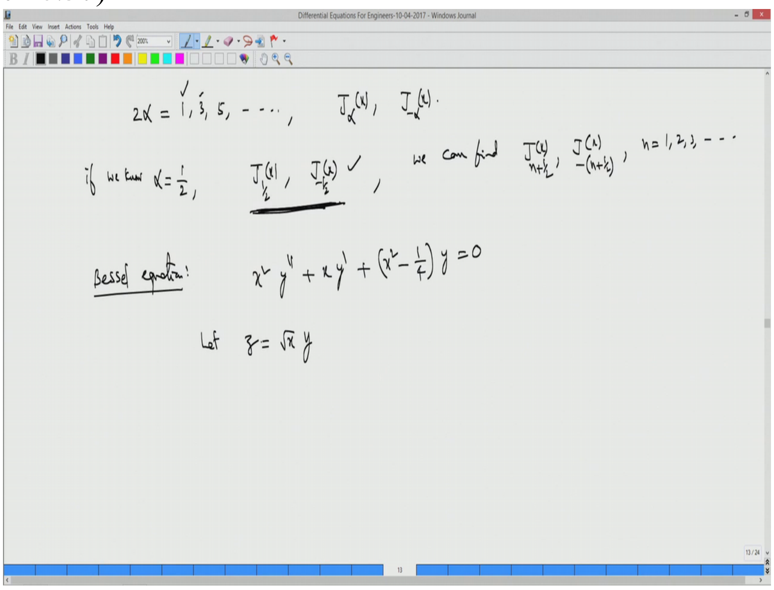

I am changing to z, z of x, Ok. If I use that, z of x, so if I use this transformation, what you have is d y by d x equal to, d y by d x is, d y by d x, d y by, d y by d x, this is d z by d x. I want d z by d x, Ok. So d z by d x, you can write d z by d x, so how do I transform this?

So you simply replace d d x of z is root x y, so this you have root x d y by d x plus 1 by 2 root x into y, this is what you replace, d z by d x, Ok. Then what

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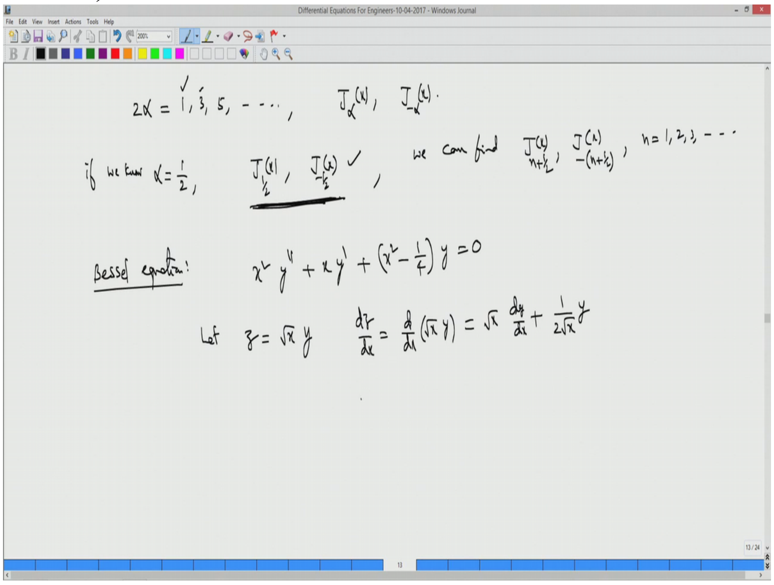

you need is d square z by d x square. This is equal to d d x of, d z by d x is, we already found so that is d y by d x plus 1 by 2 root x y.

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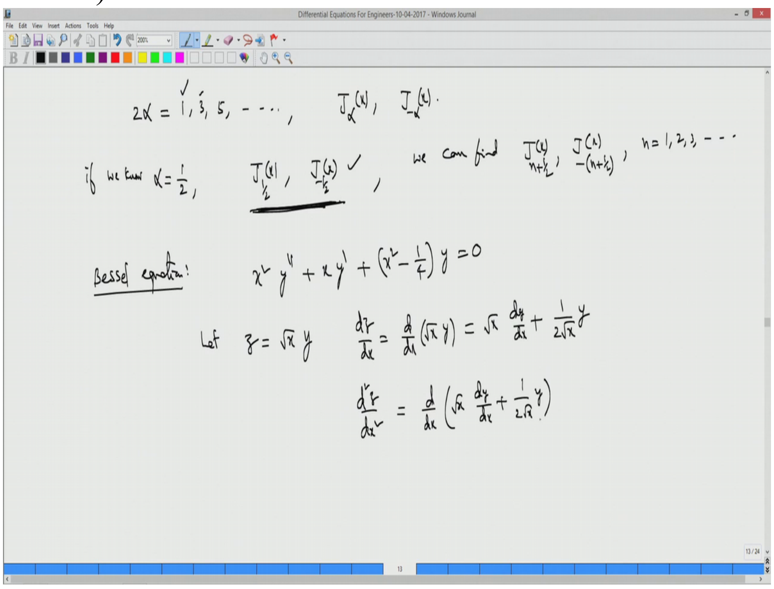

This is equal to 1 by 2 root x d y by d x plus root x d square y by d x square plus d y by d x, 2 root x d y by d x, Ok plus 1 by 2 into x power

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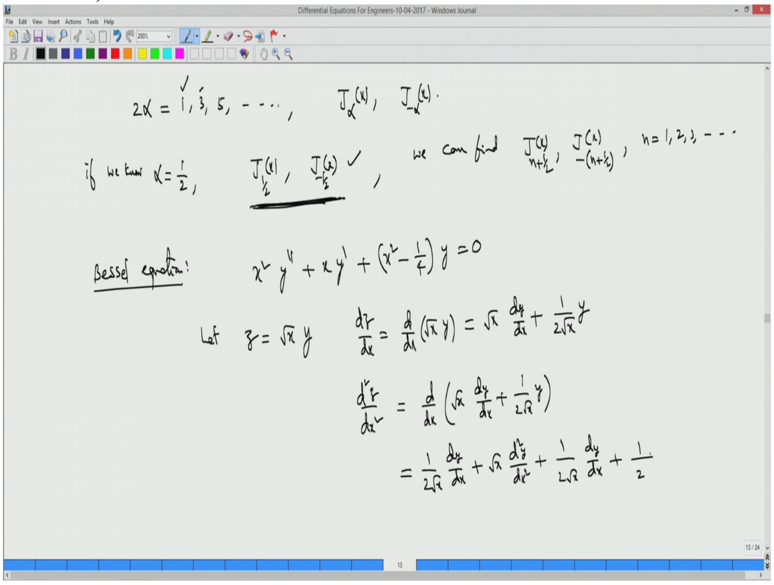

minus half that is, what is the derivative of x power minus half, that is minus half x power minus 3 by 2.

So this is going to be minus, so 1 by 2 into 1 by 2 square, that's going to be 4, x power 3 by 2. This into y, this is what you have, so substitute

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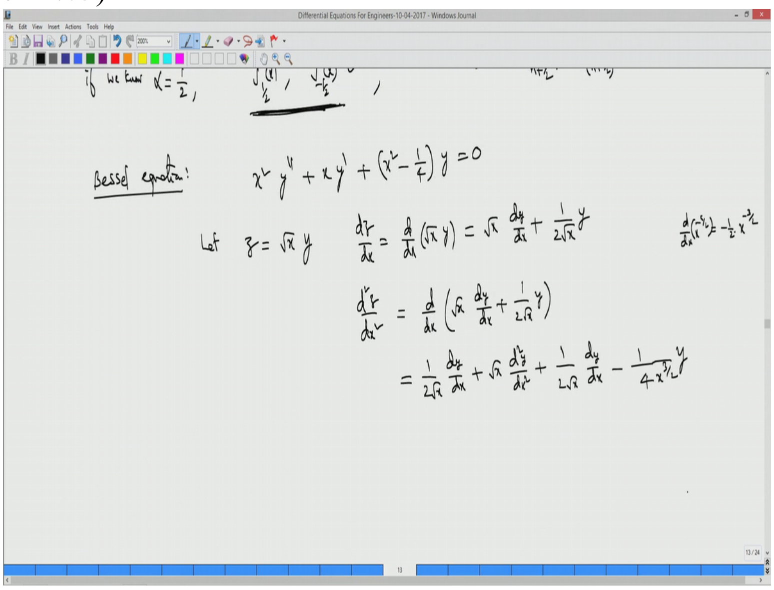

for d y by d x. From this you can calculate d y by d x. So this is 1 by root x d y by d x plus root x d square y by d x square minus 1 by 4 x cube plus 3 by 2 y.

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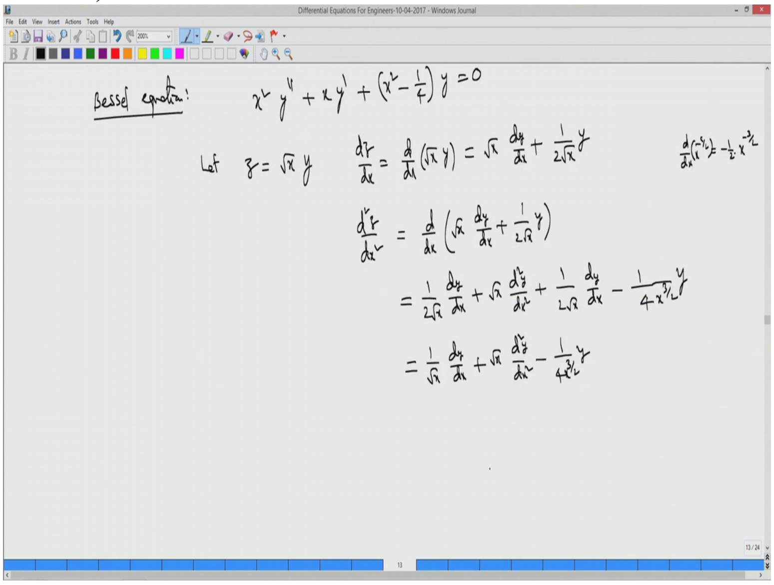

