Differential Equations for Engineers
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Lecture No 30
Properties of Bessel Functions

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So in the last video we have seen how to solve the Bessel equation when 2 alpha is non-zero integer, non-zero integer we have seen, that is corresponds to the case 2. In case 2 itself we have a special case. You don't have to calculate the second linearly independent solution by the method of Frobenius. You can see that the first solution itself will give me the way we just replace alpha by minus alpha will give you.

So alpha, so 2 alpha, so n is alpha equal to n, n is, when you have alpha is 2 alpha, so the difference between the 2 alpha is non-zero integer. So you have seen when they are odd integers, when they are odd integers, you have seen that, we already, you can get linearly independent solutions as J alpha and J minus alpha,

Ok and when 2 alpha, the difference between the roots of the indicial equation is even integer that is same as alpha is equal to integer, alpha equal to n, n is from 1,2, 3 and so on, you have seen how to calculate the second linearly independent solution other than J alpha of x. Ok so what is the, what is left is only for the case when alpha is repeated. alpha is repeated, the difference is 2 alpha that is alpha and minus alpha, they are repeated only when alpha equal to zero. So that is the case we see.

So that we solve this case 3 now. Case 3, you consider this

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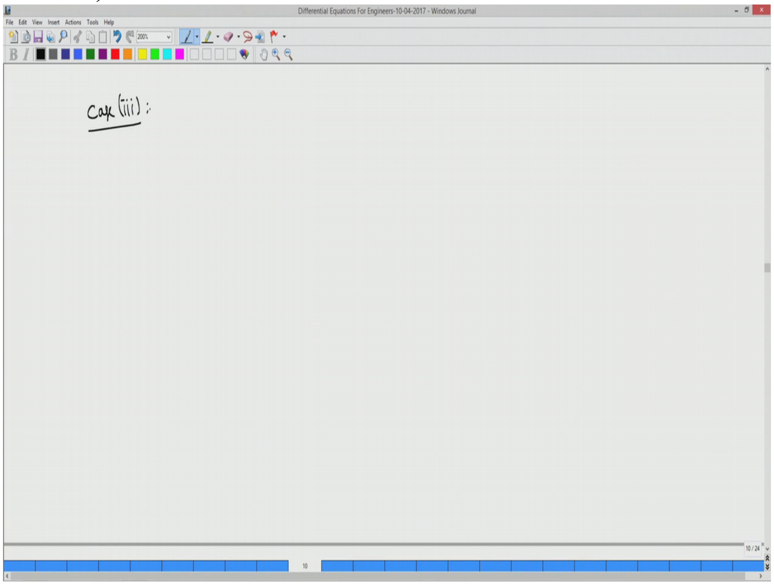

case when, what we have seen is you have already seen that in this case, when alpha equal to zero, so we have alpha equal to zero, so the equation, equation is simply x square y double dash plus x y dash plus x square minus alpha square, alpha is zero y equal to zero.

So this is the equation, x positive, so

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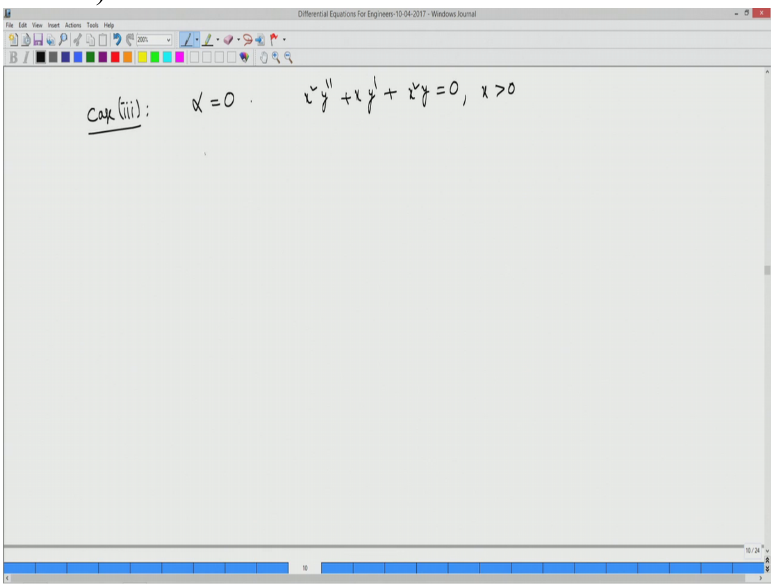

you have already seen that this solution is J, y, y 1 of x is this solution. One solution is this, Ok. This is one

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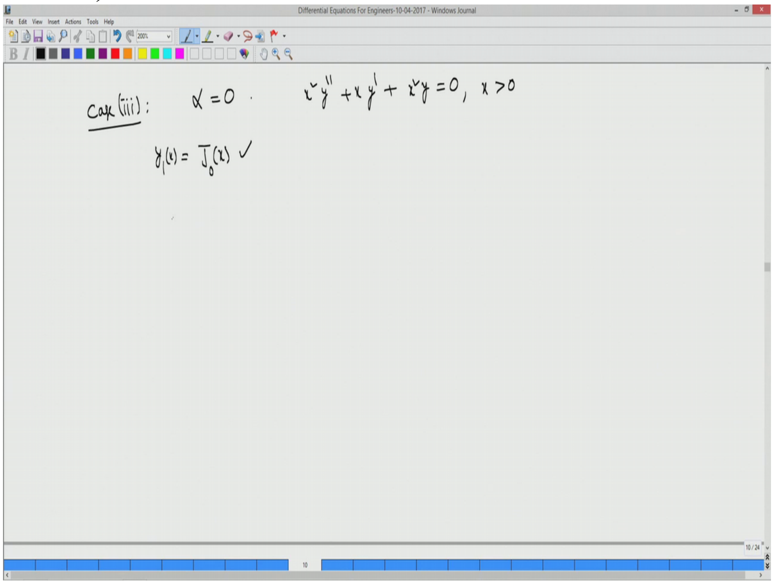

solution. So to get the other solution, you know when the repeated roots, when the

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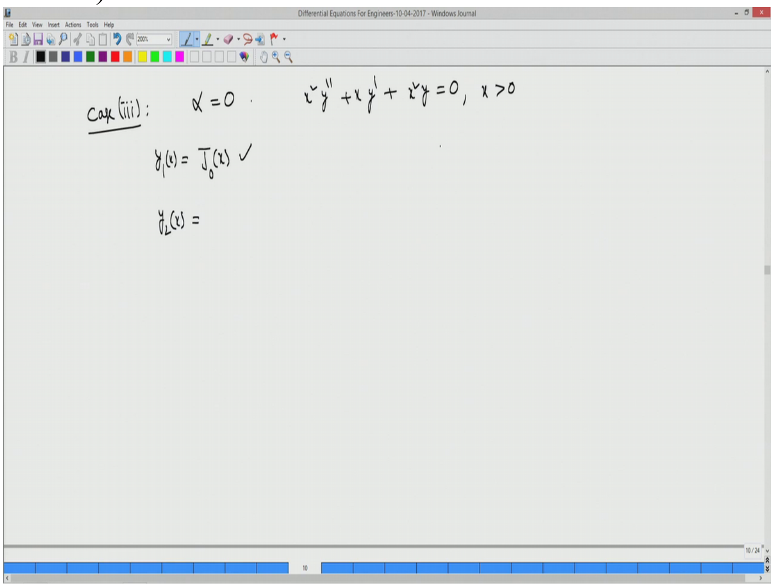

roots are repeated, Ok, so we have seen that, when you look at k square minus alpha square equal to zero. So

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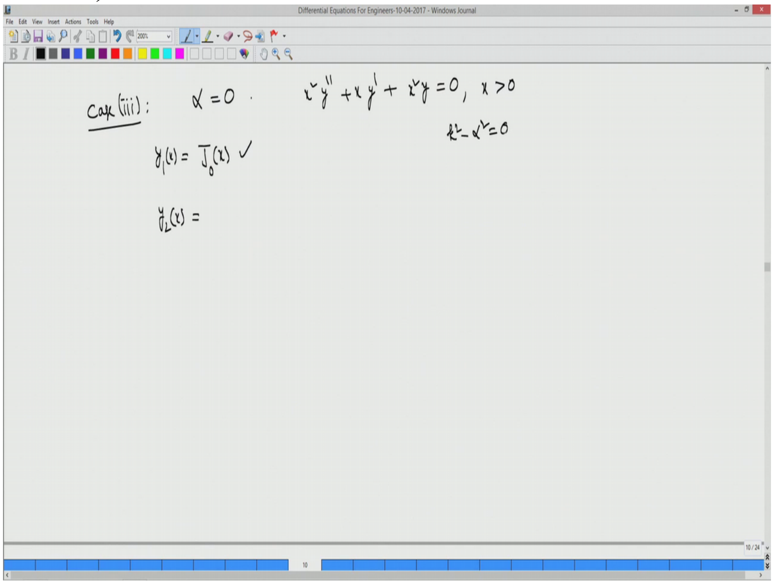

k equal to plus or minus alpha.

So, when they are repeated roots if alpha is equal to zero, Ok. So in this case, second linearly independent solution is,

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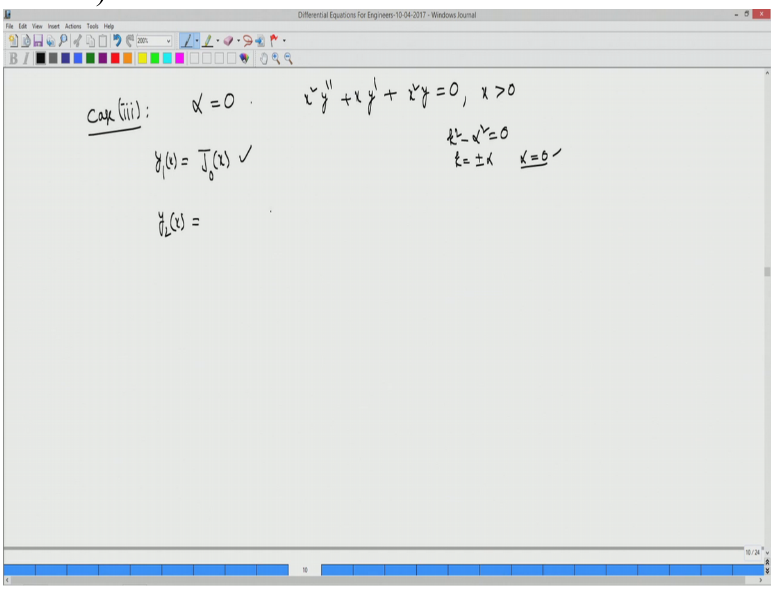

there is no arbitrary constant here, so it is simply J naught, that is y 1 of x into log x plus so x power zero, x power r, x power other root minus alpha, x power minus alpha, alpha is zero so that is simply 1. You have n is only from 1 to infinity some d n x power n.

So this is how you should look for second solution

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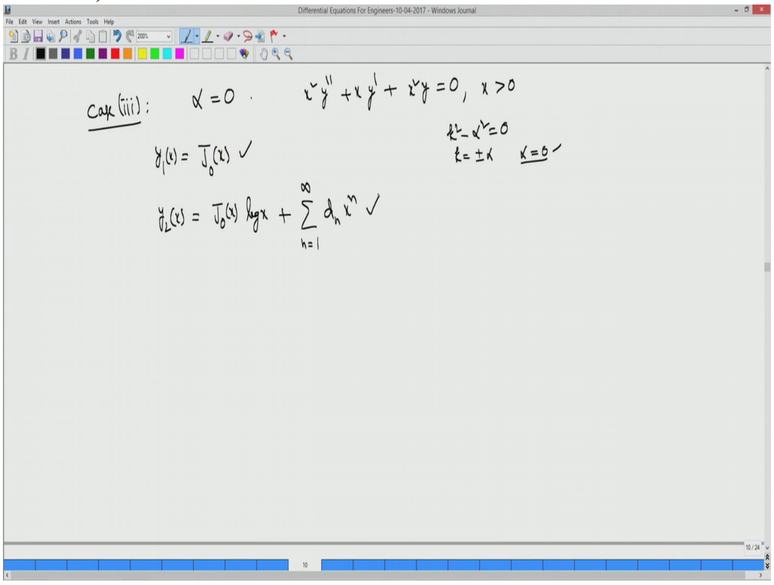

as in the previous, as in the Frobenius method. So you simply solve, try to substitute this equation into the, into the Bessel equation. So substitute this form of solution, Ok, look for y 2 in this form and substitute there, so to get that, you need the derivatives to substitute into the equation.

So you have J naught of x by x plus log x J naught dash of x plus here, if you, if you repeat, if you differentiate, n i s from 1 to infinity, d n, n d n x power n minus 1,

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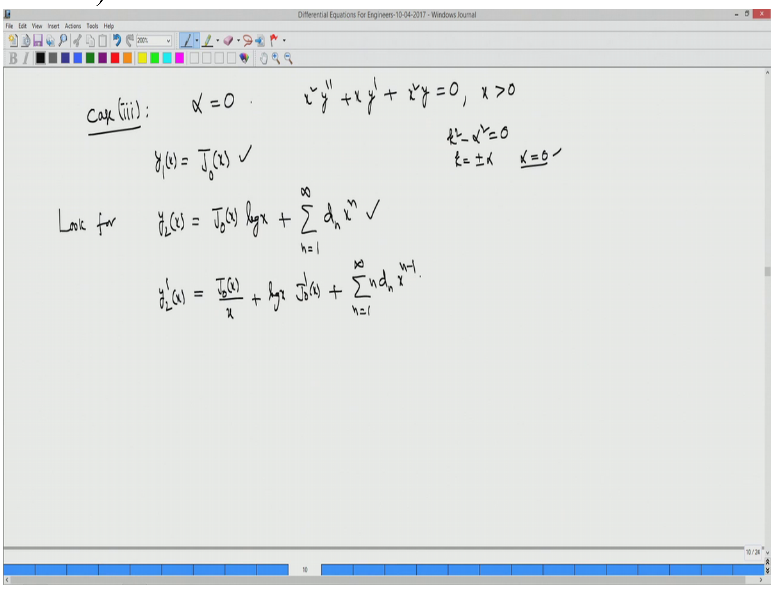

this is what is the derivative. Now if you do this y 2 double dash of x, again minus J naught of x by x square plus J naught dash of x by x, Ok plus if you differentiate this, you get J naught dash of x, so you have repeated, so twice, so you have J naught dash of x, J naught double dash of x into log x. This is, so you see these are the 3 terms you get if you differentiate these 2 terms. And then you have n into n minus 1 d n x power n minus 2. Now this is running from, x power zero so constant. If you differentiate it will go, so starting from 2 to infinity.

Ok so these 3,

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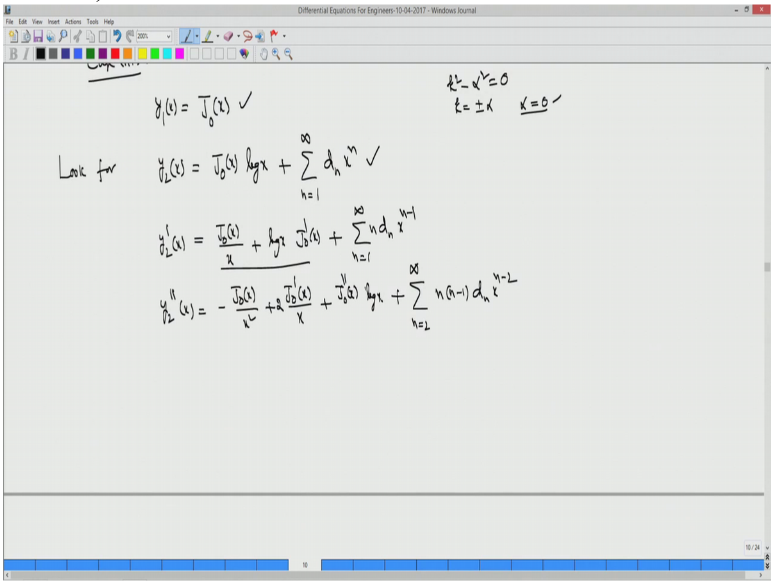

you substitute in the equation, x square y double dash, that will give you minus J naught of x plus 2 x J naught dash of x plus x square J naught double dash of x into log x. Plus n is from 2 to infinity t n into n minus 1 d n x power n. So it starts from n is 2 onwards, Ok.

This is the

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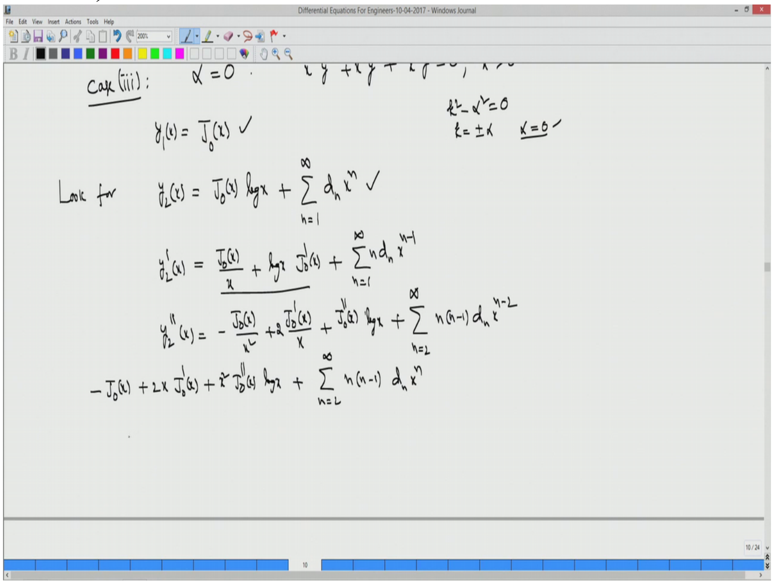

x square y dash, x square y double dash plus x into y dash, x into y 2 dash that is J naught of x x log x J naught dash of x plus you simply have again 1 is from and d n x power n. Our remaining is x square, x square y, so that is x square into y is, so x square J naught of x log x plus x square so this you have and you multiply this x square d n x power n plus 2.

So you see that

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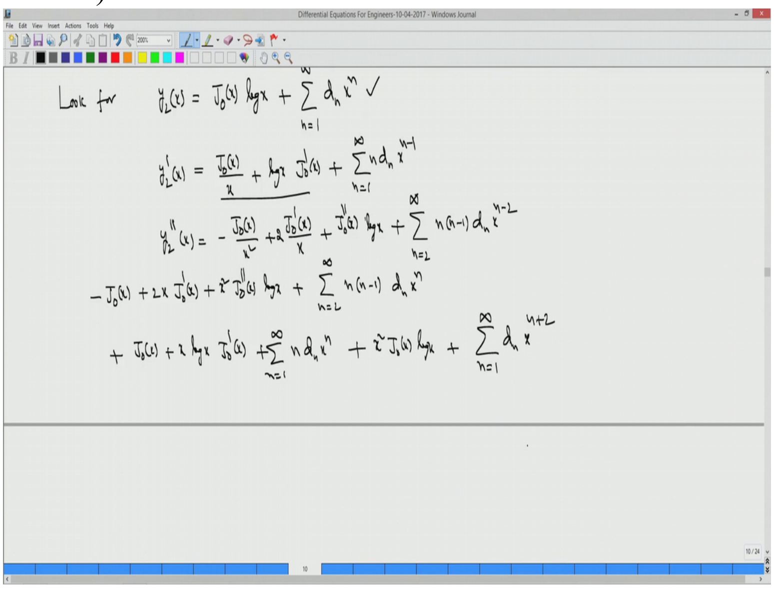

this is what you substituted and equal to zero. This is what is the equation. This, this goes

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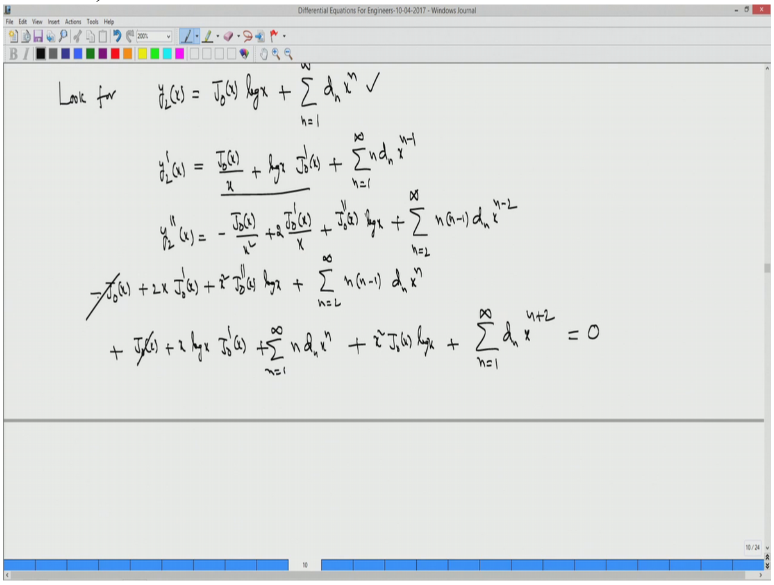

and you see that all the powers of series are x power n so here if you make it n equal to n minus 2, if you make n equal to n minus 2, so then becomes n d n minus 2. So if you replace n equal to n minus 2, this starts from

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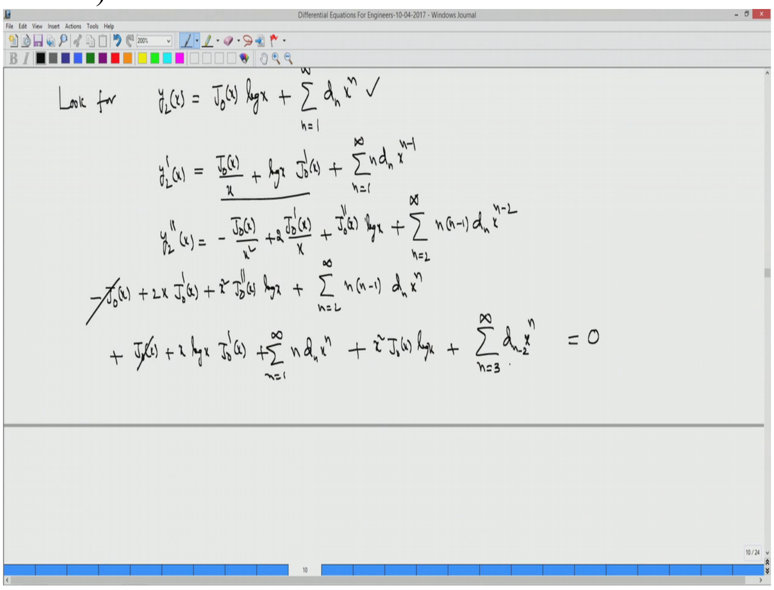

3 to infinity. So you can see that, now look at these 3 terms.

So this one, this one and this one, all the coefficients of log x, if you see these together, J naught is since J naught is

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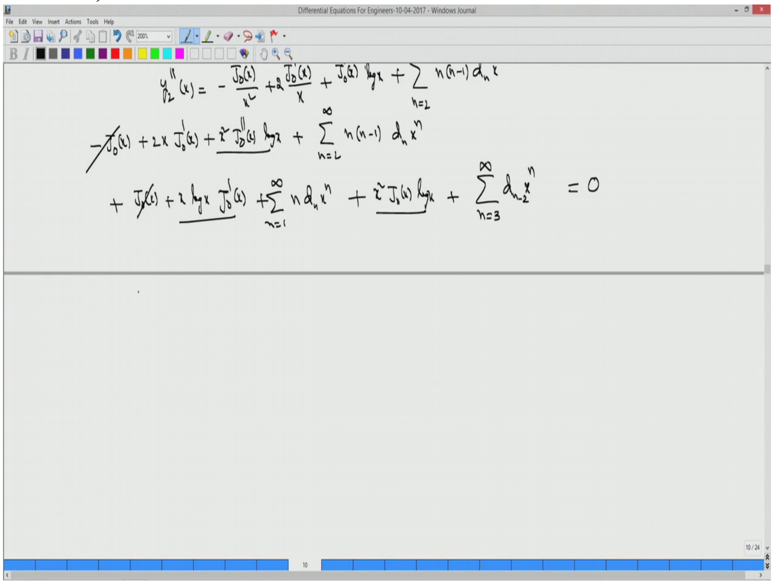

satisfying the equation, so you can see that log x times, so x square J naught double dash plus x J naught dash plus x square J naught of x is equal to zero. So because this is equal to zero, because J naught satisfies the equation, Bessel equation, this has gone.

So plus, now you have 2 x J naught dash of x then plus, now you have, you can put it together, so it starts from, n is from 1 to infinity, here n 3 to infinity, this is 2 to infinity. So common thing is from 3 to infinity, so n is starting from 3 to infinity, so you have n into n minus 1 d n, Ok plus n d n, Ok plus d n minus 2 into x power n

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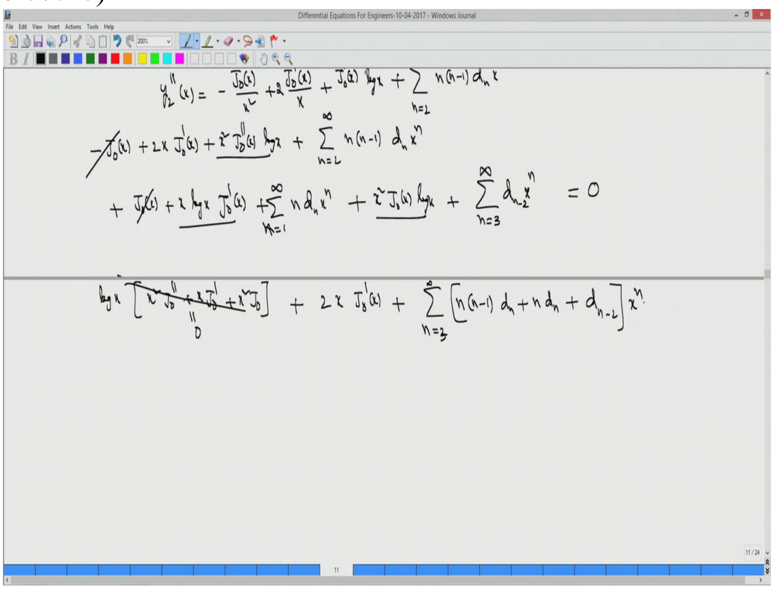

and whatever the remaining terms you missed, here n equal to 2, so it will give me 2 d 2 x square, here n equal to 1 and 2, so here n equal to 1 and 2 will give me, n equal to 1 you have d 1 x plus 2 d 2 x square equal to zero. So this is what you have.

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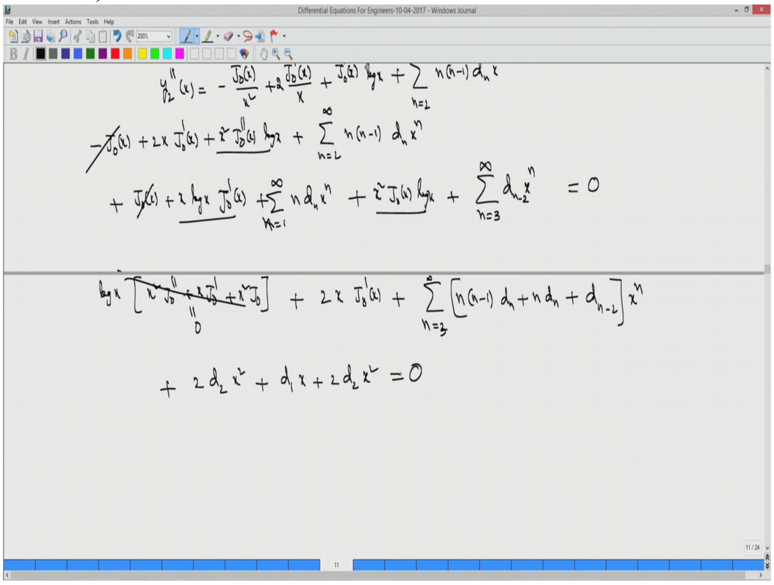

So if we rewrite, so d 2 d 2 x square, so that is going to be 4 d 2 square d 1 x plus plus 2 x J naught dash of x plus this is running from 3 to infinity so you see that here, n into n minus, n square minus n plus n, so that is going to be simply n square d n plus d n minus 2, Ok so into x power n equal to zero.

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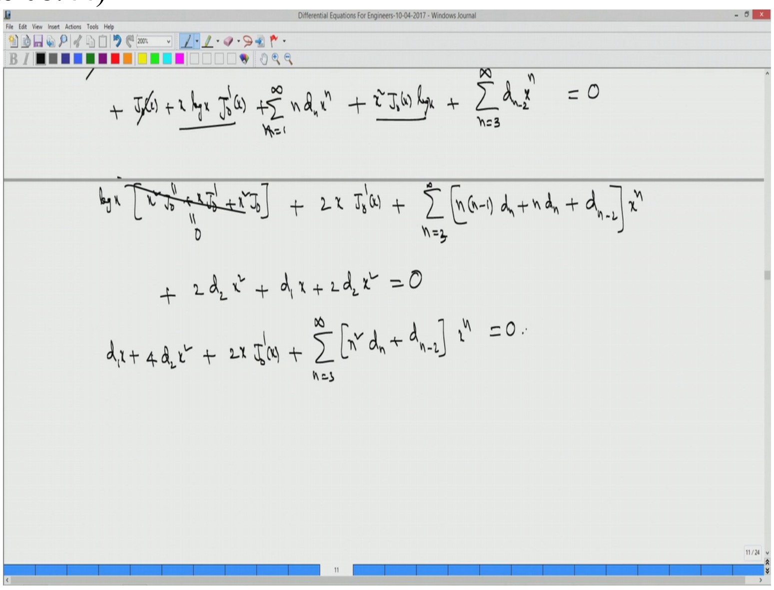

So what is your J naught, J naught of x is actually equal to, so you can rewrite, so that is, you have already seen what is J naught, J naught is 1 plus sigma m is from 1 to infinity minus 1 power m x power 2 m plus alpha alpha is zero, so divide by 2 power, so you have x by 2, x by 2 power 2 m plus alpha, so you have 2 m m factorial gamma of m plus 1, that is one more m factorial that is square.