Rather, so instead of these calculations, we can consider,

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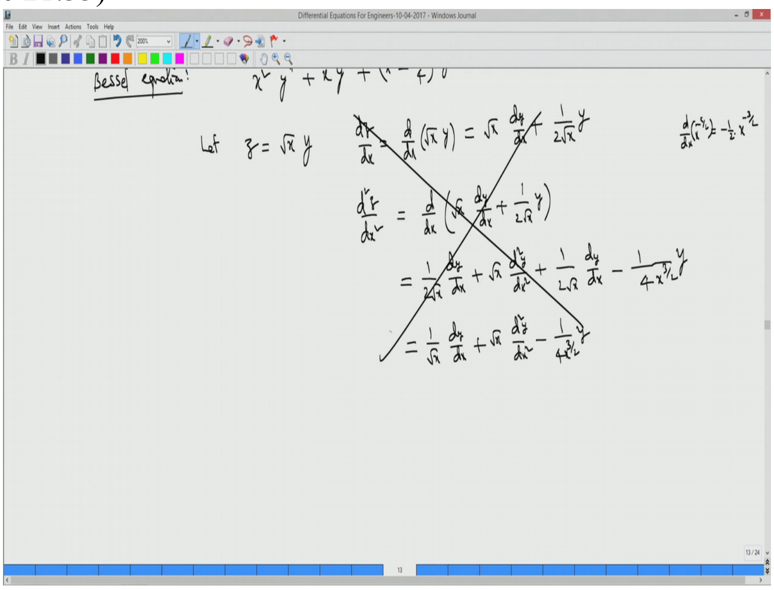

you can consider, if you choose this transformation, I can get d y as z by root x,

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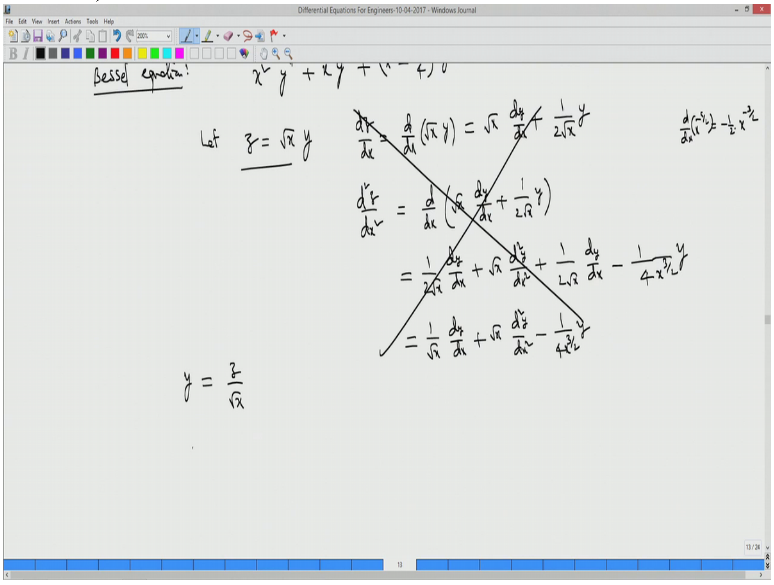

so that you can calculate d y by d x as directly.

So you have z minus half x power minus 3 by 2 z plus 1 by root z, root x d z by d x. Similarly we can replace, we can get this d square y by d x square as 3 by 4 x power minus 5 by 2 z plus minus, minus 1 by 2 x power minus 3 by 2 d z by d x plus or rather minus half x power minus 3 by 2 d z by d x plus 1 by root x, d square z by d x square.

So once you calculate this d y by d x and d square y by d x square, substitute

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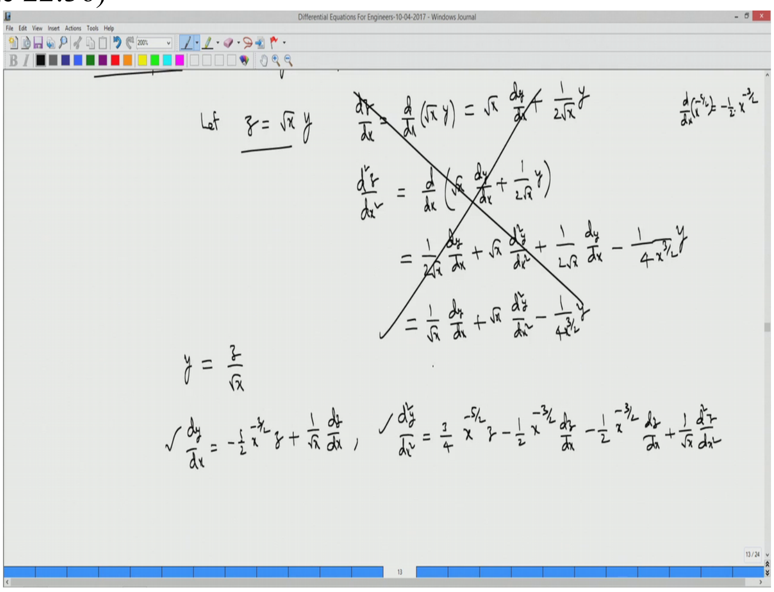

into the equation. So if you substitute into the equation, you get x square y double dash that is x square into this part. So you get 3 by 4, x square will give you 1 by root x, 1 by root x z, z by root x Ok, minus this, this together will give me minus 1 that is 1, minus 1 x power minus 3 by 2 plus 2 that is root x, root x times d z by d x plus x power 3 by 2 times d square z by d x square.

So this is what is x square y double dash plus x times, x times d y by d x you can write this part, for this you multiply x, so you have minus 1 by 2 root x z plus root x d z by d x and then plus what you have is x square minus 4, x square minus 1 by 4, y is z by root x equal to zero.

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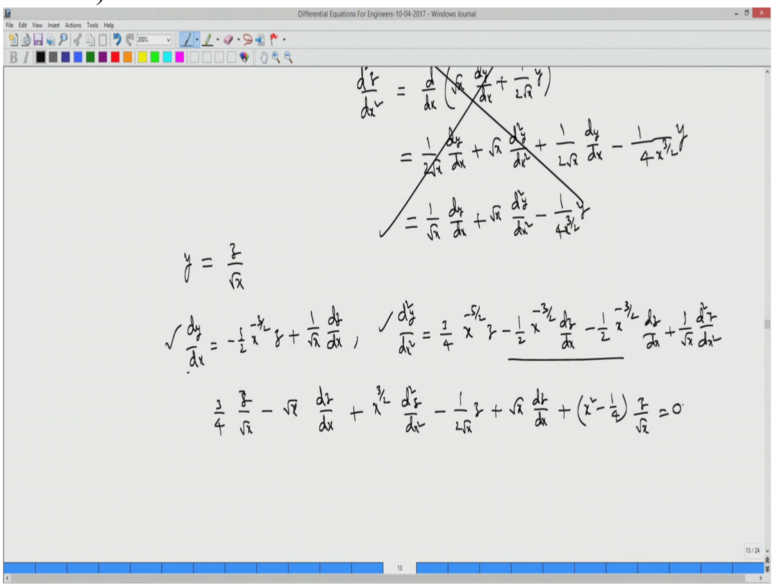

So you see that nicely this transforms into simpler equation.

As you see, multiply so you can see that x root x d z by d x cancels. And this part, minus 1 by 4 z by root x Ok and 3 by 4,

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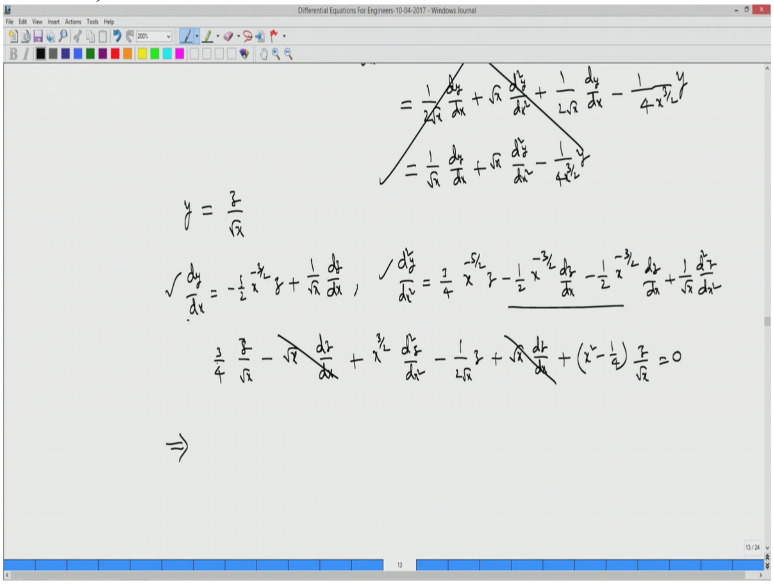

3 by 4 z by root x and minus 1 by 2 z by root x, this becomes zero, so 3 by 4,

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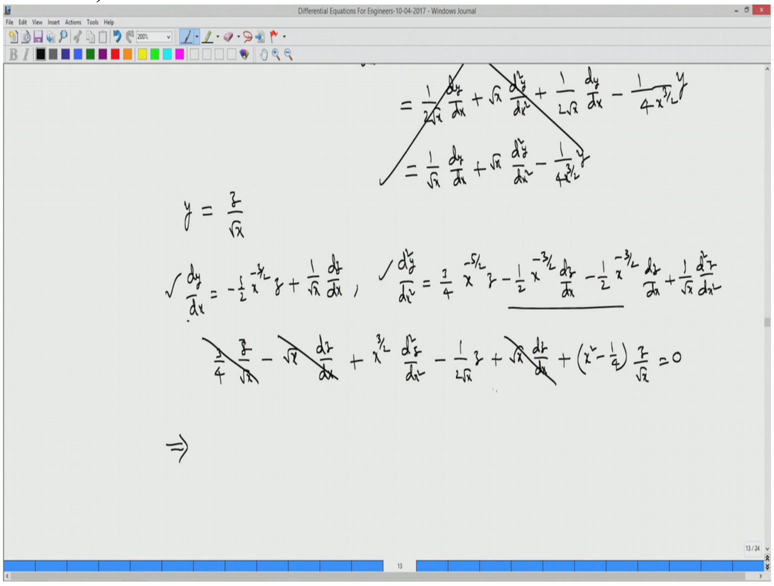

it becomes minus half minus 1 by 4 z by root x. This part will be zero. So what

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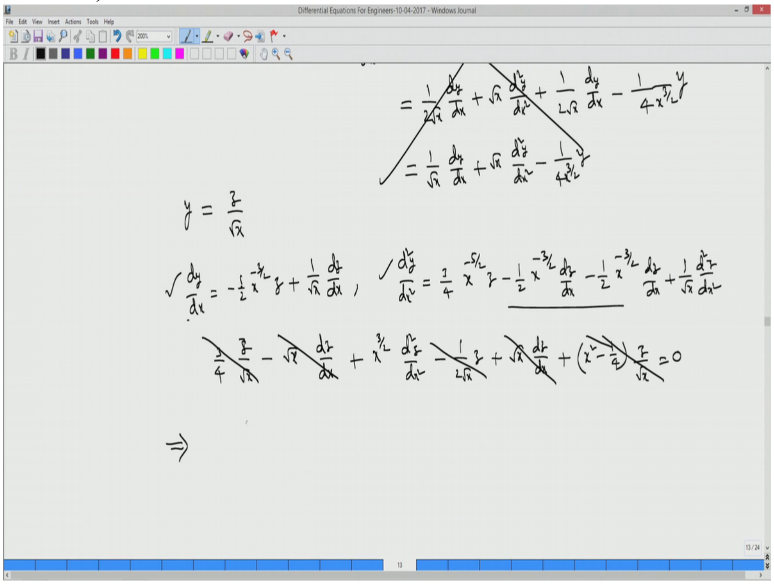

you are left with is, simply x 3 by 2 d square z by d x square plus x 3 by 2 because x square divided by root x into z equal to zero.

So this will give me, now

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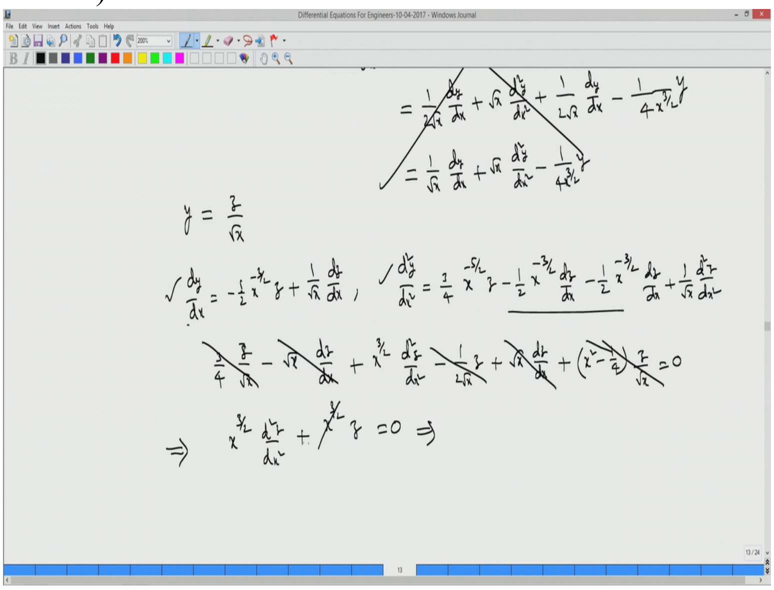

x 3 both sides it will cancel, you get z double dash plus z equal to zero. So if you use this transformation,

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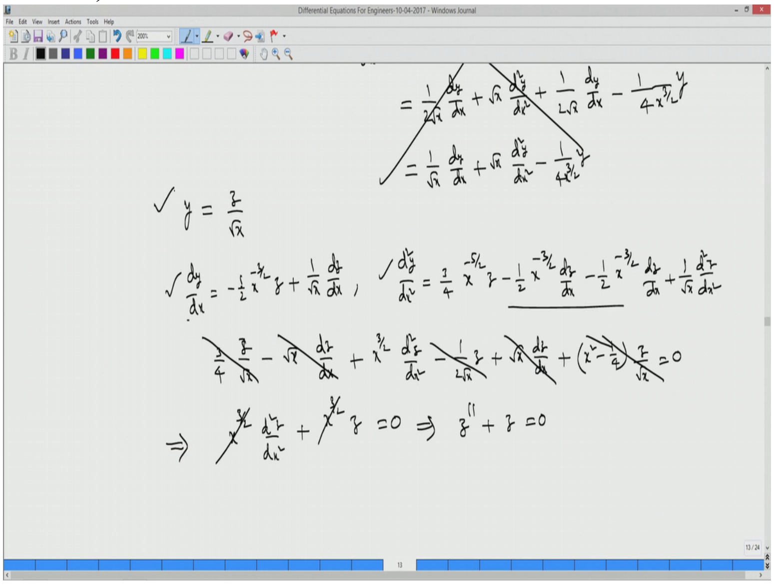

the equation given, Bessel equation when alpha equals to half becomes simpler. So this we know z x as the general solution of this equation is c 1 cos x plus c 2 sin x. So once I know

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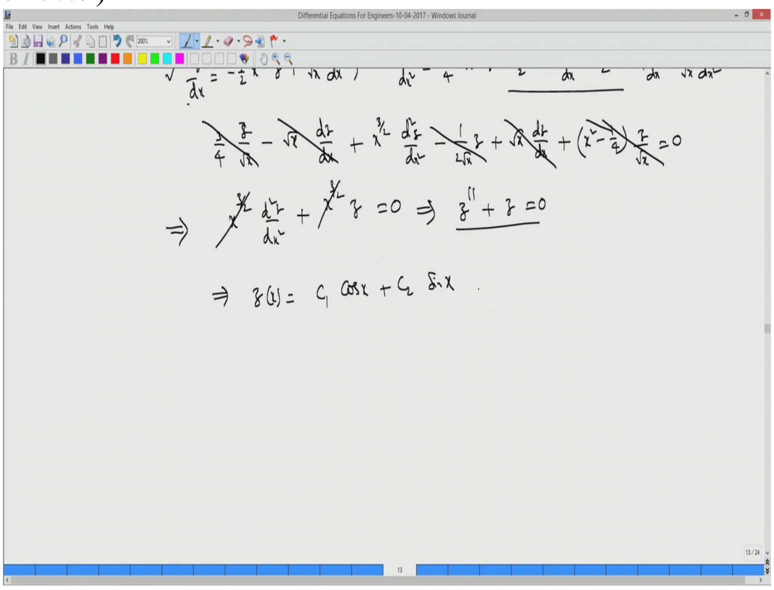

z what is my y x, simply c 1 cos x by root x plus c 2 sin x by root x

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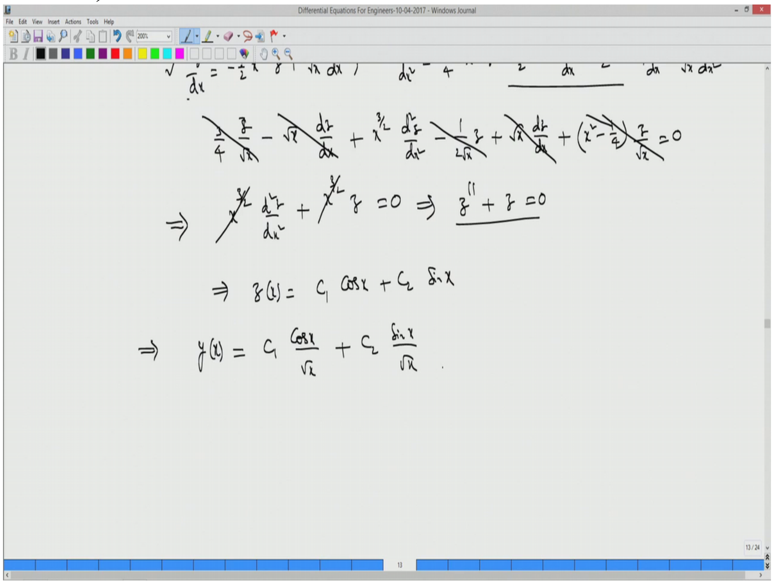

because z x by root x, z by root x is my general solution y for the Bessel equation.

So this is the, is the general solution, fine. So this is the general solution of

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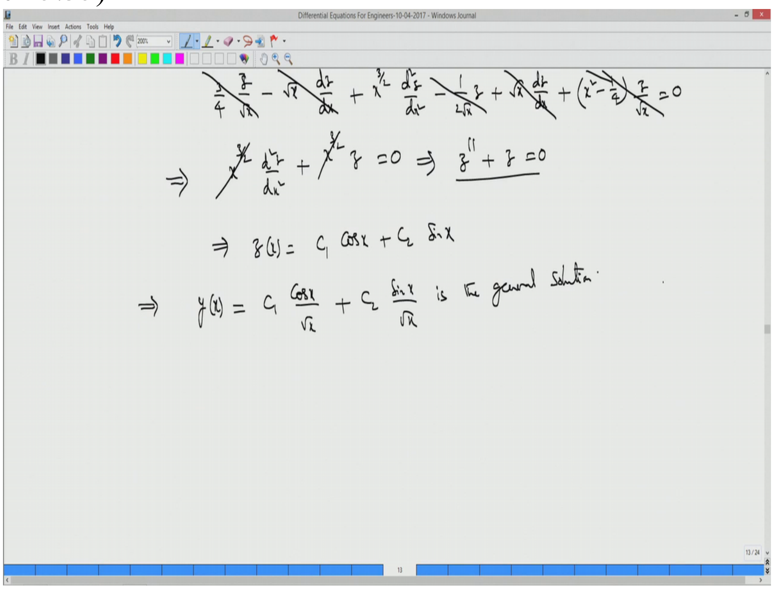

the, what you found so far is the general solution

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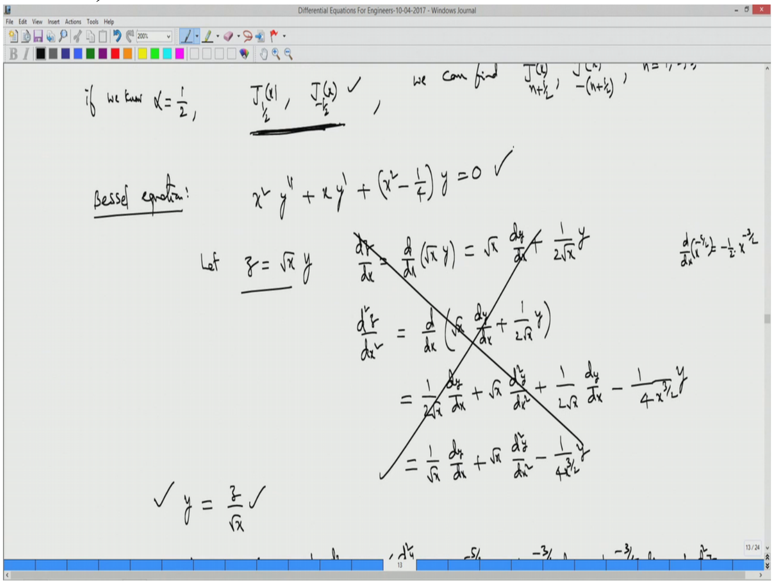

of the Bessel equation. But we already know that J half and J minus half are the two linearly independent solutions, Ok. J half and J minus half are two linear independent solutions.