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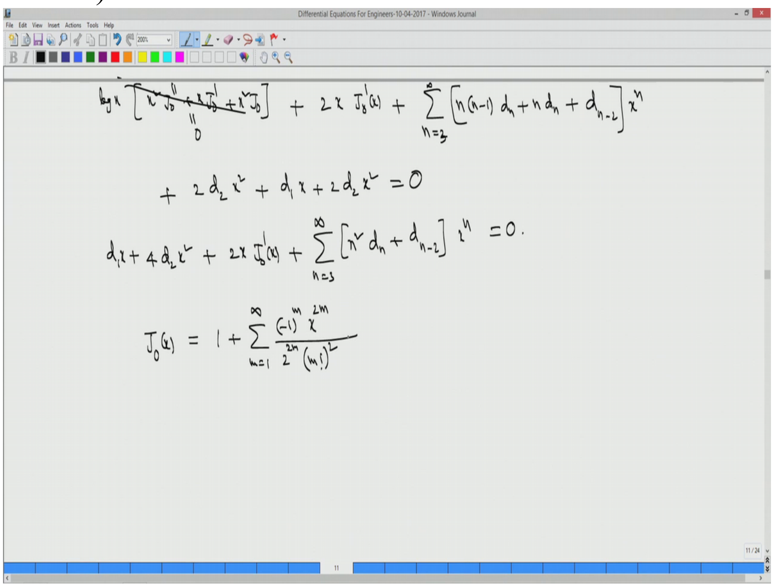

So this is what is your J naught of x. So you need J naught dash of x, so that will give you, that is zero. Here this will give me simply, m is from 1 to infinity minus 1 power m x power 2 m minus 1 Ok divided by 2 power 2 m into

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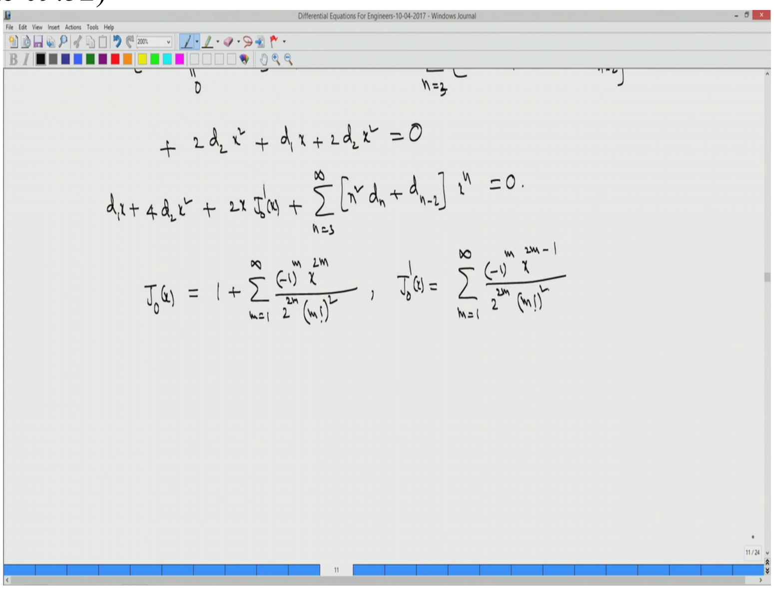

m dash square. This is what your J naught dash. So go and substitute into the equation wherever here J naught dash, so you get, so it starts from x onwards. So x, x and so if you see this, this series starting from x cube onwards, this is x, x square this is starting from x

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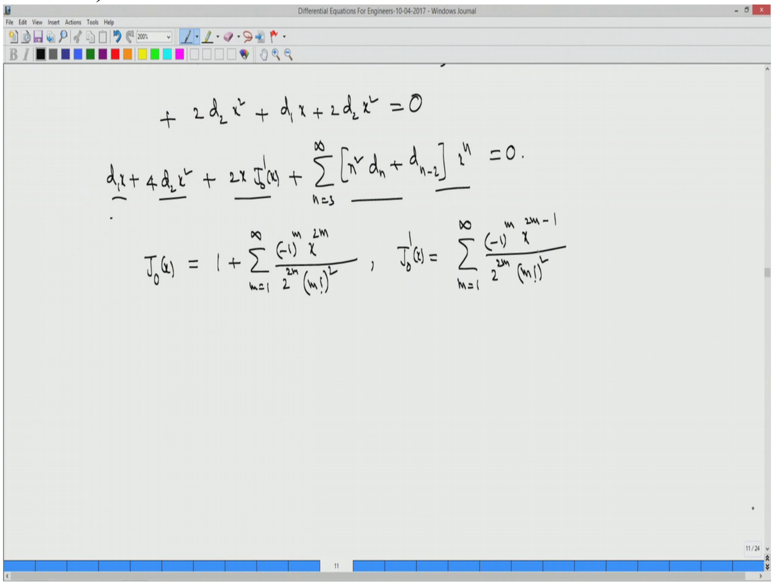

Ok so x, x cube, x 5 and so on.

So J naught dash is actually x powers of odd. So finally you get this d 1 x plus 4 d 2 x square plus 2 x, 2 x you multiply so two times, so if you do this, if you replace m is from 1 to infinity minus 1 power m x power 2 m divided by 2 power 2 m minus 1 m factorial square. This is what is 2 x J naught dash plus n is from 3 to infinity n square d n plus d n minus 2 into x power n equal to zero. So

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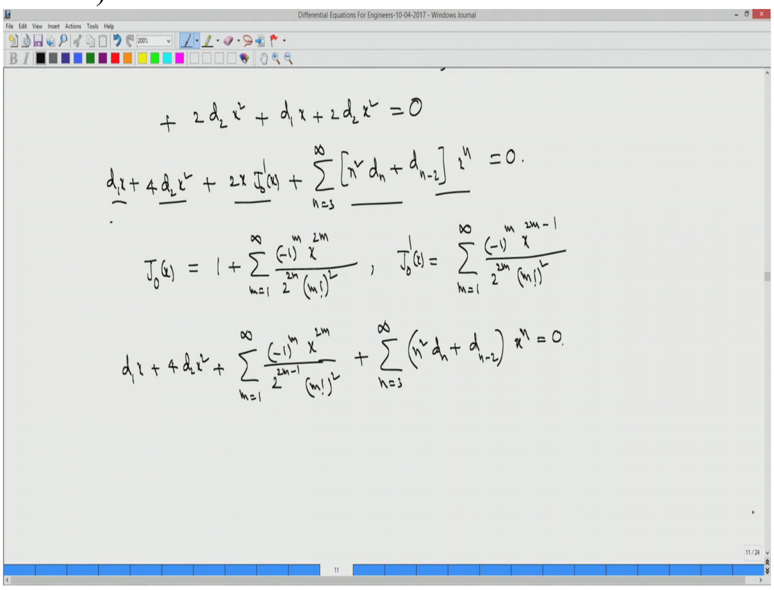

now you can equate, these are odd. These are now only even terms, so x square, x 4 and so on, so this is running from x 3 onwards. So equate the coefficients of lowest, coefficients of starting with x equal to zero will give me, so d 1 equal to zero, Ok. d 1 is the coefficient

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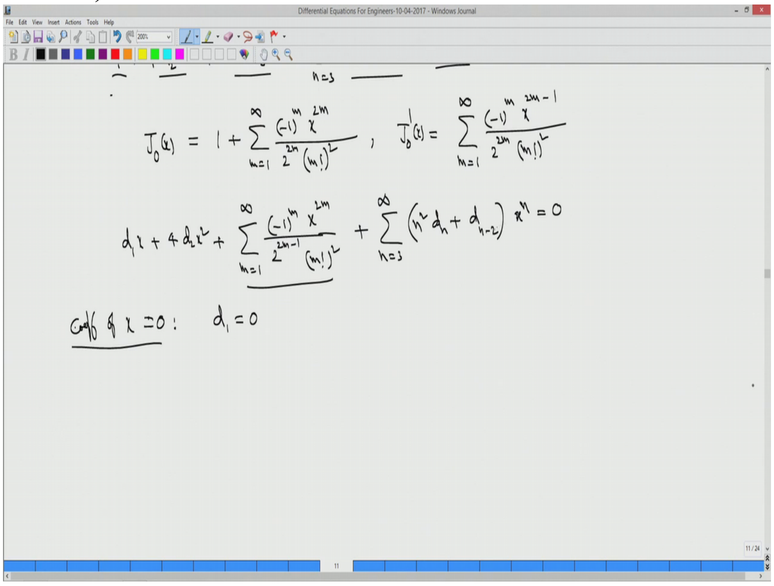

of x, so coefficient of x square will give me, what is the coefficient of x square, 4 d 2 and here m equal to 1, so that corresponds to minus 1 by, m equal to 1, so you have 2 and you have m equal to 1, so this is simply half, Ok plus and this is zero so that is going to be zero. So this will give me d 2 equal to 1 by 8,

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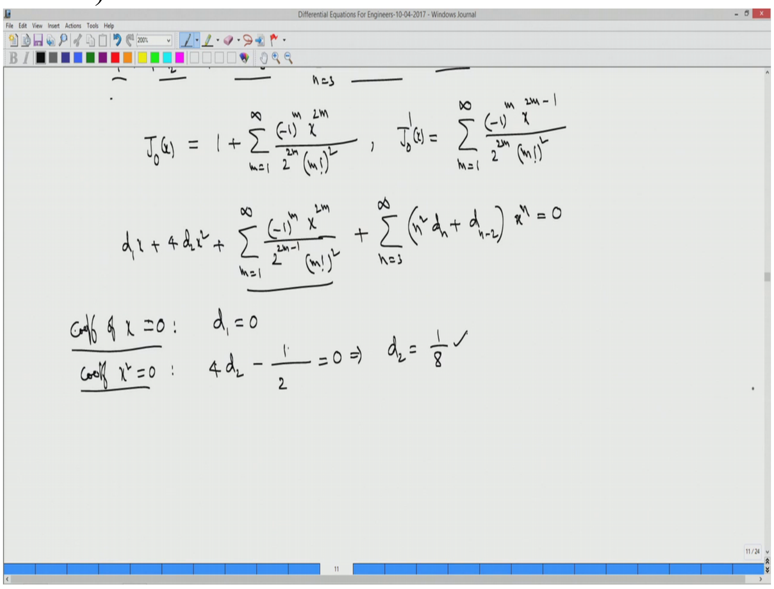

Ok.

Similarly coefficient of x cube equal to zero. So this will give me x cube, what is x cube here, so you have, there is no, these are all even. So you have only here so that will give n square d 3 plus d 1 equal to zero. Because d 1 is zero, d 3 has to be zero. So implies d 2 n plus 1 equal to zero for every n, 1,2, 3 and so

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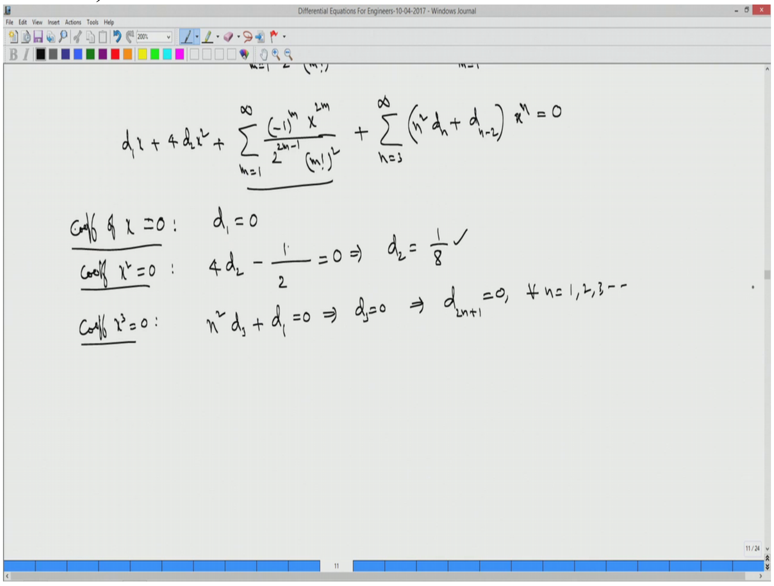

on, because, because of this. These are all odd. These are all the odd coefficients, Ok, x power of odd numbers. So that coefficient has to be zero.

So from this, because d 1 is zero, d 3 is zero. d 3 is zero, next time d 5 will be zero and so on. So you will get all the odd coefficients will be zero. Only now we have to consider coefficient of x power, while say 2 n, 2 m equal to zero will give me, so start with, if you want you can say simply x power 4, x power 4 will give me, you get m equal to 2. So you have 1 by 2 power 4 minus 1 so 3, 2 factorial square, Ok plus n equal to 4 is 4 d 4 plus d 2. Ok, so 4 square d 4 plus this is equal to zero. So this will give me d 4 equal to minus d 2 minus 1 by 2 cube 2 factorial square divided by 1 by 4 square.

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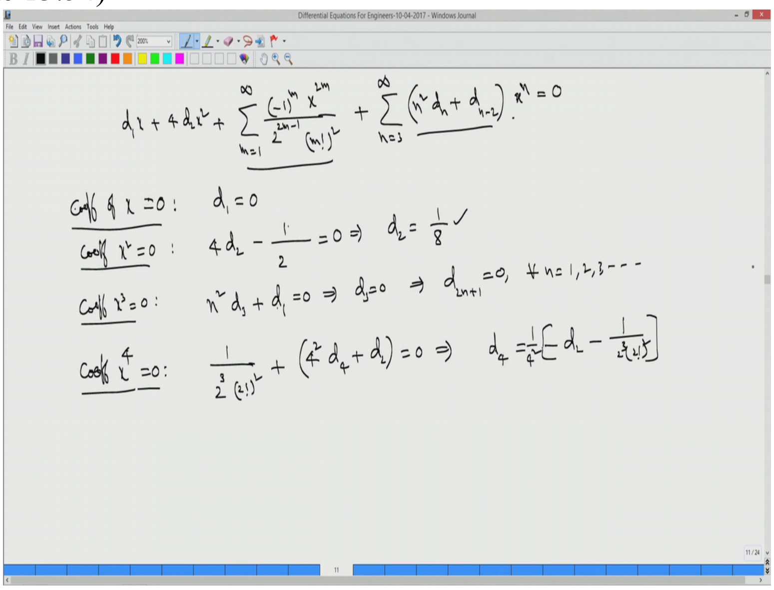

So what is d 2, d 2 I know already, so 1 by 8, so you have 1 by 4 square minus comes out, so 1 by 8, 1 by 2 cube into 1 plus 1 by 2 factorial square. This is what is your d 4.

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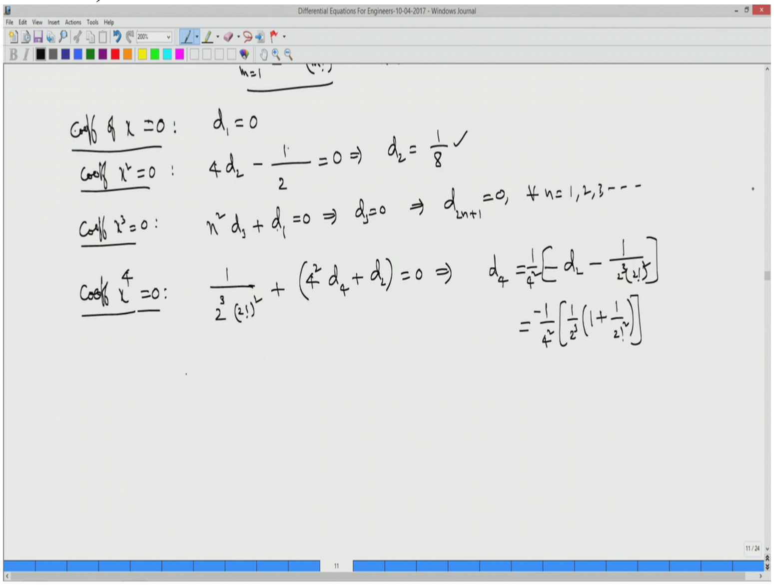

So like this you can get coefficient of x 5 zero, if you make it equal to zero, you see that d 5 will be zero. So that we have already seen here. So you make it equal to 6 equal to zero, so you see that d 6 will be, so for this you have to use m equal to 3. m equal to 3, so you have minus 1 by 2 power, m equal to 3, 5, 3 factorial square plus 3 square, this is m equal to 6, so x power 6 you need, so n equal to 6 here in this series. So n equal to 6, so 6 square plus d 6 plus d 4 equal to zero. So this

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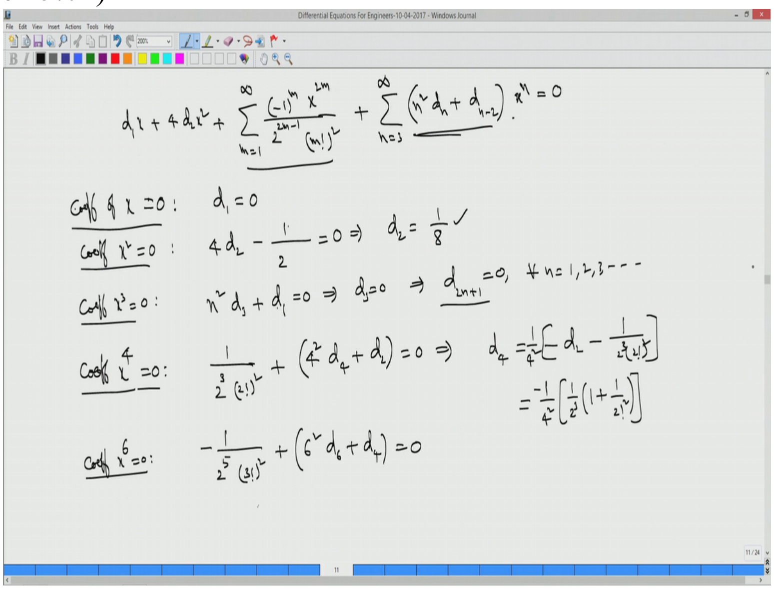

will determine d 6 in terms of, again you know what is d naught, so substitute and get your d 6. It's going to be 1 by 6 square minus d 4 minus, minus minus plus now, so 1 by 2 power 5 3 factorial square.