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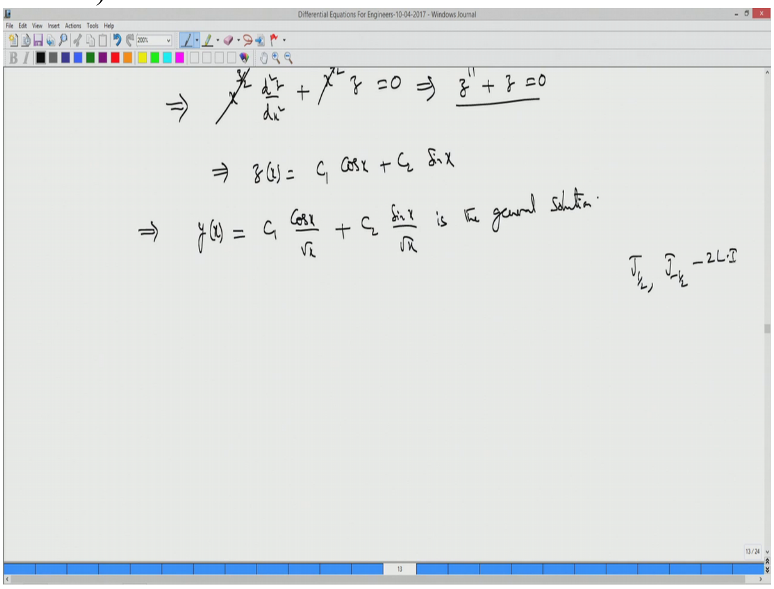

Here now I found that cos x by x sin x by x root x are two linearly independent solutions.

Because

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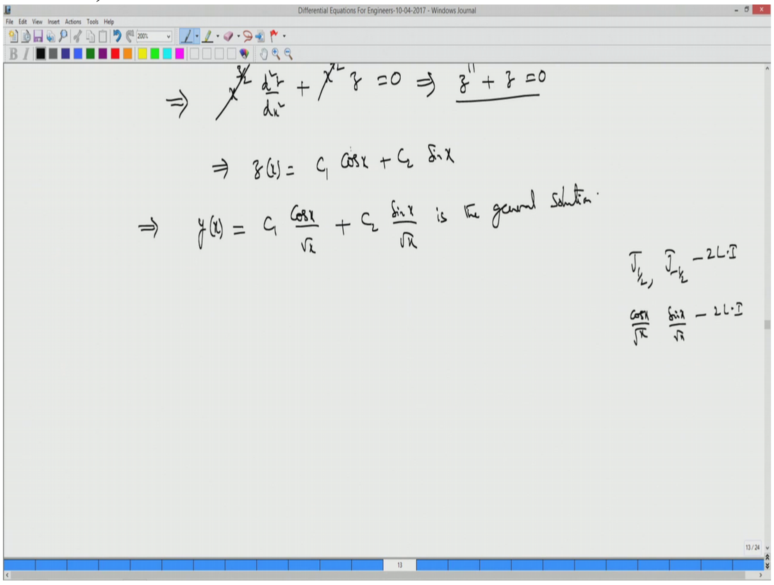

these 2 are linearly independent, so this is the general solution. So this is how you get. So I found these 2 are linearly independent. So that means this must be one of them

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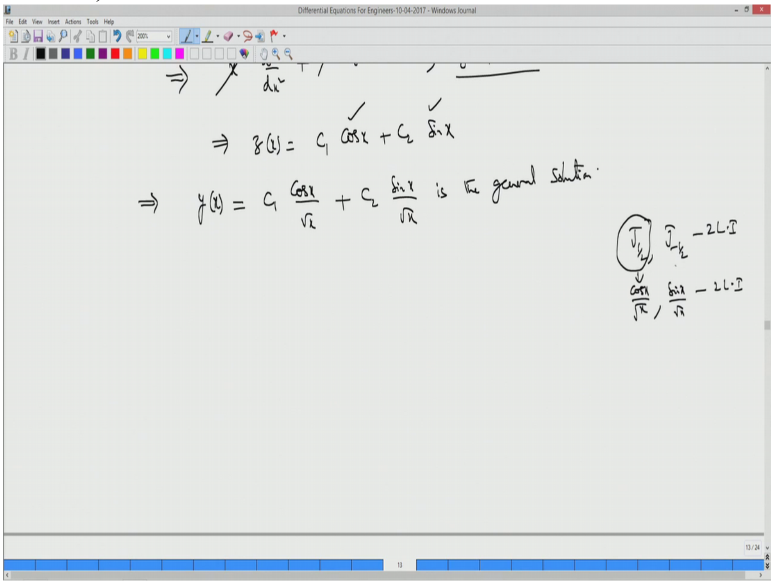

with some constant multiple, Ok. So this, I should be able to write this in terms of this and this in terms of this, Ok.

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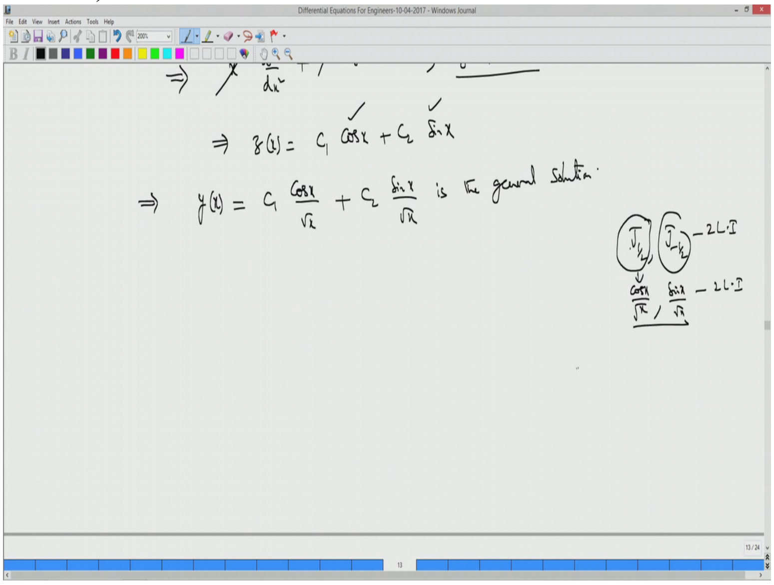

So let us do one in terms of the other, Ok. What we do is, so we know that x half, so that is what is, right, so you know that y is, so you know that y is z by root x, right so y, so we see, let us see y of x is, the transformation is z of x by root x, Ok. This is the transformation. That means my y x, if I, I know that y x, y is satisfying the Bessel equation so that is J alpha, J half of x is one solution, Ok.

So this is the solution for y which is equal to z. Some z will be, so if I choose my y x, sorry, I will expand properly, so you have, this implies z x equal to root y x into root x, Ok. So if I choose y

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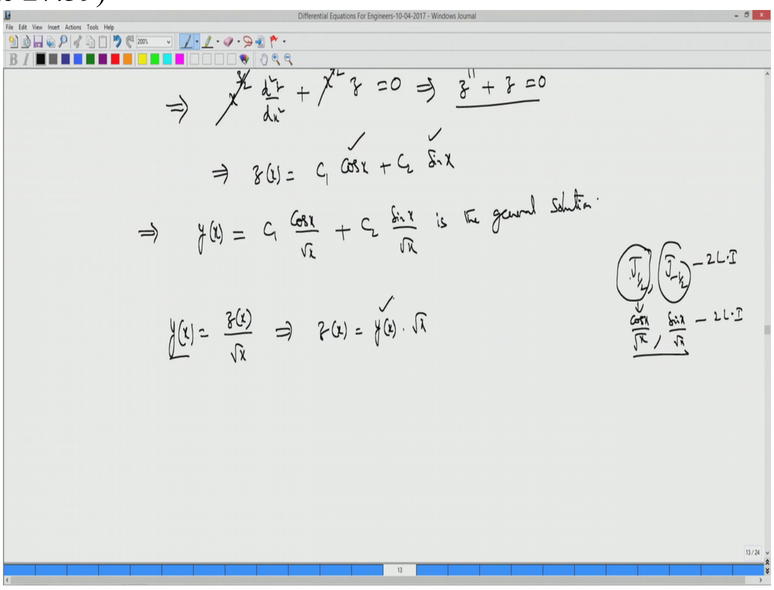

as J half of x into root x, this is also the solution, Ok. That means z x

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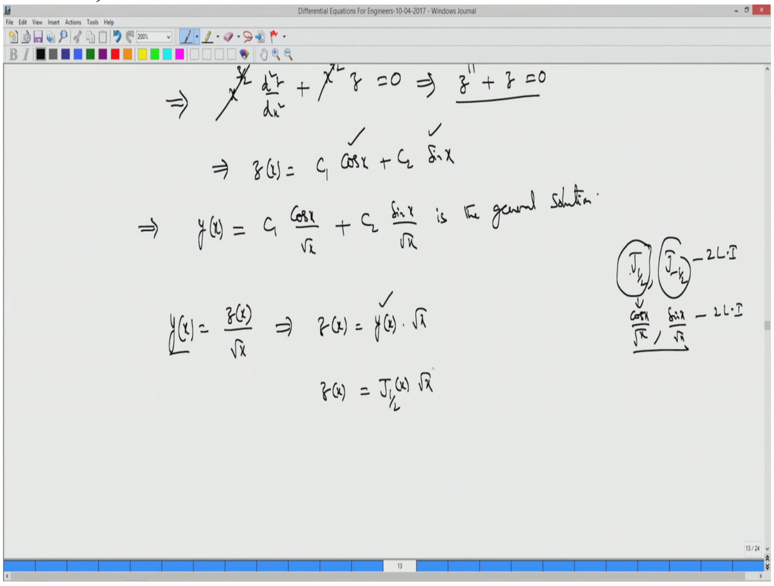

will be like this. If I choose in the, for y if I choose as, I know that J half is the solution,

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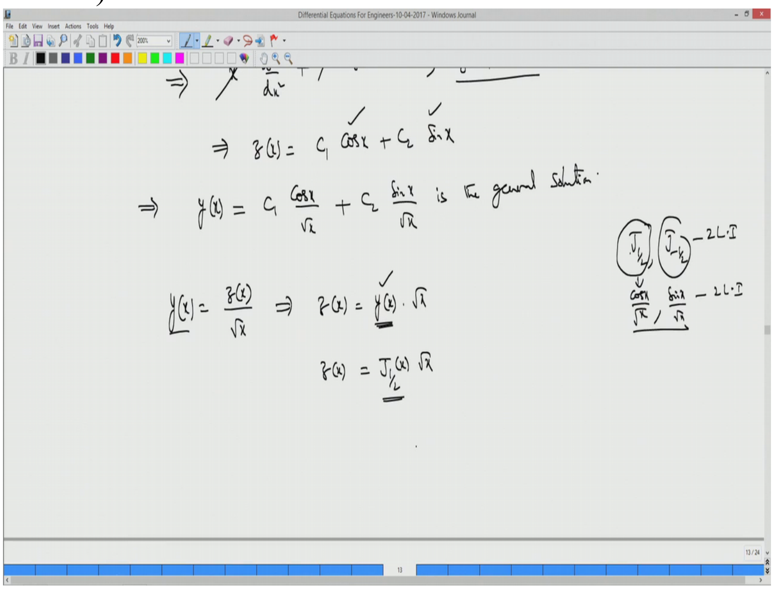

so once I choose this, root x into J half is the solution of this transformed equation, Ok. This satisfies z double dash plus z equal to zero.

So that means

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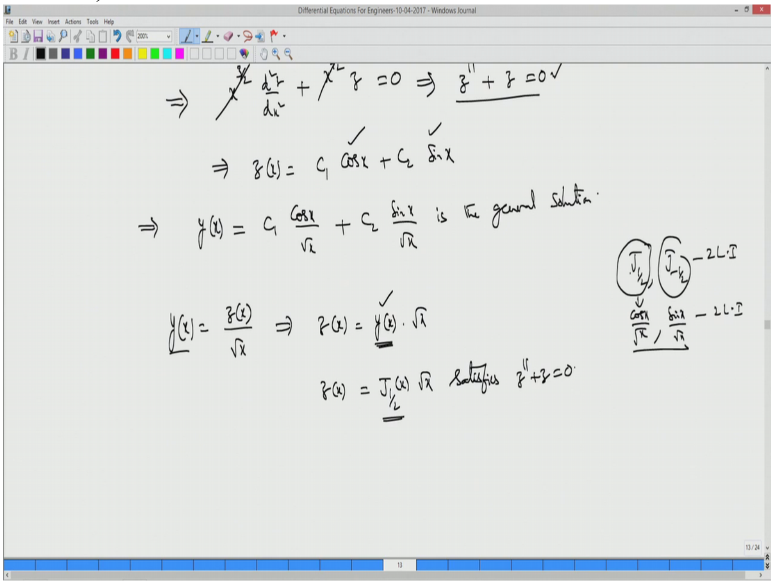

this is the solution of this equation, one solution. So that implies J half of x into root x is equal to some linear combination of some d 1 times cos x by root x, because these are the solutions, not root x, what are the solutions of z double dash plus z x, cos x plus d 2 sin x. Put x equal to zero,

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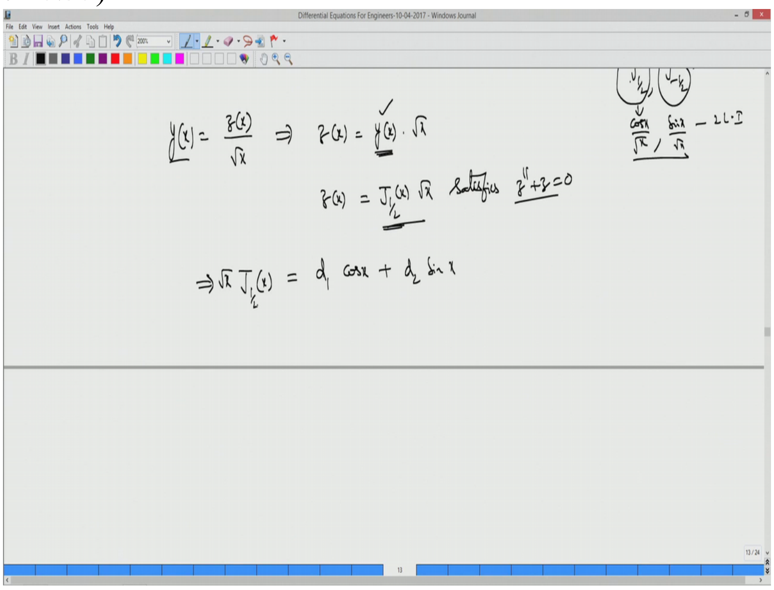

that will give me d 1 equal to zero, Ok. So that means this is not true.

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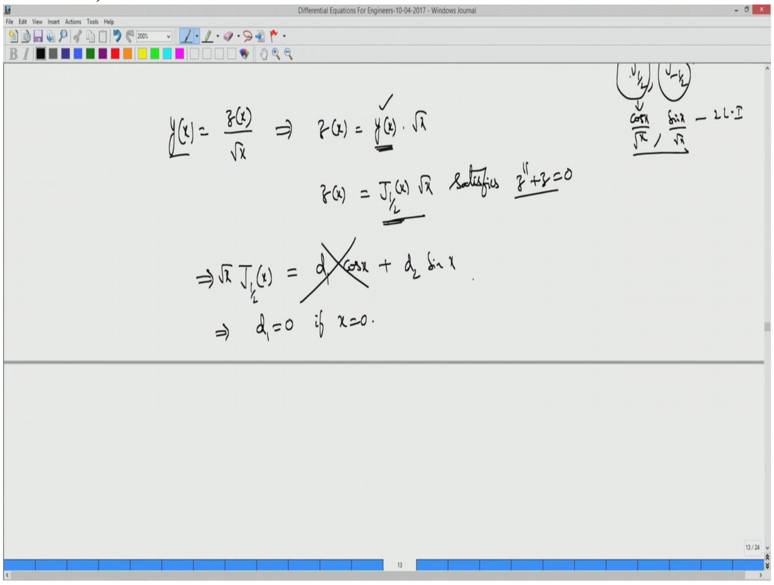

So this implies root x J half of x is actually equal to d 2 sin x.

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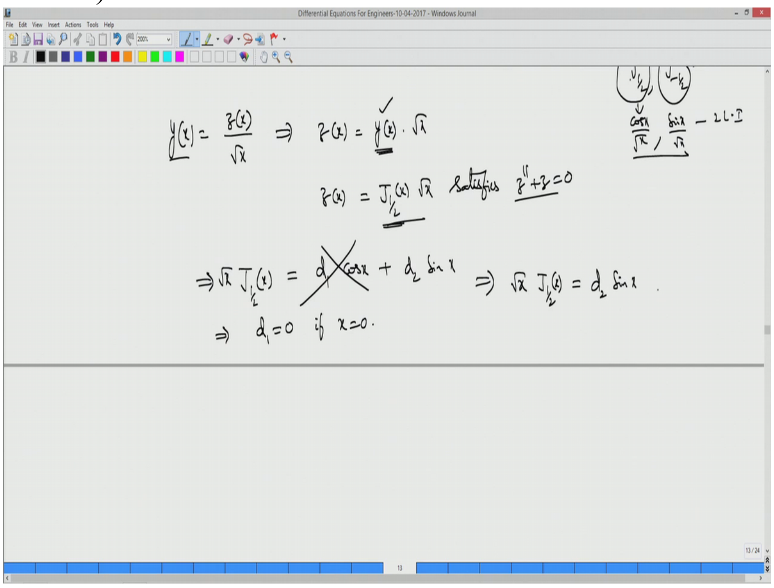

So how do we find my d 2? So that I can get my J half in nice expression. So this implies you write simply d 2 as root x J half of x divided by sin x, Ok. If d 2 is constant, now if I can take this limit x, this should be true, this should be a constant. Even at x equal to zero or any other value it should give the same value, Ok. So at x equal to zero if I take this is well-defined at x equal to zero, well-defined,

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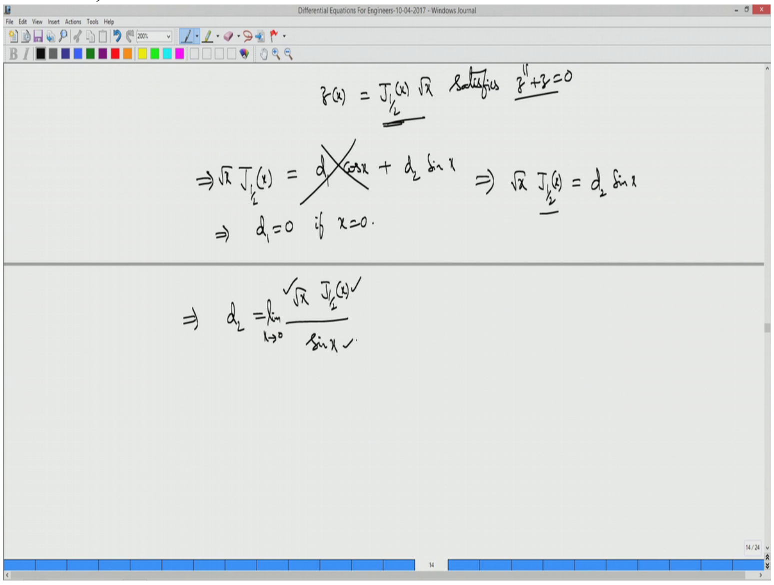

well-defined at x equal to zero. So I can consider this limit. This is equal to, limit x close to zero, this I write sin x by x, so that if I write sin x by x, I should have, I should have J half of x by root x, Ok.

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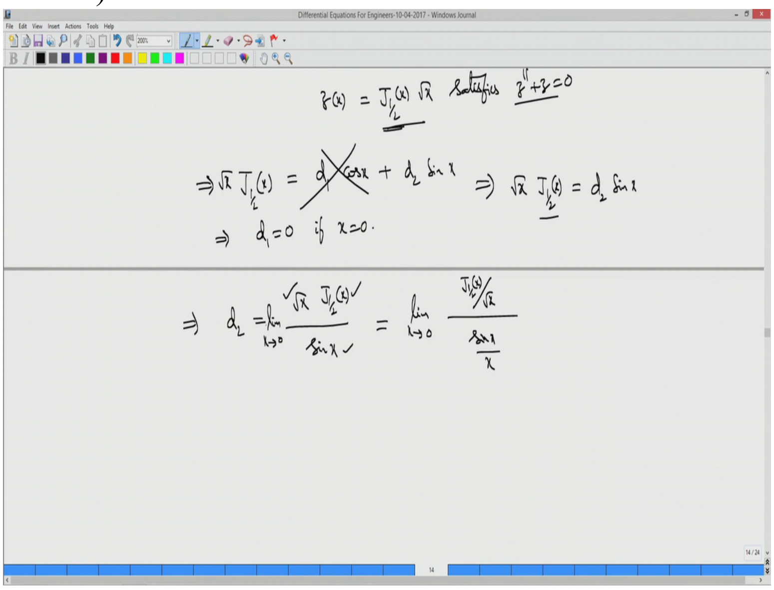

These two are same.