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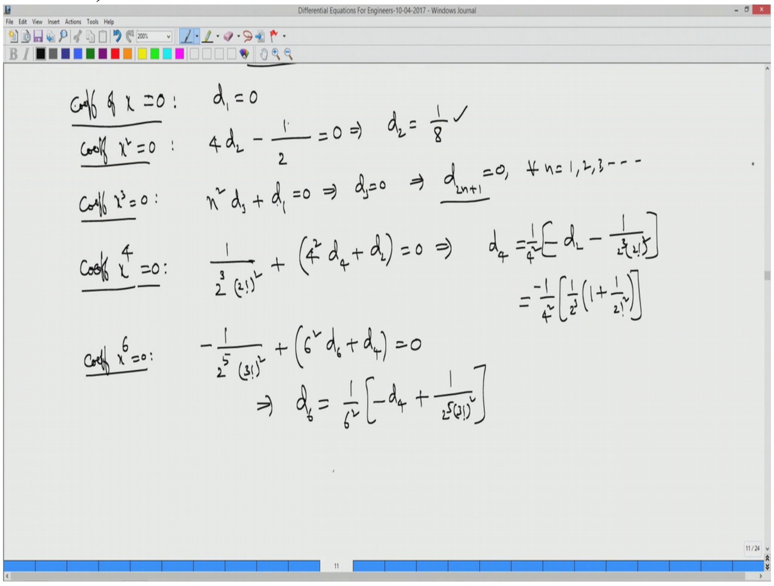

So d 4, so now d 4 you substitute, so you can see that some number, Ok, minus, my minus d 4 is 1 by 4 square into 1 by 2 cube 1 plus 1 by 2 factorial square, Ok, that is what is my d 4 plus 1 by 2 power 5 3 factorial square, something like this. So you get this d 6 and so on, and so on, Ok.

And so on, you get

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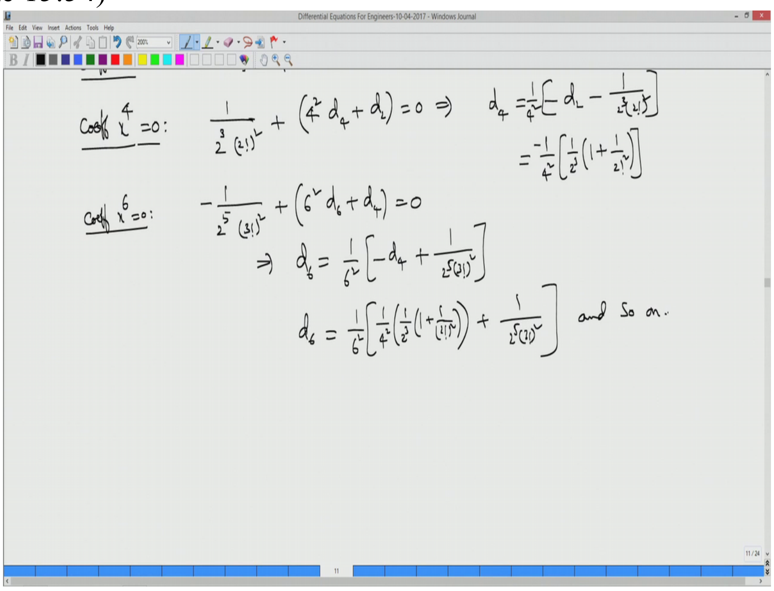

all other coefficients, so go back and substitute here, all coefficients into your second solution which is in this form. So what are the unknowns? Only to get d ns, which you have found,

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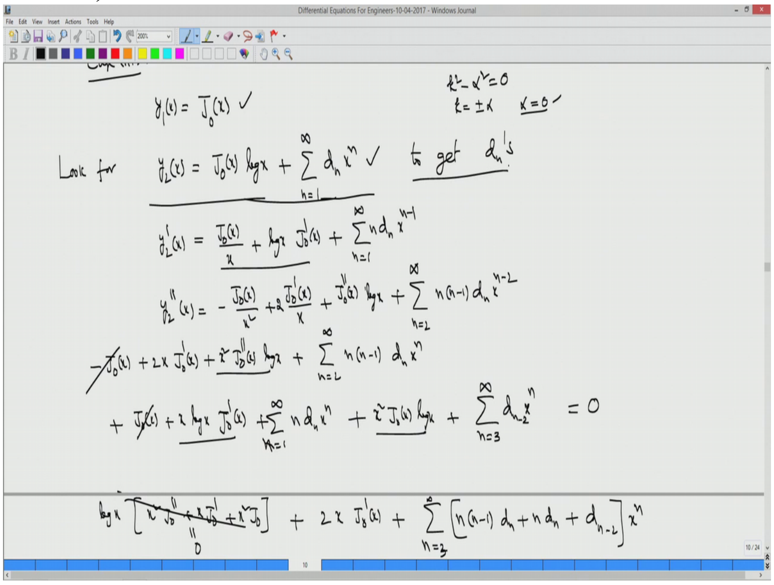

so that is your second solution. Again here you can define whatever you get here, so after getting these coefficients, that is actually whatever you get, so y 2 of x is actually Bessel function of second kind, second kind of zeroth order, because alpha is zero, Bessel function you can write y zero if you want, y of zero.

You have seen earlier y n of x and alpha equal to n. Now you see that y n of x by definition, this is your form y 2 which is given here,

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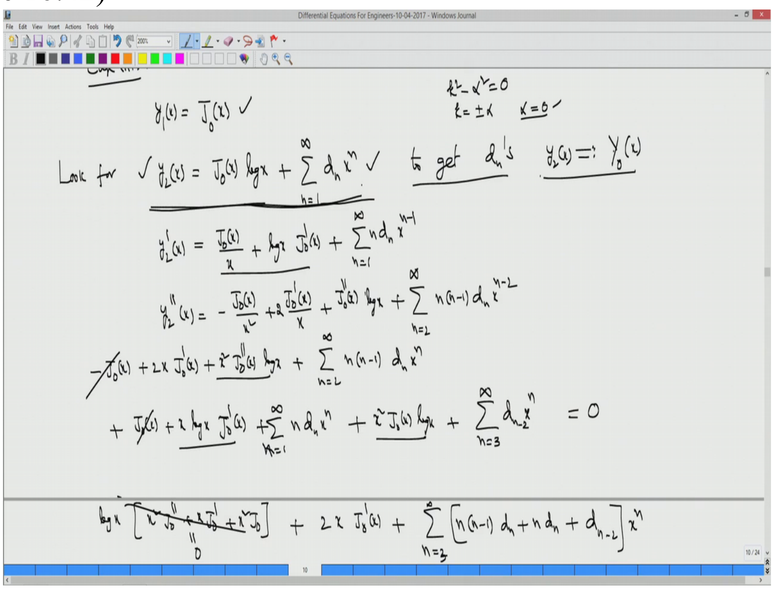

Ok with all the d ns which you calculated here. So that is how you define Bessel function of second kind with order alpha equal to zero, Ok. Bessel function of zeroth order of second kind, second kind Bessel equation of order zero.

So this is how we get this

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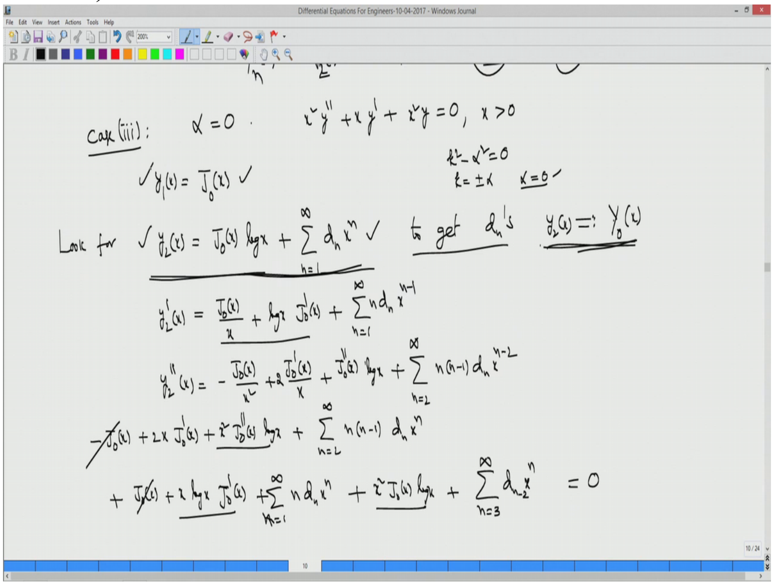

y 1 and y 2, you simply apply the Frobenius method to find the general solution. So general solution of the Bessel equation in this case is y of x, this is the case corresponds to alpha equal to zero, y 1 is simply c 1 y 1 x. y 1 we know already, this is J naught of x plus c 2, some arbitrary constant that y zero of x, that is your y 2, Ok, just calculated. So this is your

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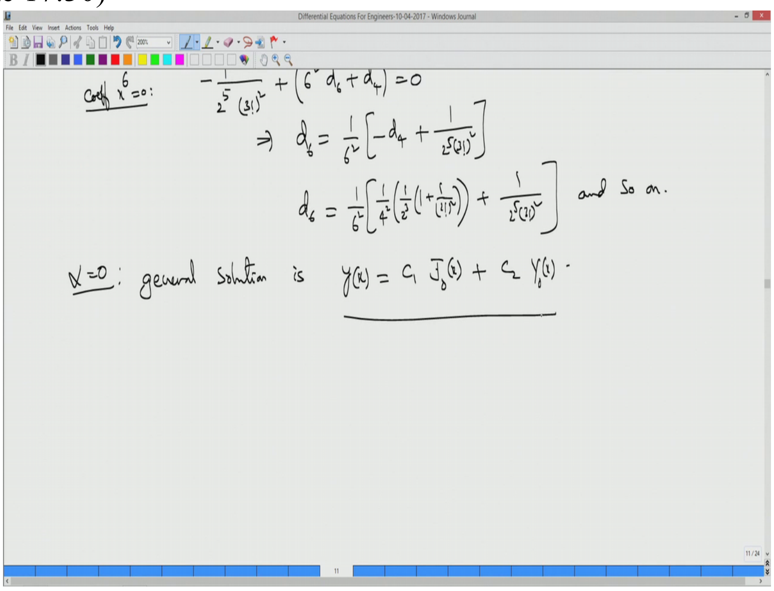

general solution of the Bessel equation in the case when alpha equal to zero.

So we have seen the special case when 2 alpha is odd integer, odd non-zero integer. That means 2 alpha is 1, 3, 5 and so on. In this case, general solution we have seen that c 1 times J alpha of x plus c 2 J minus alpha of x is what you have already seen, Ok. This is the special case.

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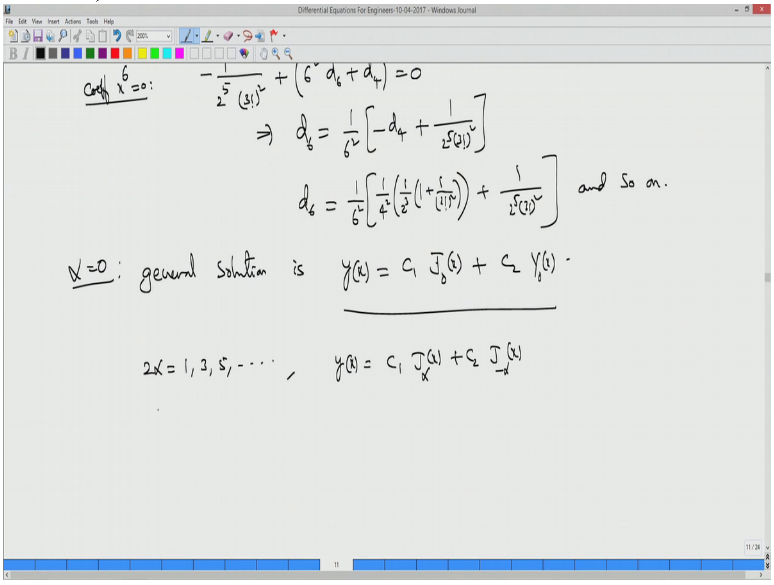

Only when alpha is, 2 alpha is 2,4,5 and so on, 2, 4, 6 and so on, when it is even, that is same as saying this is, alpha is 1,2,3 and so on. You have seen the general solution is c 1 times J alpha of x plus c 2 times y alpha of x, this is unbounded. You can see that the second solution

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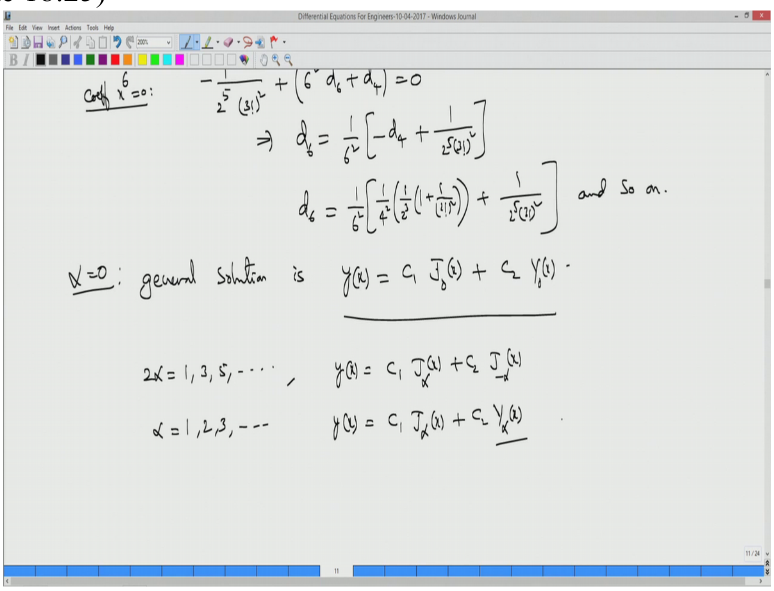

because of log x, log x term y x is unbounded. Here J alpha of x is also unbounded because of x power minus alpha.

Now what else you have seen? So this is, this is what when 2 alpha is integer. So when alpha equal to zero, so we have already seen this one. This is the case. This we have seen, so when alpha is non-zero, so when 2 alpha doesn't belongs to Z, any integer, in this case, again it is the same type. So that is like this type. c 1 J alpha of x plus c 2 J alpha of x, Ok, so these are the, this is how

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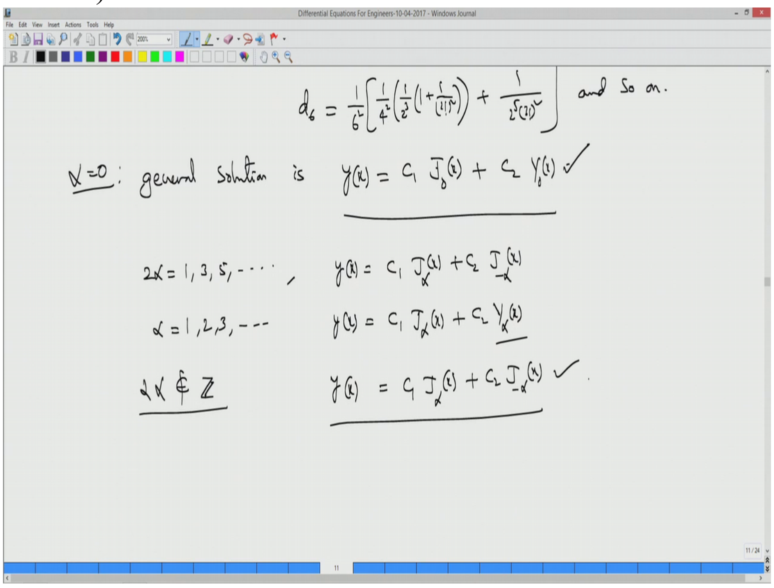

you get, whatever may be alpha value, depending on the difference between the roots of the indicial equation, you have the general solution of the Bessel equation.

But in this special case, you have this one. These are, this you can calculate in a different way, that we will using the properties of these Bessel functions, Ok. So you will have certain properties of Bessel functions.

Start with, so you

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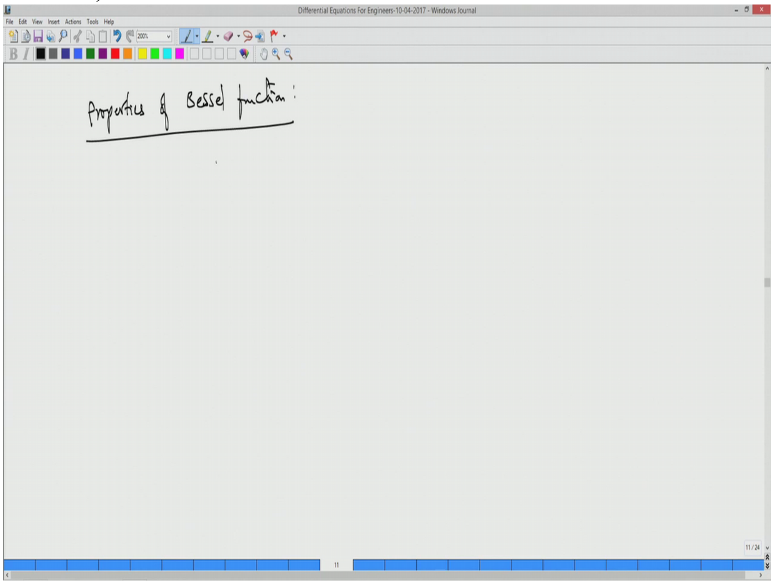

know what is J alpha of x, so let us write J alpha of x and then you can just write the properties, Ok. So having known what is J alpha of x, 1 plus sigma, m is from 1 to infinity, minus 1 power m, x by 2 power 2 m plus alpha divided by m factorial gamma of m plus alpha plus 1, this is what you have, right. So property 1 is you differentiate this, if you differentiate, actually so you have x power alpha, corresponds to m equal to x power alpha here. So x power alpha, this we can combine it and say m is from zero, so if we put m equal to zero also, you see that 1 minus 1 power m zero, so x power 2,2 m plus alpha. When you have m equal to zero, what you have is simply x power 2 alpha. So this is what you have, divided by m factorial gamma of m plus alpha plus 1.

First property is

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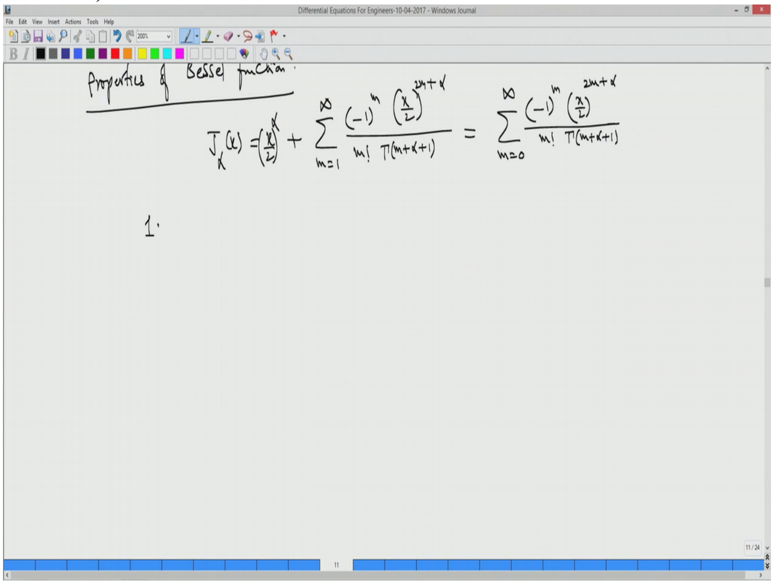

if you differentiate, I will differentiate this, multiply x power alpha and you take this J alpha of x. What you get is x power alpha J minus alpha of x.

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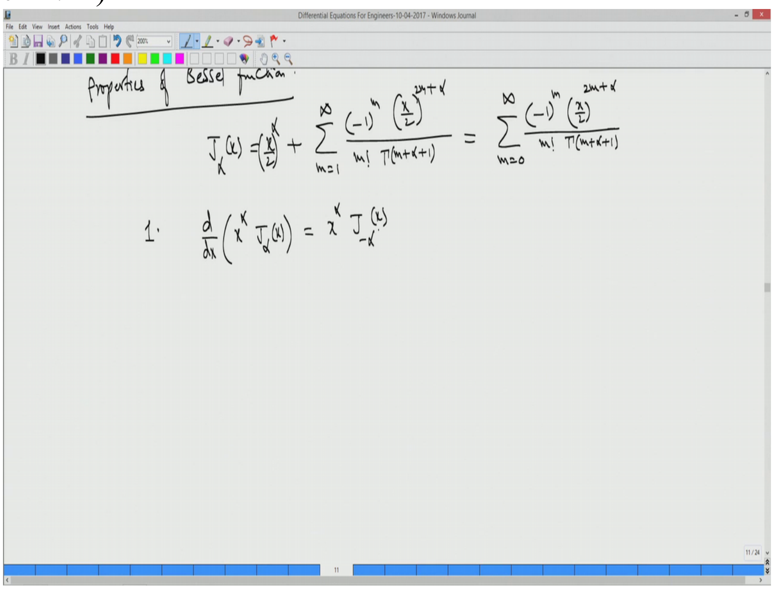

Second one is if you replace alpha by minus alpha, you take the difference, multiply x power minus alpha, J alpha of x. This becomes x power alpha, in this case what happens? x power minus alpha and you have a minus here, J alpha plus 1 of x.

I will first see these 2 properties and then

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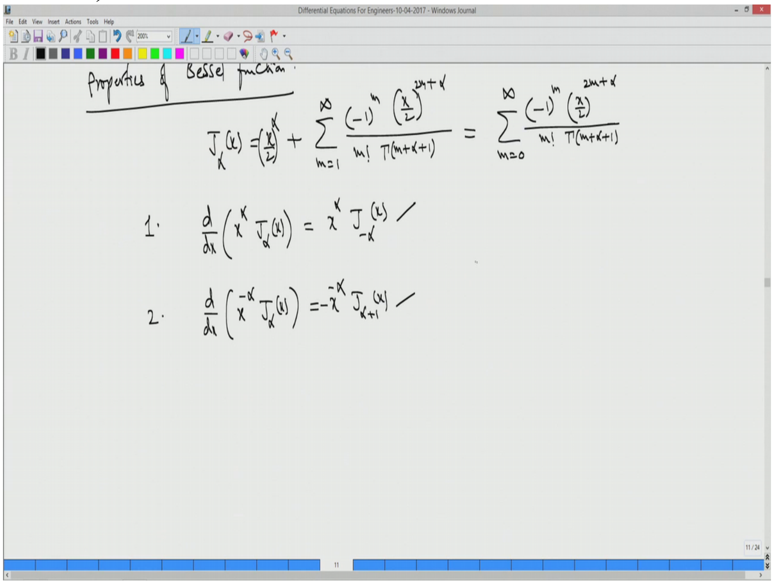

you expand this side, left hand side and if you add and subtract it will give you 2 more properties. So these are important, first we will prove these properties. Simply multiply the first one, to get the first one you consider this. So we will prove the first one. So proof of the first one is you consider this x power alpha J alpha of x. What happens, this will be equal to, m is from zero to infinity minus 1 power m x by 2 power 2 m plus 2 alpha divided by m factorial gamma of m plus alpha plus 1.

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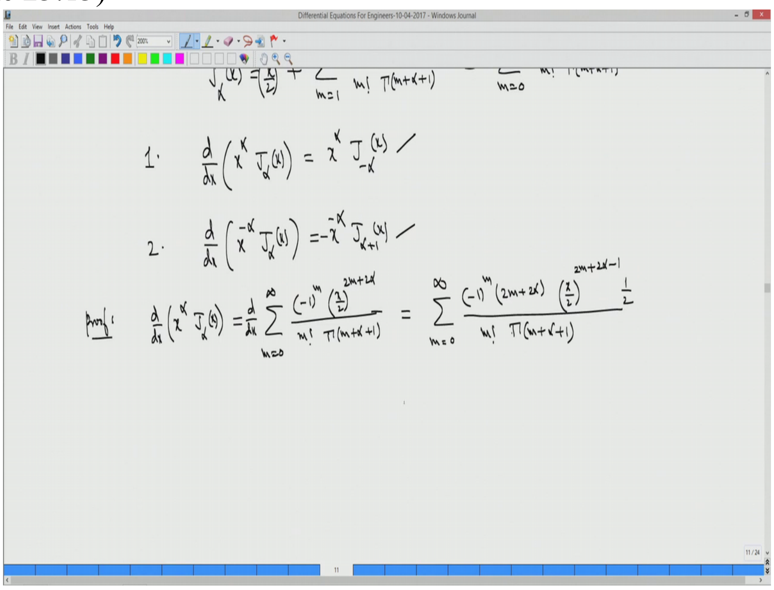

So if you differentiate this, so you have, you have to differentiate this, this will become simply, m is from zero to infinity minus 1 to power m, 2 m plus 2 alpha into x by 2 power 2 m plus 2 alpha minus 1, Ok. What is the derivative? So derivative is simply 1 by 2, derivative of x by 2, so I have 1 by 2 divided by m factorial gamma m plus alpha plus 1, so take this x power alpha out. x power alpha if I take it out, so who what we have is m is from zero to infinity, minus 1 power m, m plus alpha so what we have is, this 2, this 2 goes here, so you have m plus 1, this you can rewrite gamma m plus alpha plus 1 you can write as m plus alpha into gamma m plus alpha, so this you can cancel so

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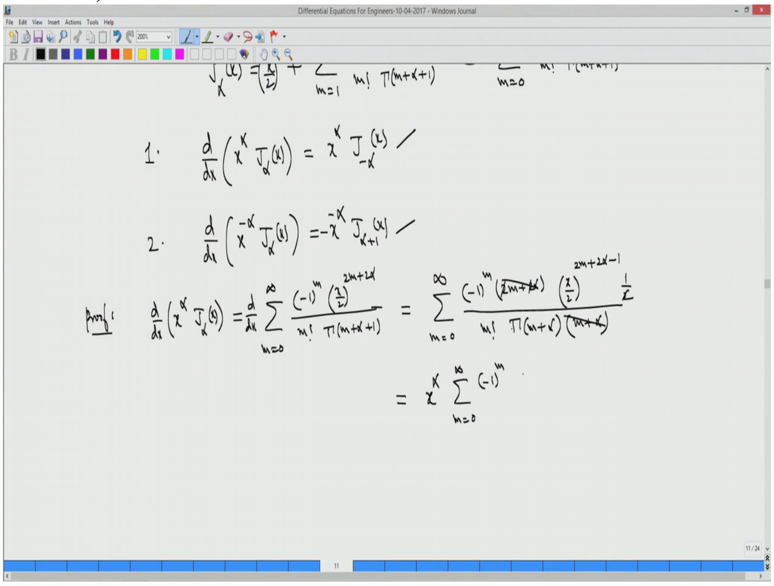

you have x power 2 m plus alpha minus 1 divided by 2 power m factorial gamma m plus gamma, gamma m plus alpha minus 1 plus 1 I can write and what is left is, 2 power 2 m plus 2 alpha, so 2 m plus 2 alpha minus 1. So when I multiply x power alpha, so it is actually only here, not for the full, so this is only x power this,2 power 2 m

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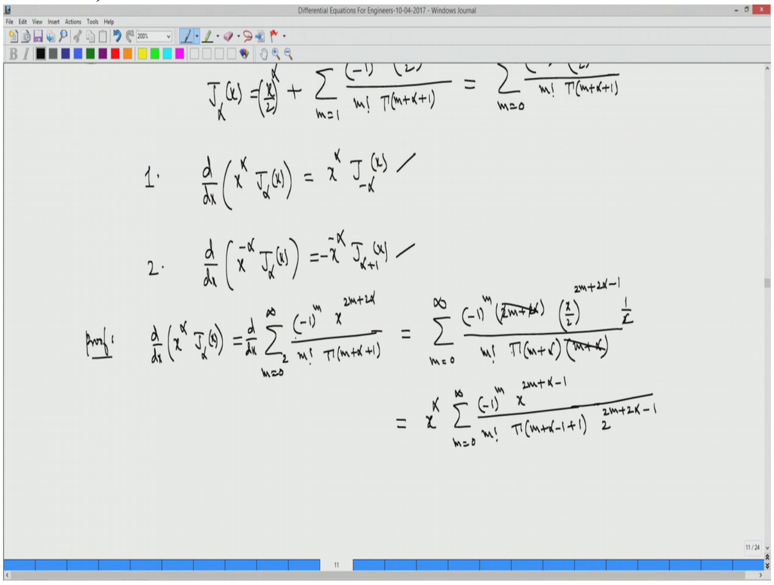

plus alpha, so only 2 m plus alpha, Ok.

So you have only, this is only for x. So you have only x, 2 power 2 m plus alpha, Ok. So if you differentiate here, so 2 m plus alpha, so minus 1, so this is what you get and you have 2 power 2 m plus alpha. So this is right, so you take x power alpha out, you have this and this you can write like this, I have 2, I have 2 m plus alpha, Ok. So this is it, so this is equal to x power alpha times this is exactly equal to alpha is replaced with alpha minus 1, wherever you have this, this shall have alpha plus 1. So this is nothing but J alpha minus 1 of x. So what you have is, this is not minus alpha, this should be alpha minus 1, Ok so that is your first property, done.

Second property

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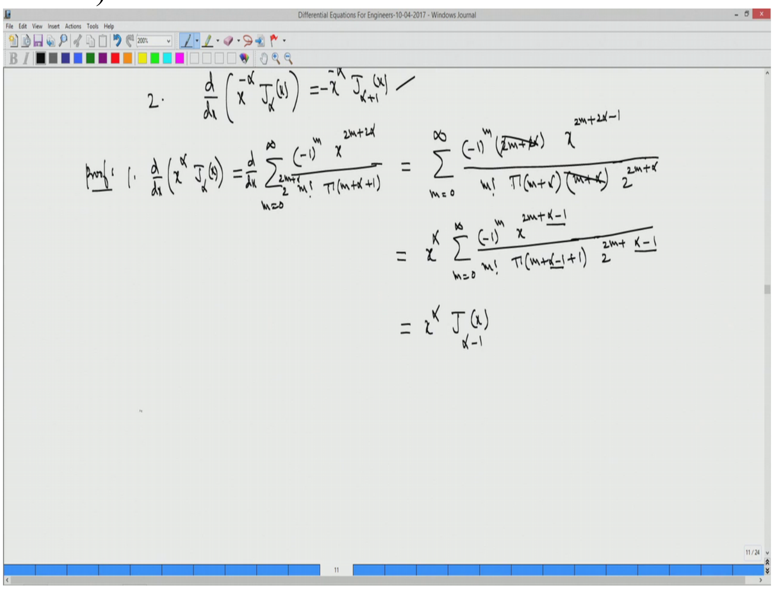

proof is also something similar. You multiply x power minus alpha to J alpha. So if you do that, and differentiate, differentiation with x power minus alpha, J alpha of x, what you get is d d x of simply m is from zero to infinity, what you have is minus 1 power m x power 2 m plus alpha. When I multiply with x power minus alpha this goes divided by 2 power 2 m plus alpha Ok because that is in the J alpha n factorial gamma of n plus alpha plus 1. Ok, so this you have to differentiate.

So this will give me, if you differentiate so m equal to zero, that is going, right, so what you are left with is only m is from 1 to infinity, minus 1 power m 2 m x power 2 m minus 1 divided by 2 power 2 m plus alpha m factorial gamma of m plus alpha plus 1.

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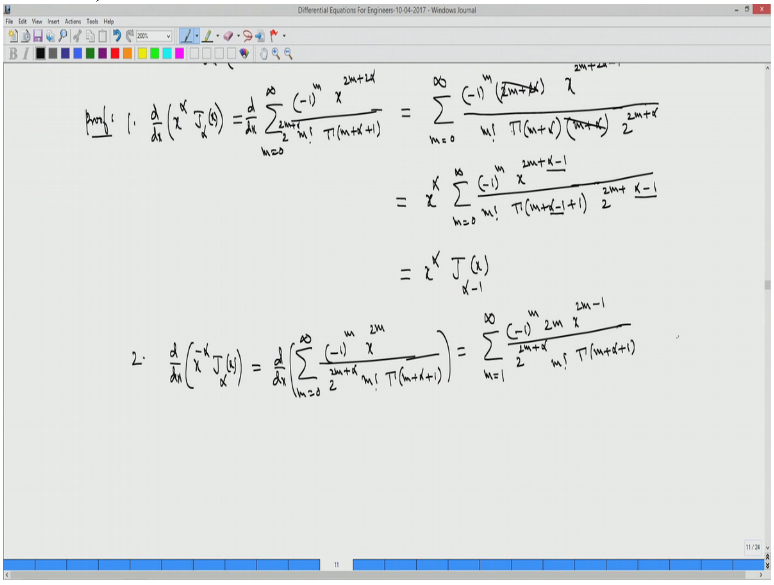

So this is it. If you change index, you are done.