So this is nothing but, now you see that bottom, this limit goes to 1, Ok. This goes to 1 and this goes to simply 1, what is the value? So this you have to calculate, Ok. So this is actually equal to limit J half of x by root x, x close to zero. So we will find what that J alpha, J half of that, Ok.

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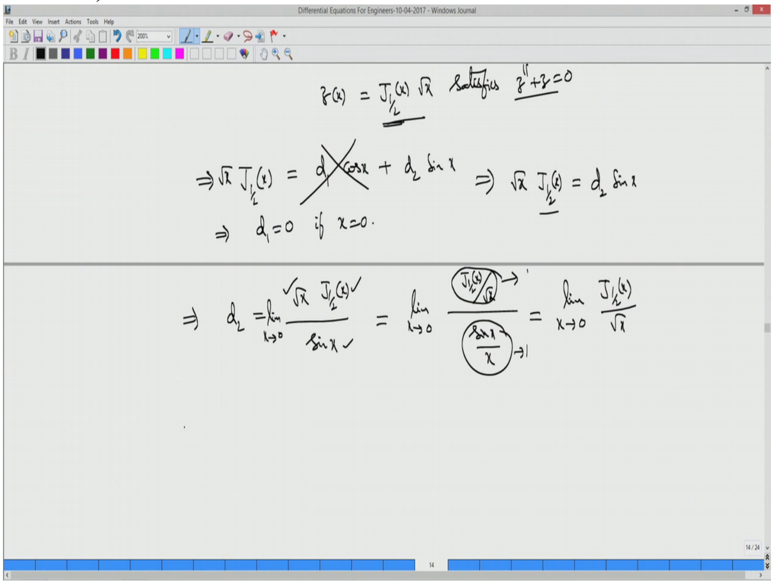

So 1 by root x, J half of x is actually equal to, what you get, you will get sigma m is from zero to infinity minus 1 power m, x by 2 power 2 m plus alpha that is half, Ok divided by m factorial into gamma of m plus 3 by 2, m plus half plus 1, this is what is the J half of x, I am dividing with 1 by root x. This is my J half.

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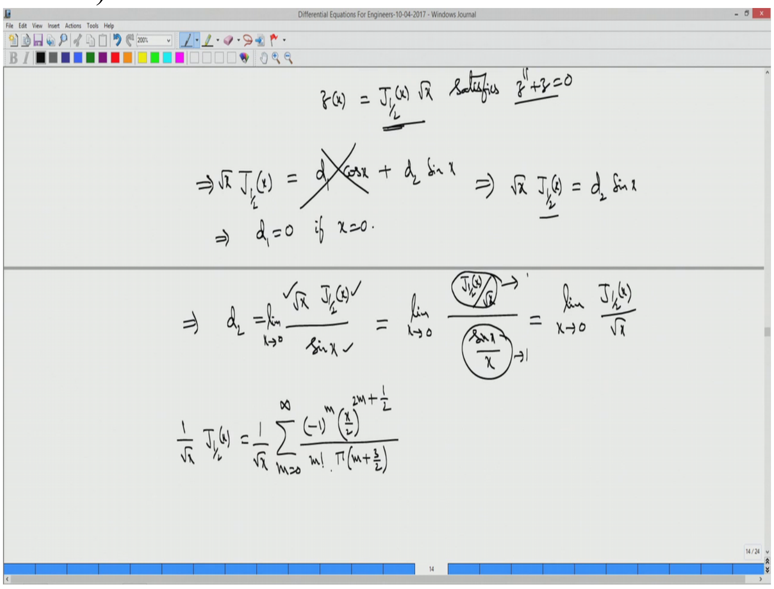

If I divide with this thing, this you will get simply m is from zero to infinity, minus 1 power m x by 2, x power 2 m plus, x power 2 m divided by, x x goes Ok, root x and I have 2 power and a root, Ok, 2 power 2 m plus half into m factorial, gamma m plus 3 by 2.

So this is what, for which I am looking for the limit, Ok. So this is, if you take the limit x close to zero, this limit x close to zero,

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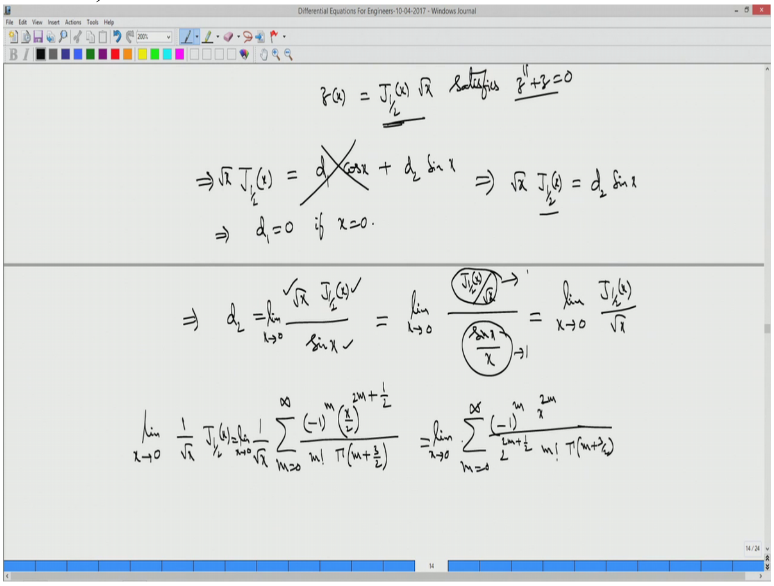

so what happens, so if I, so these are the terms starting from constant plus x square, x power 4. All the x square, x power 4 coefficients everything will be zero as x close to zero. Only that constant term that is corresponds m equal to zero, you have simply 1 divided by root 2 m factorial, zero factorial is 1, gamma of 3 by 2. This is 1 by root 2, 1 by 2 into gamma half, Ok.

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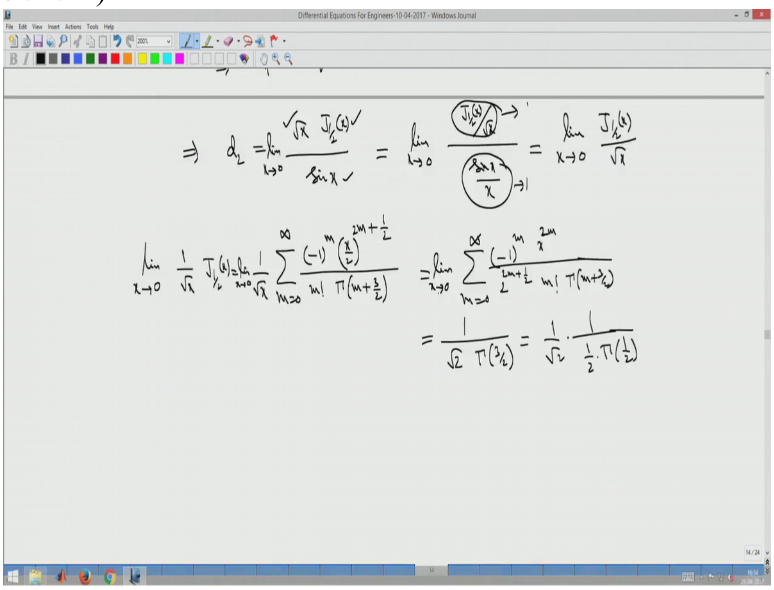

So this is nothing but, 2, 2 so this becomes 1 by root 2, that becomes root 2 plus numerator into gamma half. So gamma half is root 2 by pi. So root, gamma half is root pi, Ok, so you get root 2 by pi.

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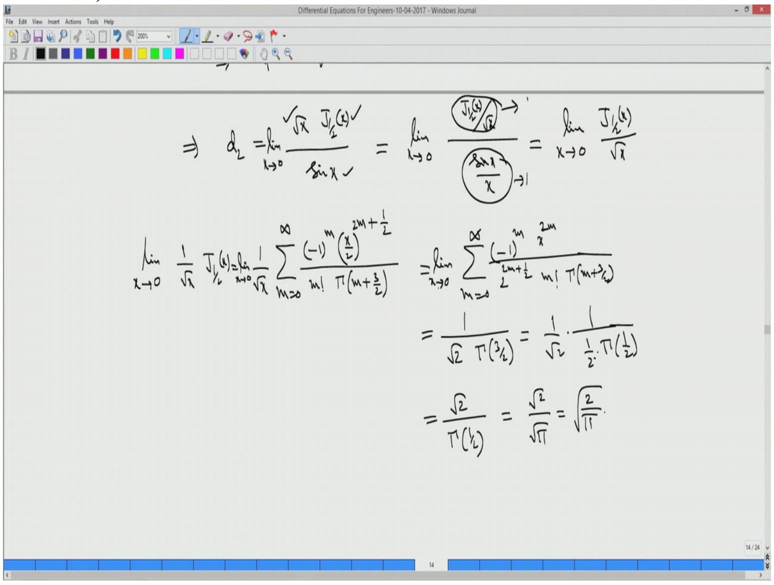

So once you get root 2 by, go and put it, so this is what you get as root 2 by pi.

So that implies, so from here, you can get J half of x, J half of x is nothing but root 2 by pi sin x by x,

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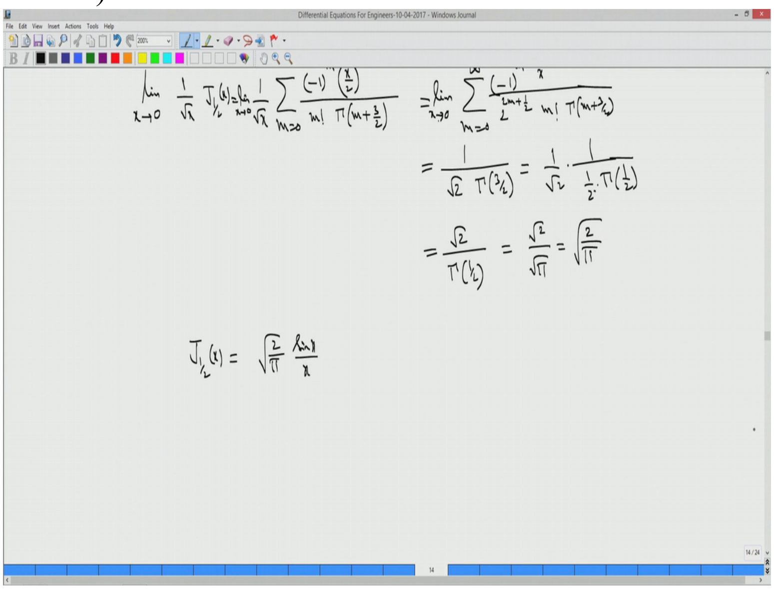

Ok. So if you want gamma of, gamma half, how to find gamma half, gamma z plus 1 is integral zero to infinity, that's what we defined yesterday, right, so we have, gamma is e power minus x, x power z minus 1, z, x power z d x. So if you want gamma half, you take gamma z is z minus 1 d x. So you take

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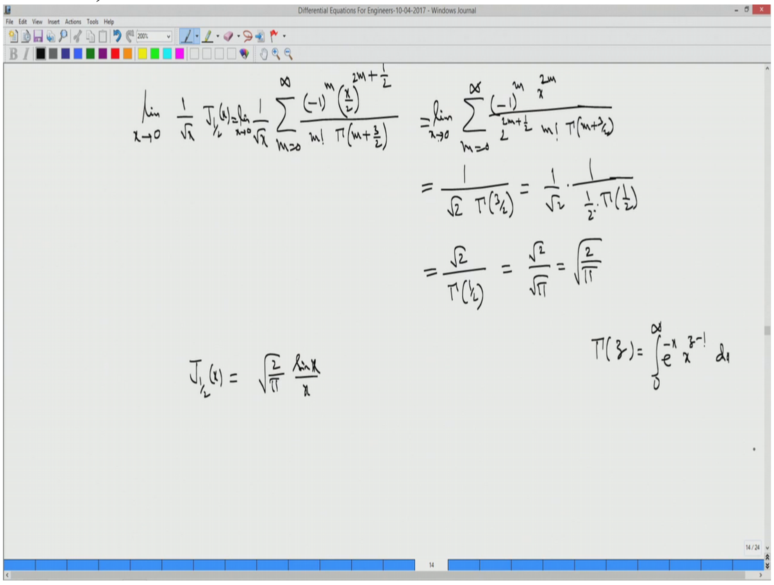

z equal to half, so gamma half, if you calculate, so zero to infinity e power minus x, x power, z is half so you have minus half d x.

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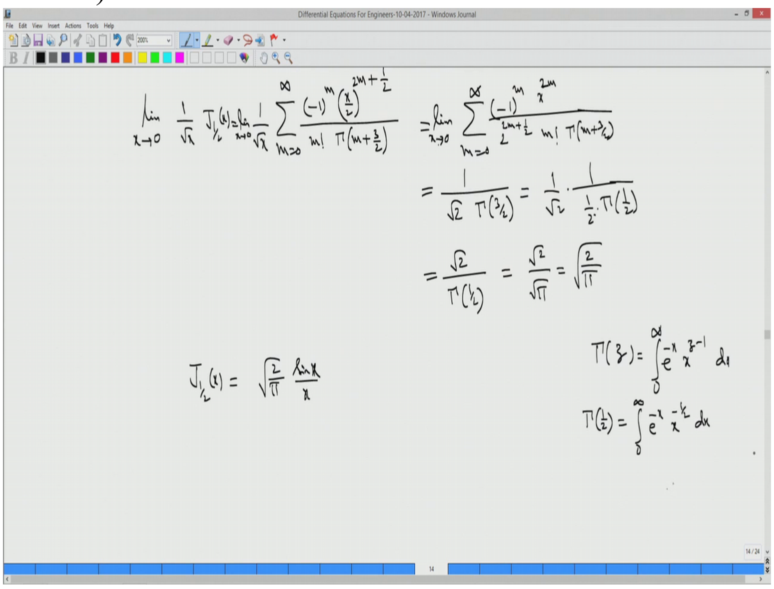

So you just do the integration by parts and you can find that its value is 1 by root pi you will get, Ok so you can do that as an exercise. So this is

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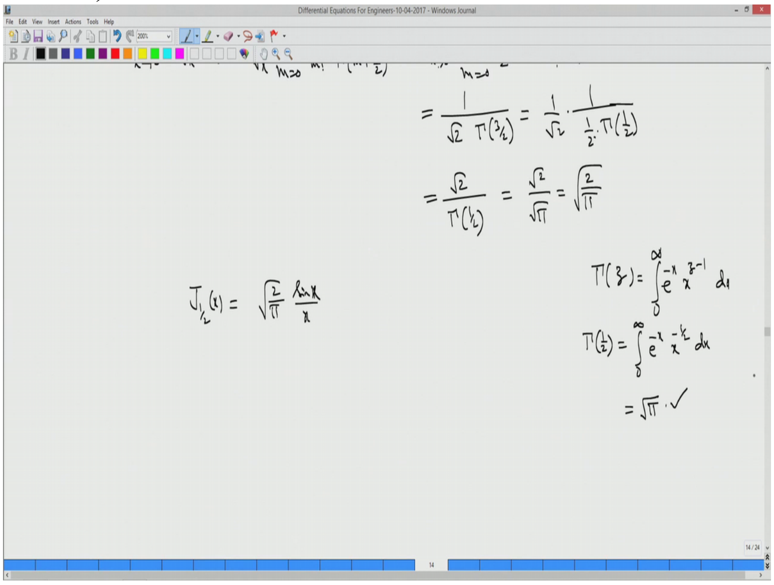

gamma half is root pi, so you will get that. So you use that to get this one.

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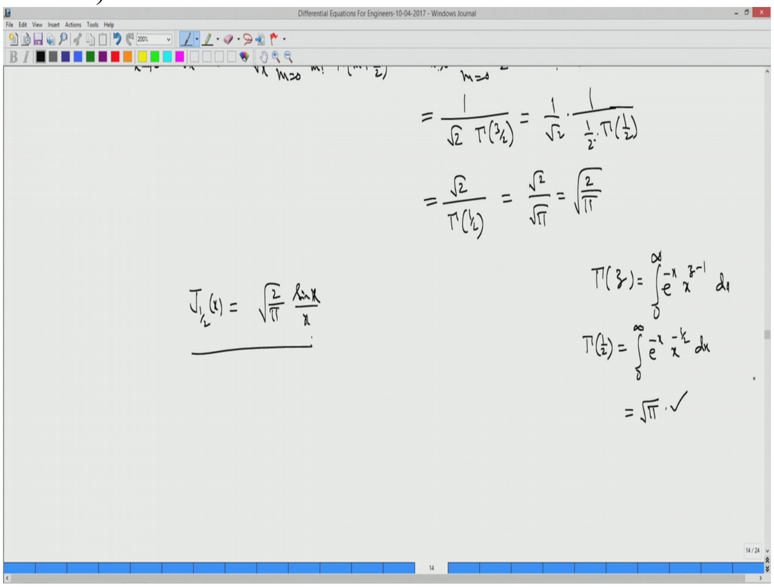

Now to find what is the, next question is J half of x. What is my J half of x? This also you can show that root 2 by pi cos x by x Ok, how do I show

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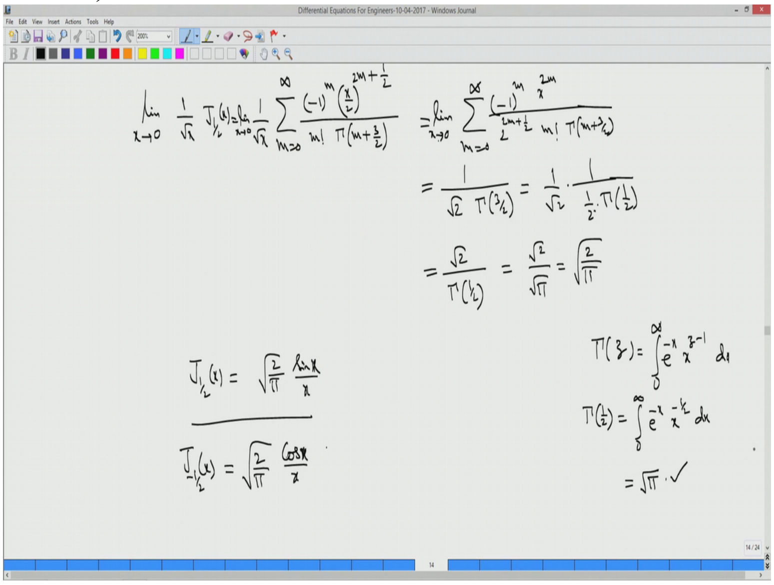

that? That is not difficult to show.