So but you need from m is from m equal to zero, even if you differentiate so you can see that x power m equal to zero, that is simply a constant. After differentiation, it will go. So x square, so you only have m equal to 1. So that is what you have so now you change the index and rewrite. So if you write the index, where is that x power minus alpha gone? Right so you have, now you take what you want is, when you multiply and differentiate you want x power minus alpha out. So you take that minus alpha out. x power minus alpha if you take it out, you have to add, so I have to add 2 m x power 2 m plus alpha minus 1 divide by 2 power 2 m plus alpha m factorial gamma of m plus alpha plus 1.

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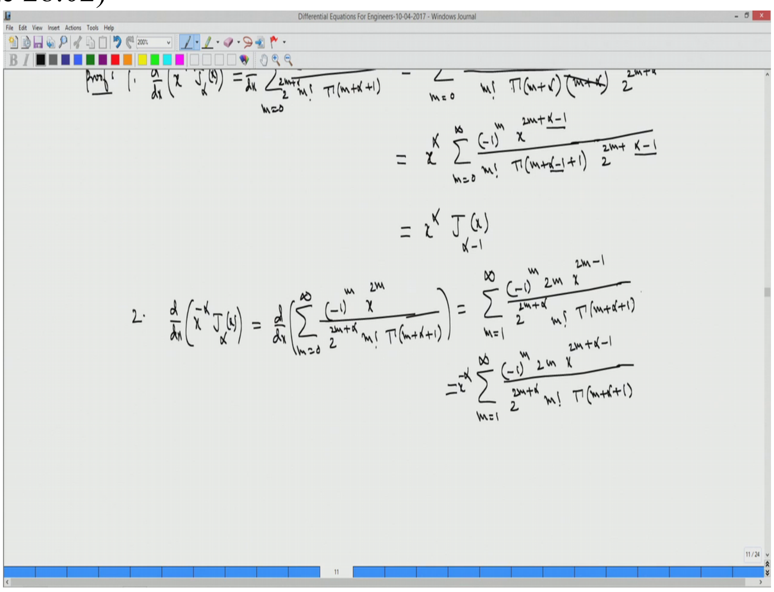

So now you change the index. If you change the index what you get is, you replace m equal to m plus 1, so it will become m is from zero to infinity minus 1 power m plus 1 2 into m plus 1, right so x power 2 m plus 2 plus alpha minus 1 divided by 2 power 2 m plus alpha plus 2, because 2 m plus, m equal m plus 1 so you have m plus 2, 2 m plus 2 plus alpha m plus 1 factorial gamma of m plus 1 plus alpha plus 1,

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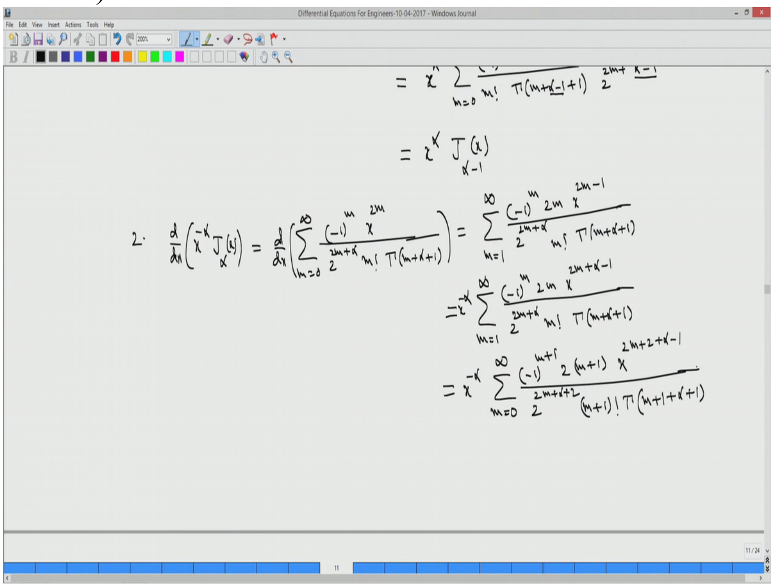

so this is what you have.

So this is equal to minus 1 power m, so in that you take minus out, so what you have is m is from zero to infinity, minus 1 power m, 2 1 goes here, so it will be 1, Ok and this goes to m factorial. So you have m factorial, Ok, x power 2 m plus alpha plus 1, Ok divided by 2 power 2 m plus alpha plus 1 into gamma of m plus alpha plus 1 plus 1,

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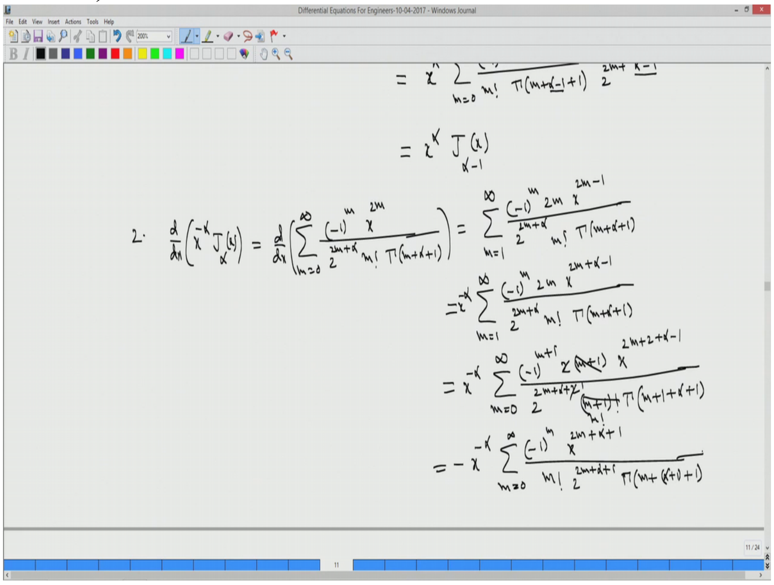

Ok. This is nothing but minus x power minus x alpha J alpha plus 1 of x. This is what you need. Ok, so this is your second property.

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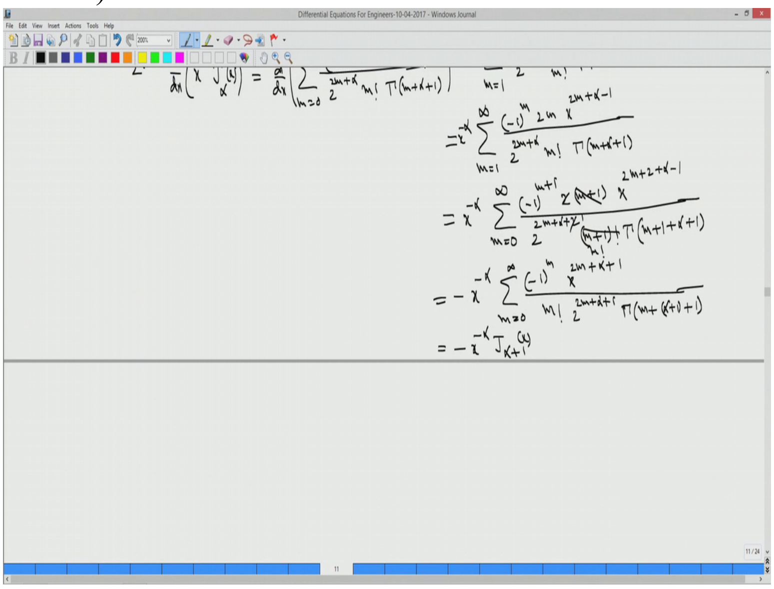

Ok so, so using this, we will show we need these properties to show that these functions, orthogonal properties of these functions.

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Just like you had, just like you had Legendre equations, Legendre functions, you had Legendre polynomials, Legendre bounded solutions which are Legendre polynomials, j, they are p n of x, n is running from 0, 1, 2, 3 onwards. They are countable number of polynomials. They form, they form, they are orthogonal to each other with respect to minus 1 to 1, Ok the dot product. So minus 1 to 1, you have seen, right?

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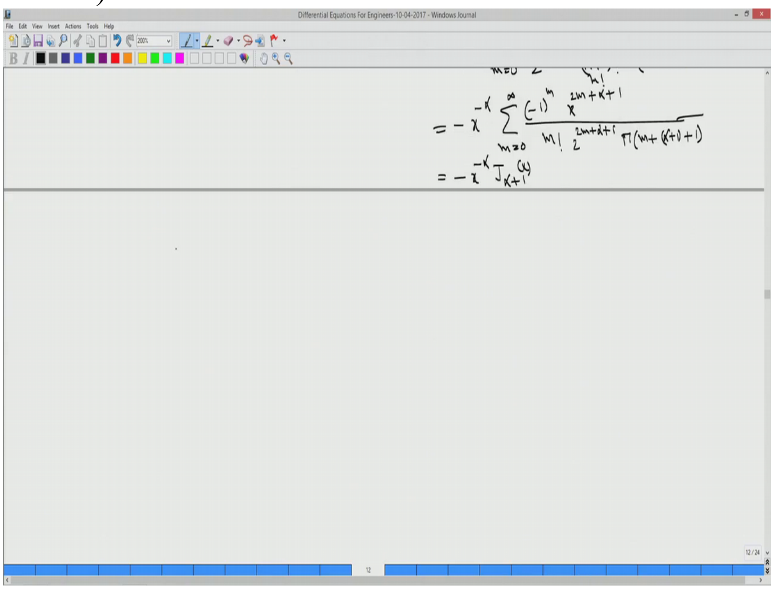

So based on these properties, we will prove this orthogonal relation which you have seen for Bessel, which you have seen for Legendre equations, so similar such thing we will do here. Before I do this, we will have Bessel equations, so Bessel equation is x y double dash plus x y dash plus x square minus alpha square into y equal to zero, so this is what is the Bessel equation. So right, so you have a Bessel equation like this.

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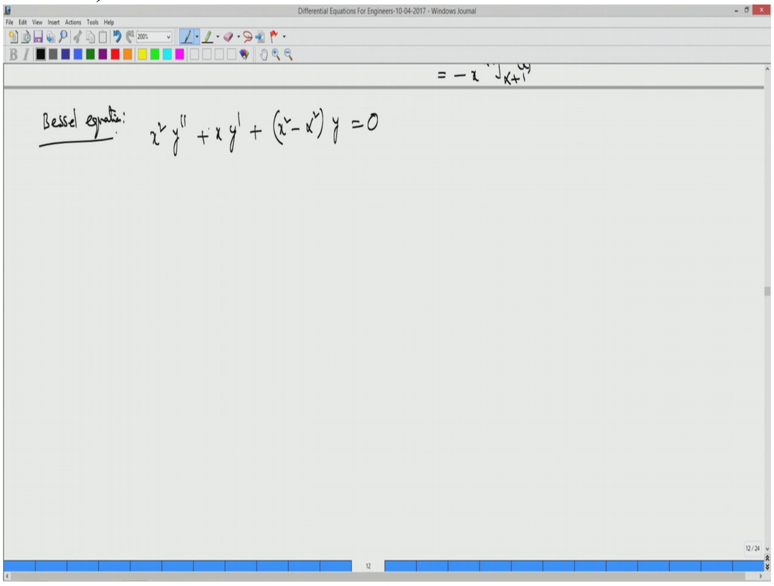

So you can modify this Bessel equation like this. If you do it, so you have, you can replace x by lambda t. If you do, let x by lambda t. If you do that, what happens? If you use that,

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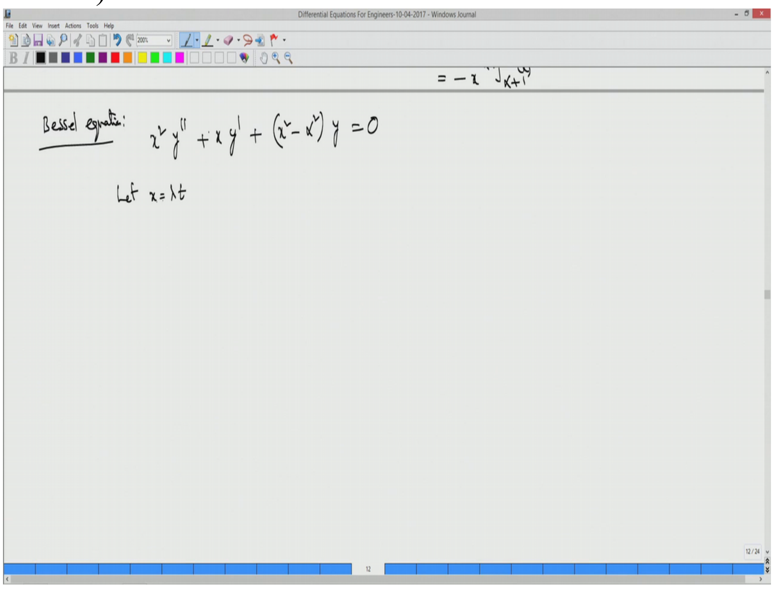

x square is lambda square t square, d y by d x, d y by d x is equal to d y by d lambda t, so this is equal to d y by d t by 1 by lambda, right?

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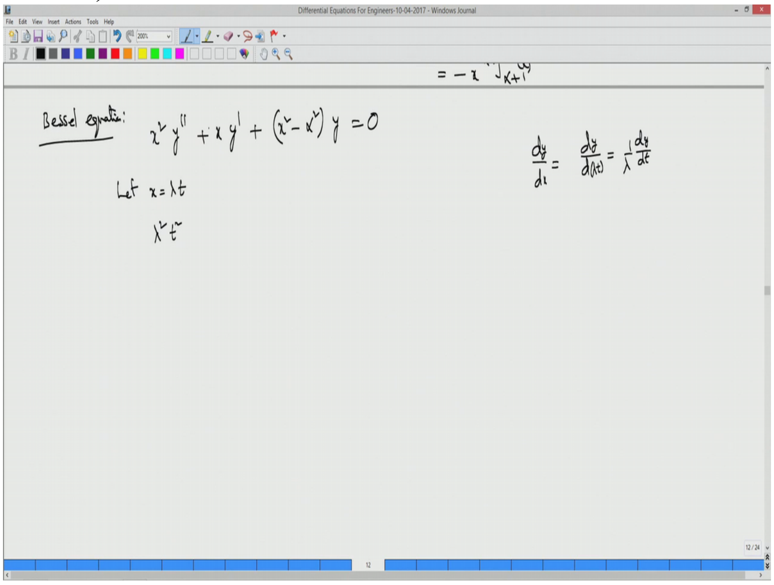

So similarly d square y by d x square is equal to 1 by lambda square d square y by d t square. So you will see that

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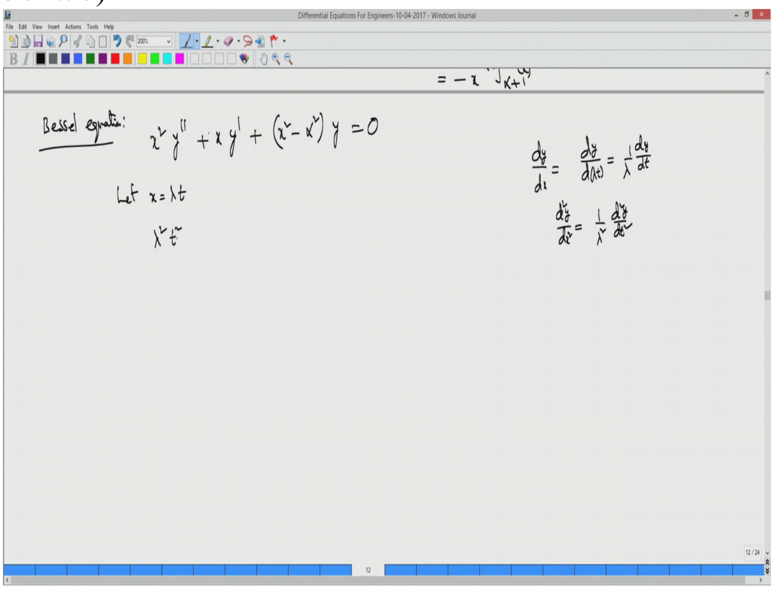

d square, this is, you have 1 by lambda square d square y by d t square plus here also, so you have lambda x, lambda x, x is replaced with lambda t, d y by d x will be 1 by lambda times d t, Ok, if you use this one, this one here plus lambda square t square minus alpha square equal to into y equal to zero,

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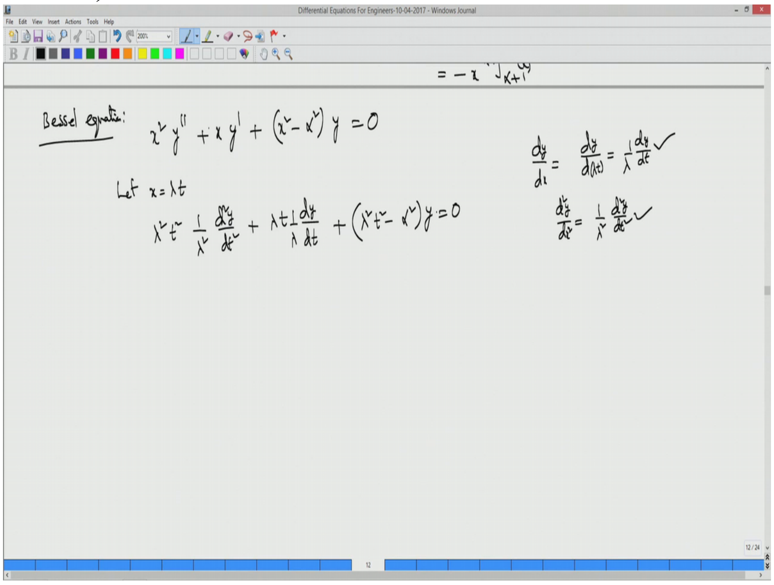

Ok.

So y is a function of t now. So this, this goes, lambda, lambda goes so what you end up is t square y double dash, now is it d, y double dash means d square y by d t square plus t y dash plus lambda square t square minus alpha square into y equal to zero. So this is what, this is called modified equation, modified Bessel equation. You can see that x is always positive, Ok so here also lambda is, again if you can take it as positive, Ok, lambda is positive and t is also positive. Lambda is fixed;

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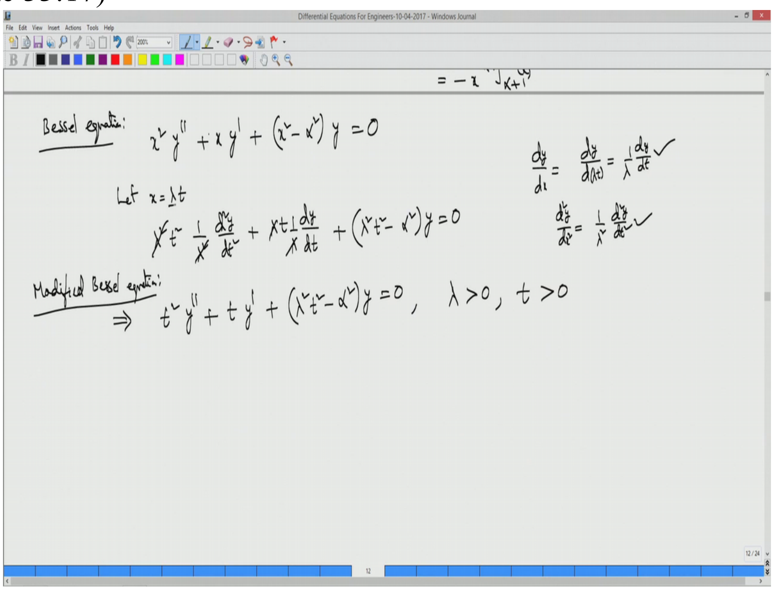

you fix this with this transformation. x is, the domain is x positive, so that makes it, t is also positive. Domain is always, this is the domain for which you have, so with this some lambda positive. If you choose your domain is t positive.

So this is the modified equation, Ok. So once you have this, what are the solutions here? The solutions are, so you know that dot (()) are the solutions here. So these are J alpha of x are solutions, right. J alpha is a solution, right, is a solution for this equation.

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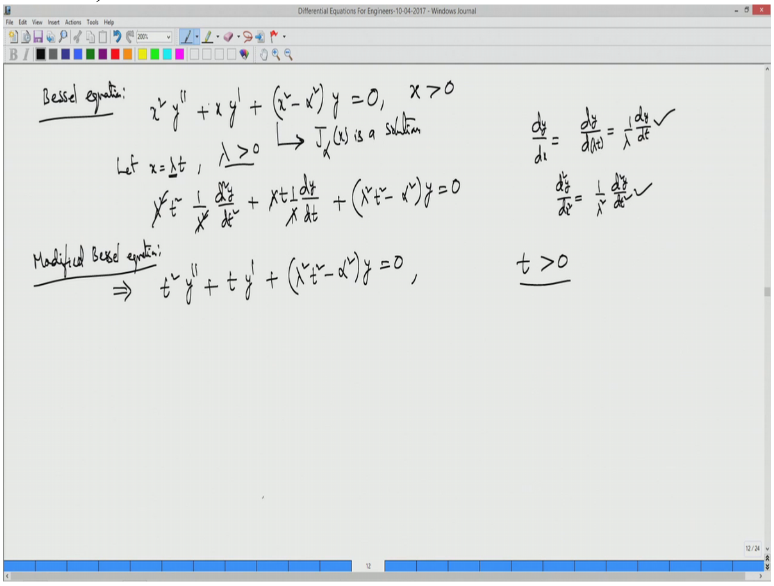

Then correspondingly I should get here J alpha of lambda t is a solution, Ok.

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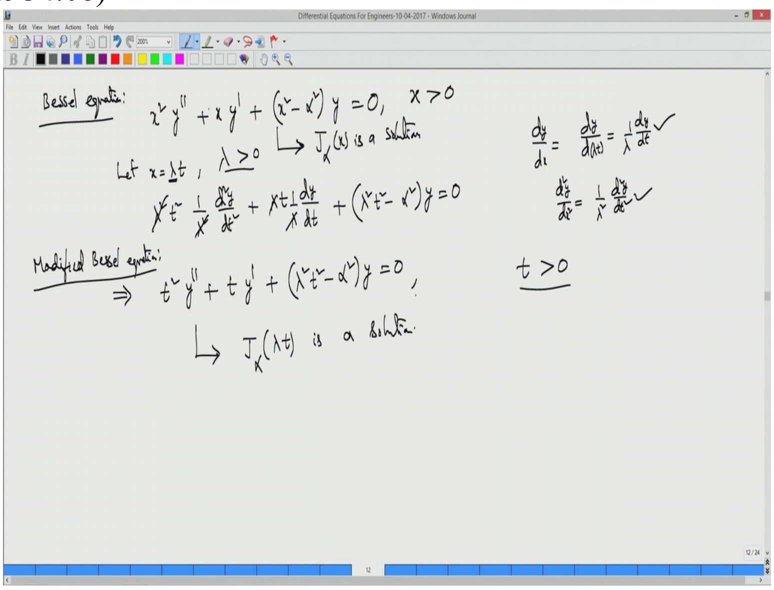

So here I fix now. So this is what I do. I fix my alpha here and I fix the domain. So then I have the domain t greater than t, I choose some l, Ok. I fix some number, fix some number l which is positive, Ok, so in the domain, I fix, I have t greater than equal to zero. This is the full real line, in that I fix this l, Ok. So if I do that, so you have j, so now you look at this x positive. So this has roots. So this corresponds to x, x positive, Ok. This corresponds to x positive. You have the roots of this J alpha are x 1, x 2, x 3 and so on. Recall this, Ok. Then what happens

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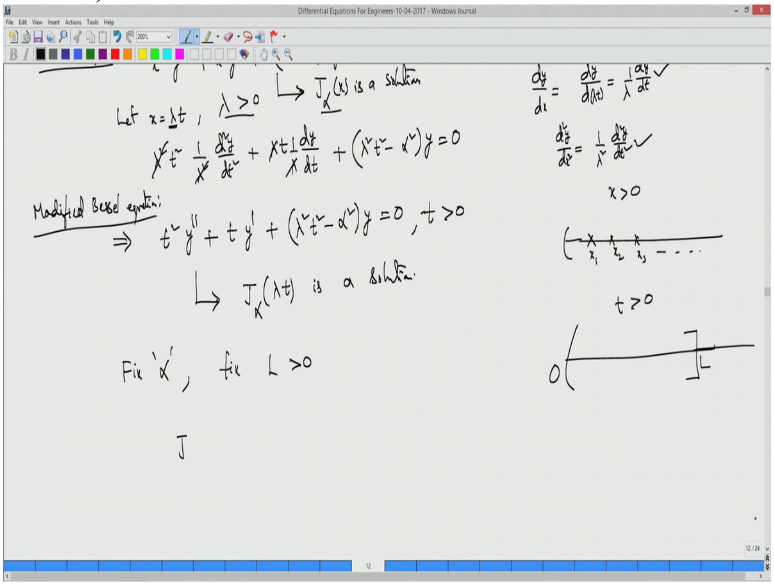

is that, once I fix this number l, J alpha of some lambda i of l will be zero, what is lambda i l? It is actually equal to x i. So x i I know that is the root,

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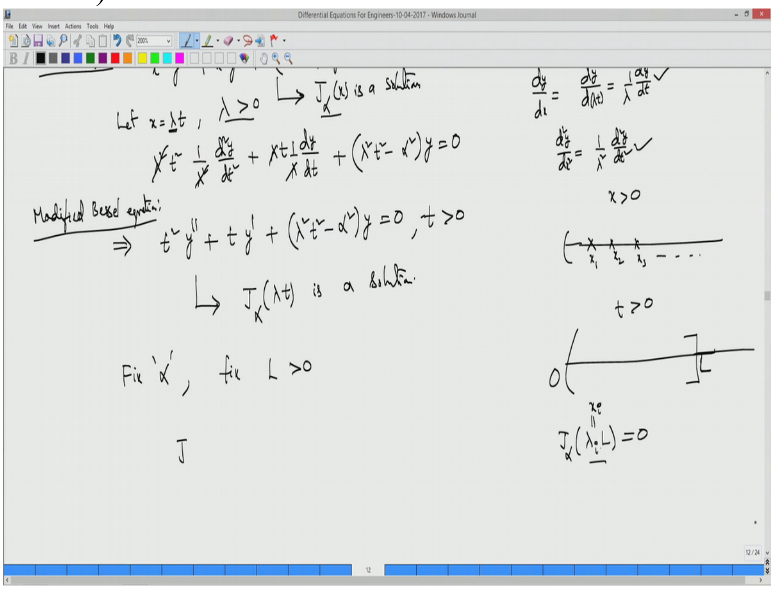

Ok, so that will make say, that will make lambda i is x i by l for every i 1, 2, 3 onwards. So if

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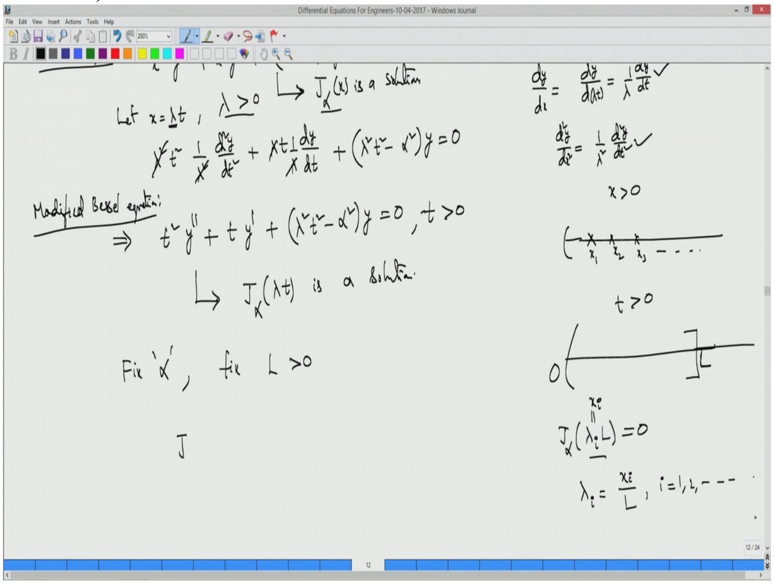

you know these roots of this J alpha, J alpha of x is are zero, then I can choose, I can fix this l and make my lambda is like this. Then I have J alpha of lambda i l equal to zero for every i 1, 2, 3 onwards, right? Because lambda is,

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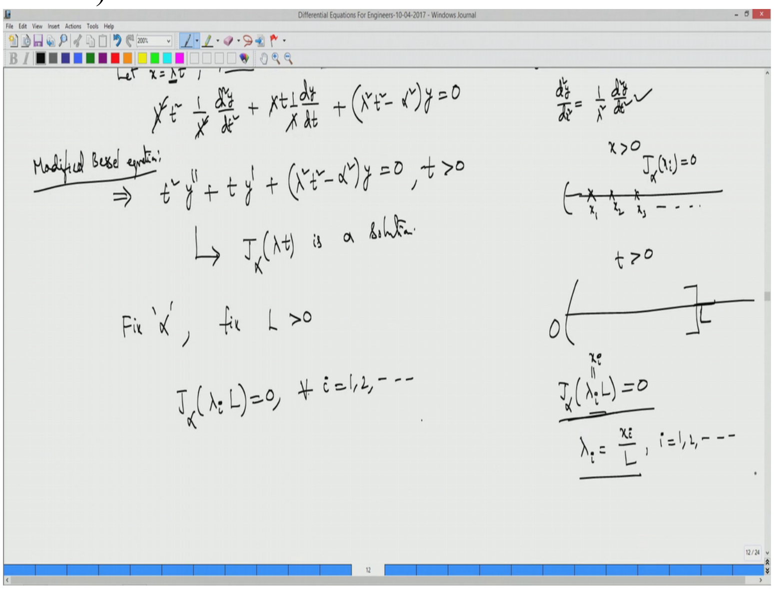

where lambda i is x is by l, Ok. So lambda is are these. So you see that if I fix the domain l, and if I fix my alpha I have infinitely many functions, Ok.