So we use the property that, what is the property, that root x right, this should be root x, sin x by root x, so here also, cos x by root x, Ok,

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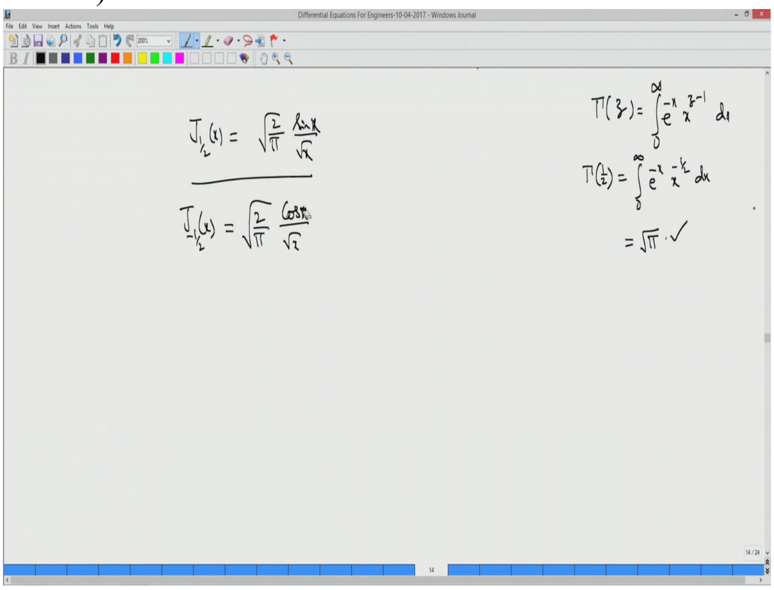

this is what you get, this is what you want to show, Ok. To show, you can also show that this one, this is,

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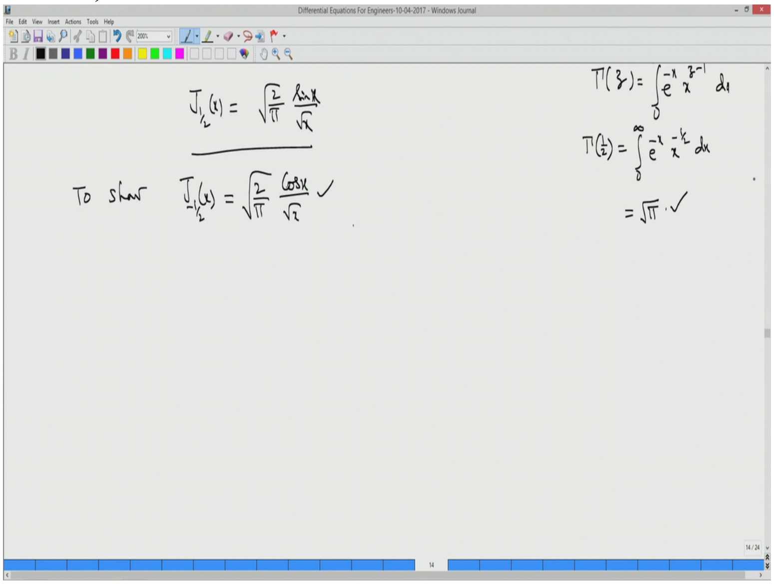

how do you show this one, so we use this property, d d x of x power half, x power alpha into J alpha, J half alpha, J alpha, Ok, this is equal to x power alpha and this is J of alpha minus 1, that is going to be minus half of x, Ok. So you want to, you want this one, so the left hand side you replace.

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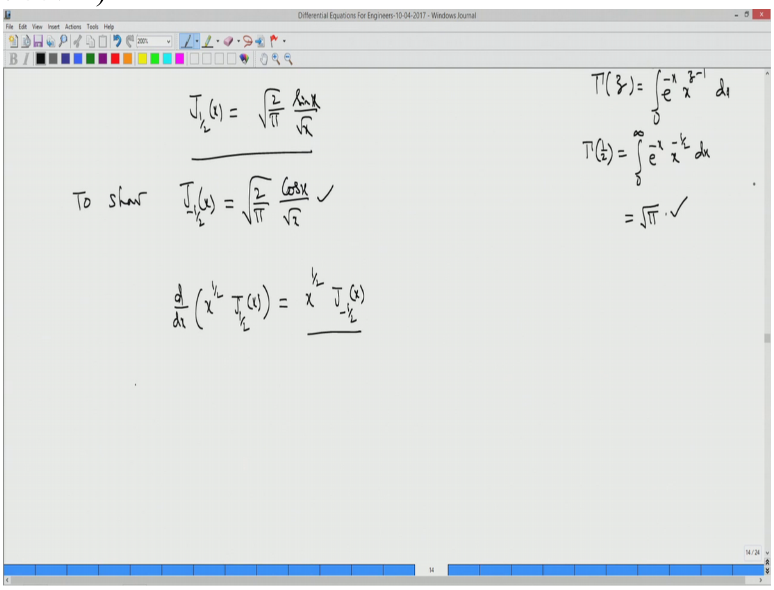

What is the left hand side, this is the derivative of, you can see that J half, this we know already, this is what we proved, Ok, root x into J half is root 2 by pi sin x which is J half J minus half of x from the property 1, Ok. This is from the property 1.

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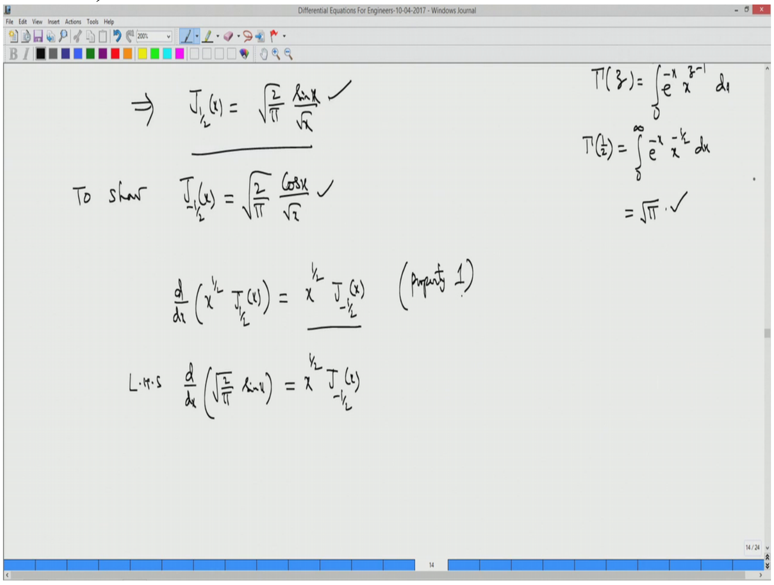

So this gives me, what happens, so right, this property 1 if you use, this is what you will get but I know that d x root half, root x into J half is, I already know from here, so if you do that this is nothing but root 2 by pi cos x equal to root x J minus half of x.

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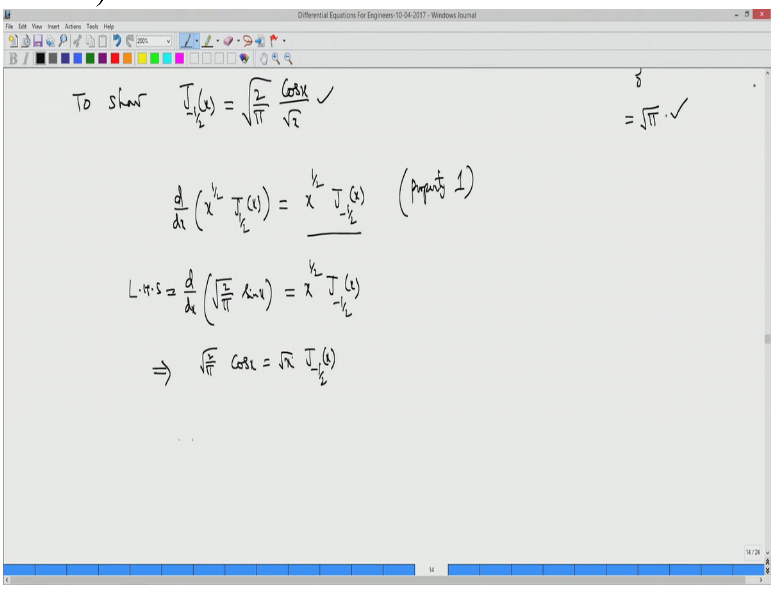

So implies J minus half of x is root 2 by pi x cos x, what you will get, Ok.

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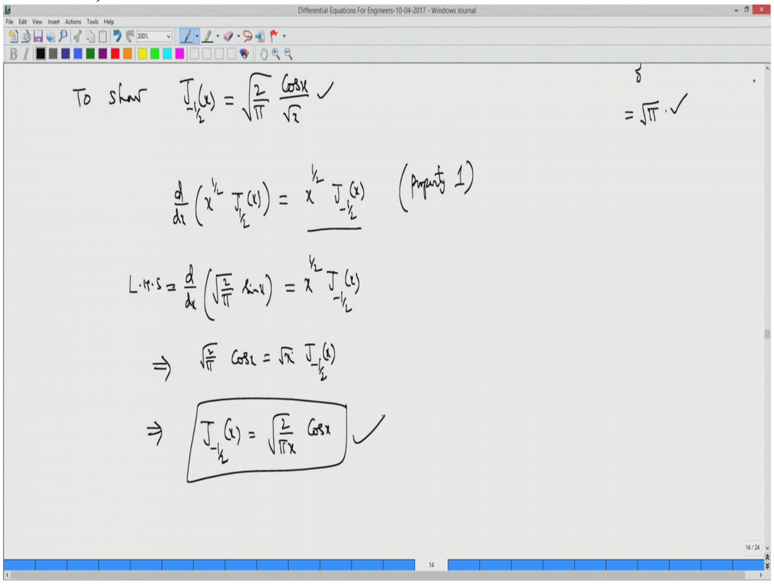

So here, we already know this one, Ok, this is what you

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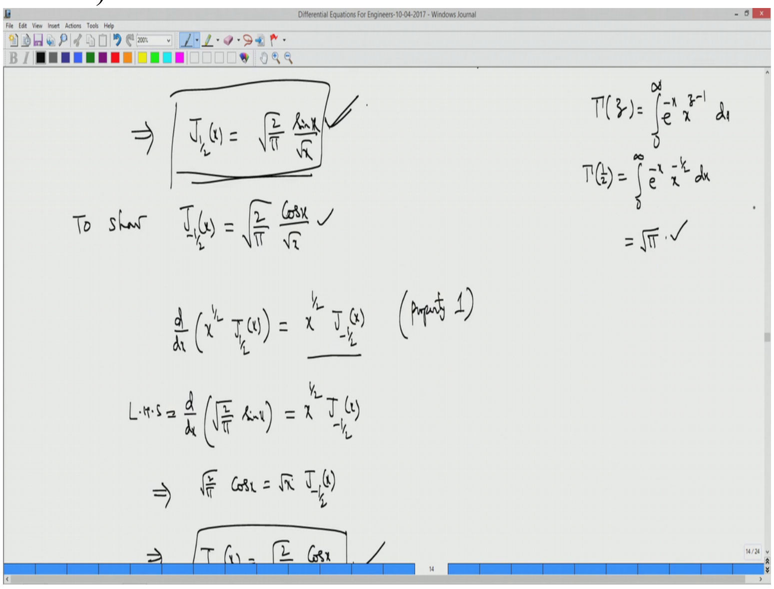

have shown. So you have nicely get J half and J minus half.

Ok, so

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we will, we have used the property 1 and property 2. You just get, you just expand property 1 and property 2 and add and subtract them to get 2 more relations. Using that you can actually get all J 3 by 2, J 5 by 2, all J, J 3 by 2, 5 by 2, or J minus 3 by 2, J minus 5 by 2 and so on, Ok. All these things you can get it as nice expressions in terms of cos x and sin x and root x, Ok, in terms of this you can express. We will see this in the next video.