So what are those functions? J alpha of lambda i x, Ok, J alpha of lambda i x for every i, 1, 2, 3 onwards,

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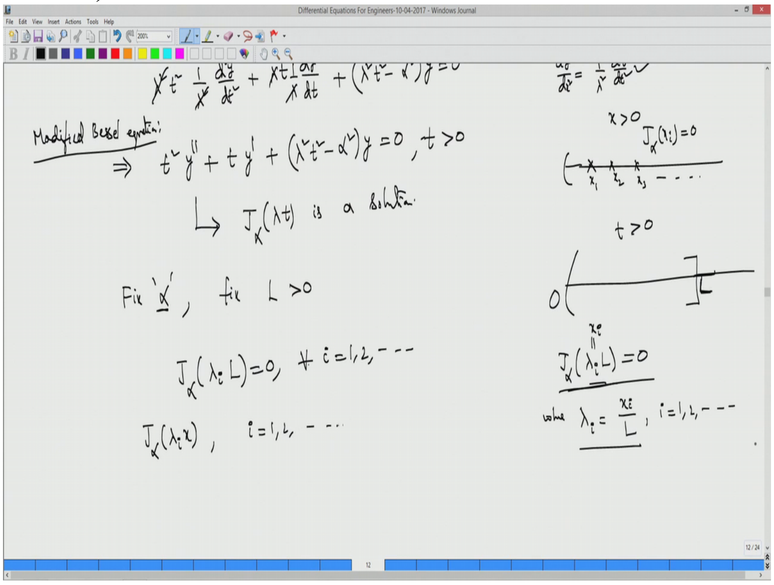

Ok. Where is these functions are defined? x belongs to zero to l, Ok in fact zero, zero is also there. So let's, because

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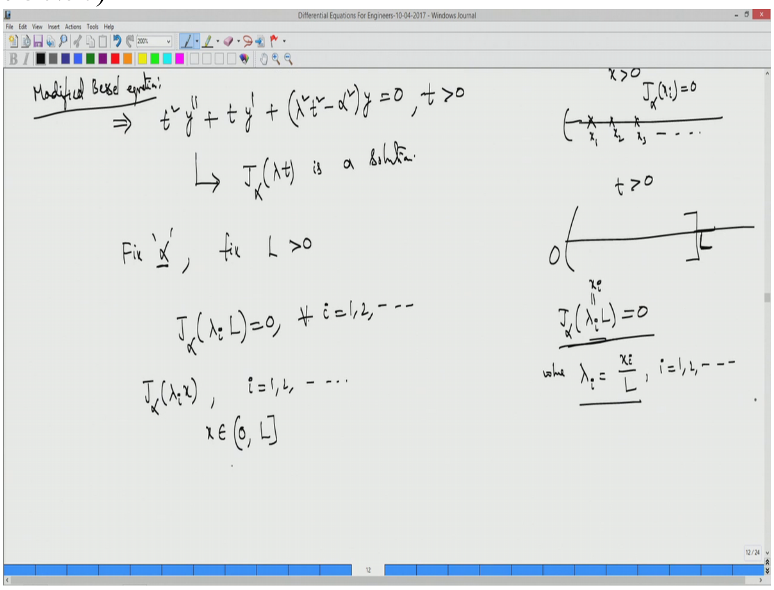

it is, as a function it is also included. zero is also included. These are the definitions, so these are the functions, ok. These functions, these Bessel functions when you fix alpha in this domain, in this domain, these functions are orthogonal, are orthogonal,

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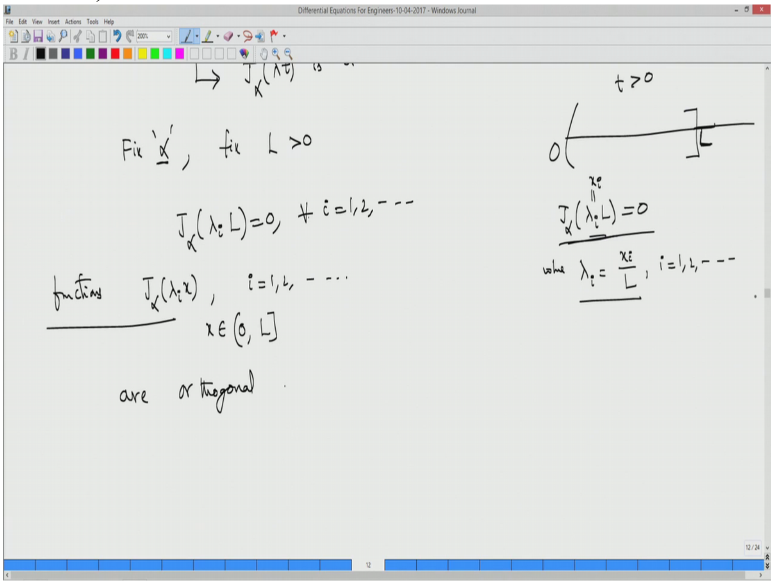

orthogonal with respect to, so orthogonal with respect to, so orthogonal means, this means they are from, what is the domain, from zero to l, you take any, alpha is fixed, lambda i x, Ok, alpha is fixed, lambda j of x. This d x with, with the weight function x, in this sense it is zero for every i in R equal to j.

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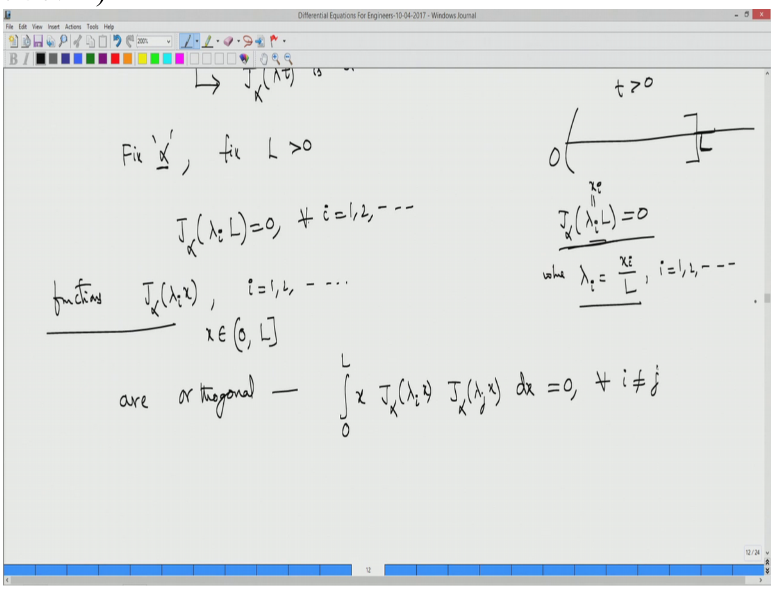

Whenever you choose these lambda is are different, Ok. So it is zero. If it is i equal to j, it is non-zero, Ok. It is something else. So what happens to that, so zero to l x J alpha of lambda is of x square d x, you will have something, Ok, something we will prove what it is, Ok, something we will see, some constant. Some constant, we will see what it is later.

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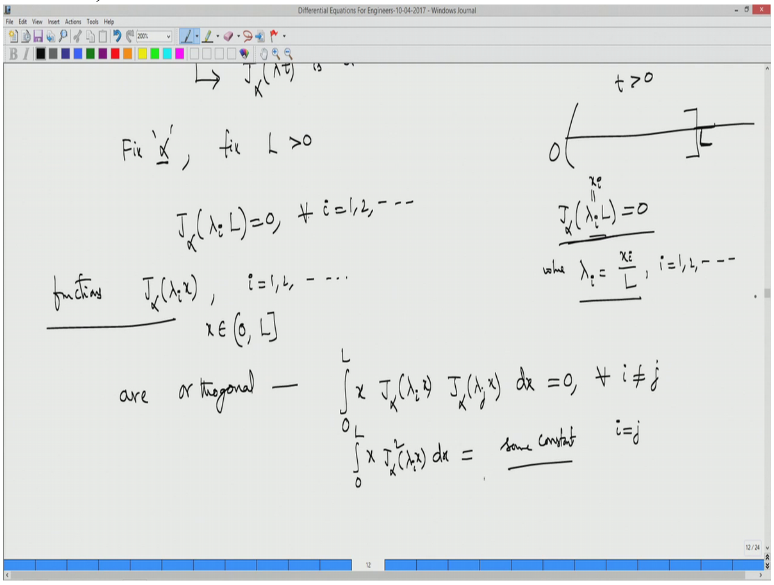

So we will first prove the first part that these functions, so I will write it as a result, the functions J alpha of lambda i x when x belongs to zero to l, Ok are orthogonal satisfy, or rather you say satisfy integral zero to l x times J alpha of lambda i x, J alpha of lambda j of x d x equal to zero for every i in R equal

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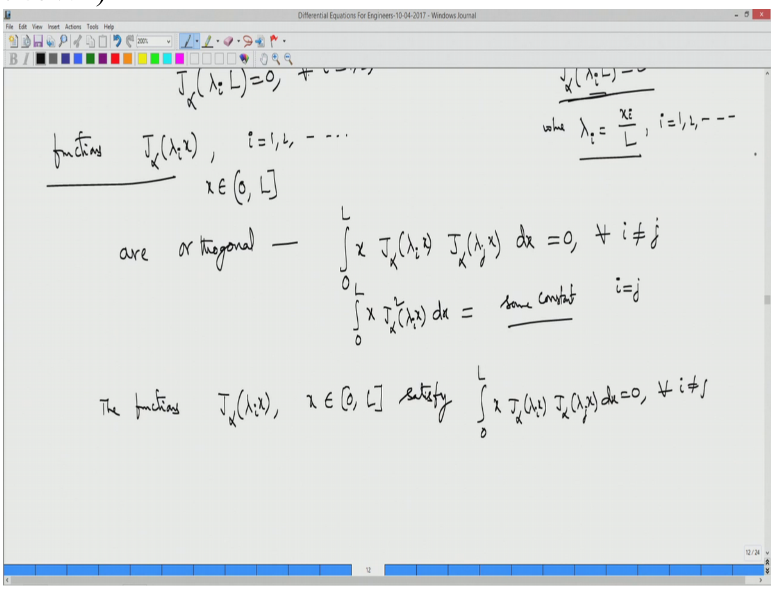

to j where alpha is fixed and l is fixed. So if you fix your alpha and

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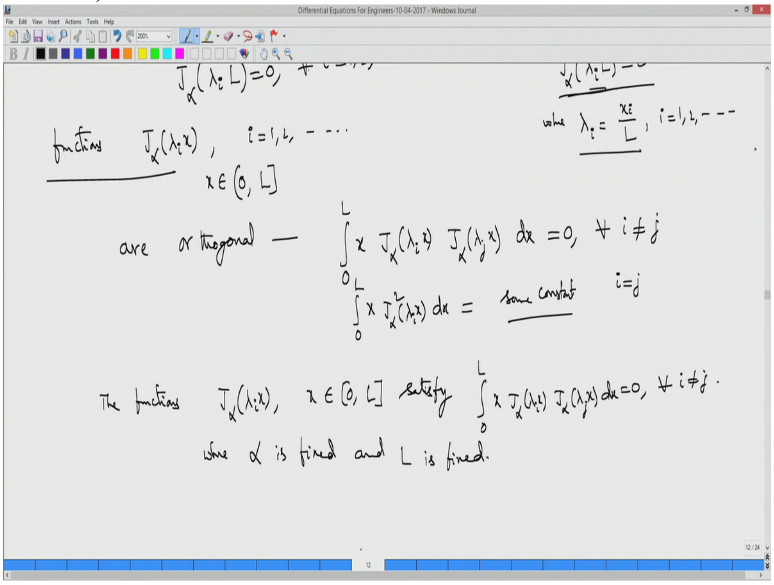

the domain some point l, you have orthogonal functions, Ok, these Bessel functions are orthogonal in that domain. So you can choose infinitely many such class of orthogonal functions, Ok, just by changing alpha and l, you can have infinitely many orthogonal functions, Ok so for each alpha and l, you can have, so you can have, because J alpha of x has infinitely many roots, for each root you have such a function. So for each root x i there is a corresponding lambda i. So for each lambda i, you have J alpha of lambda x. So for each lambda, for each root of the equation you have one function. So that means for each fixed alpha and l, you have infinitely many roots,

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many roots that implies infinitely many functions and which are orthogonal, Ok. So every fixed alpha and l, you have a class of functions, orthogonal functions. Similarly, so like that, so we will first see these functions, these orthogonal,

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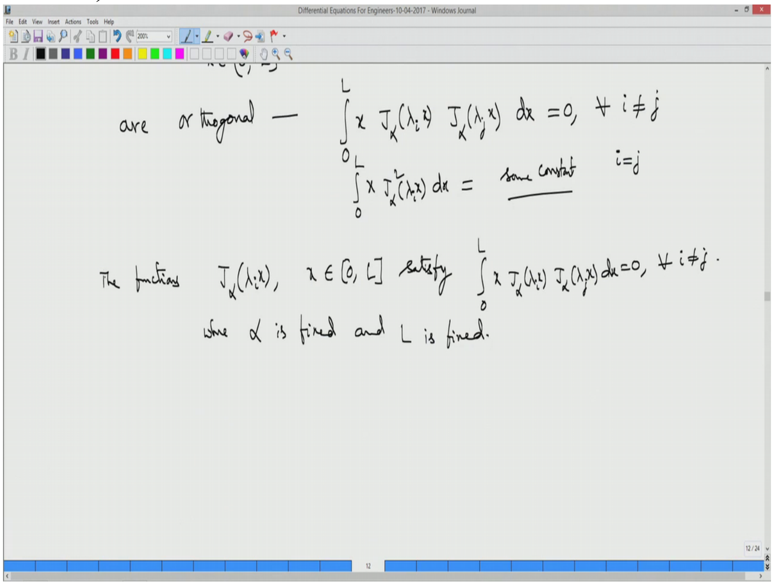

we will show that this is, these are orthogonal. So this is straight forward. What is J alpha of lambda i x? This satisfies this equation, right. So this is the equation it satisfies. So we take this one. So how do we prove this is, we simply show that J alpha of lambda i x satisfies x square, so instead of t I have x, Ok so doesn't matter, so x square J alpha double dash of lambda i x, Ok. This is what you have, right. So you see that J alpha of lambda t satisfies this equation, that is d square by d t square of J alpha of lambda t. So I have x square here, just notation change, instead of x I am, instead of t I am writing x. x square into J alpha double dash, that is d square by d x square of J alpha of lambda i of x plus x times J alpha dash of lambda i x plus x square minus, here this corresponds to lambda so this is lambda i square x square minus alpha square into J alpha of lambda i x equal to zero. So if it satisfies

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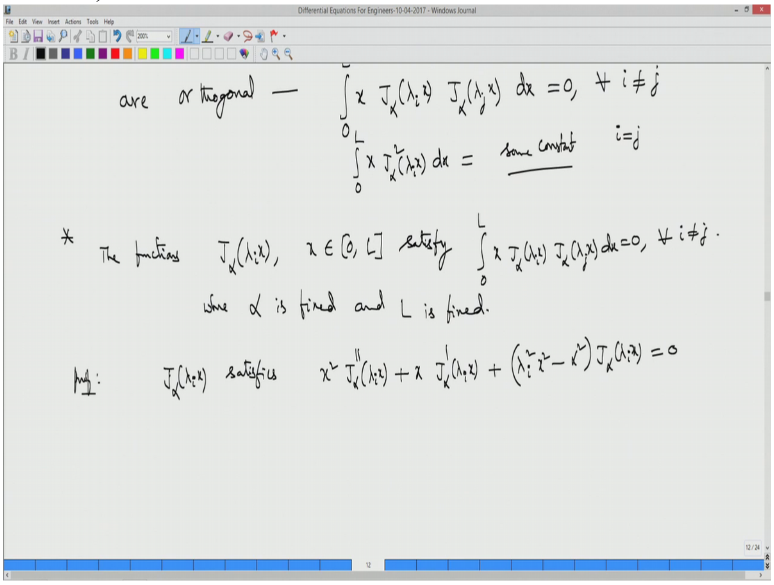

the equation here, if it satisfies this equation, this is what you have to write here. So that is what I have written. So similarly J alpha of lambda j of x satisfies x square J alpha double dash of lambda j x plus x times J alpha dash of lambda j x plus lambda j square x square minus alpha square times J alpha of lambda j x equal to zero.

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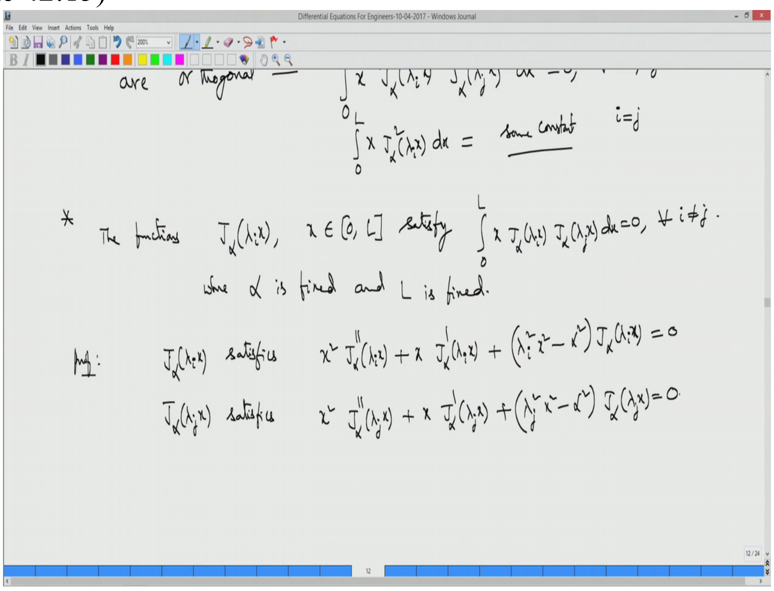

What I do is I simply multiply J alpha lambda j here, Ok. For this equation you multiply simply j both sides, J alpha of lambda j x and here you multiply J alpha, J alpha of lambda i x, Ok

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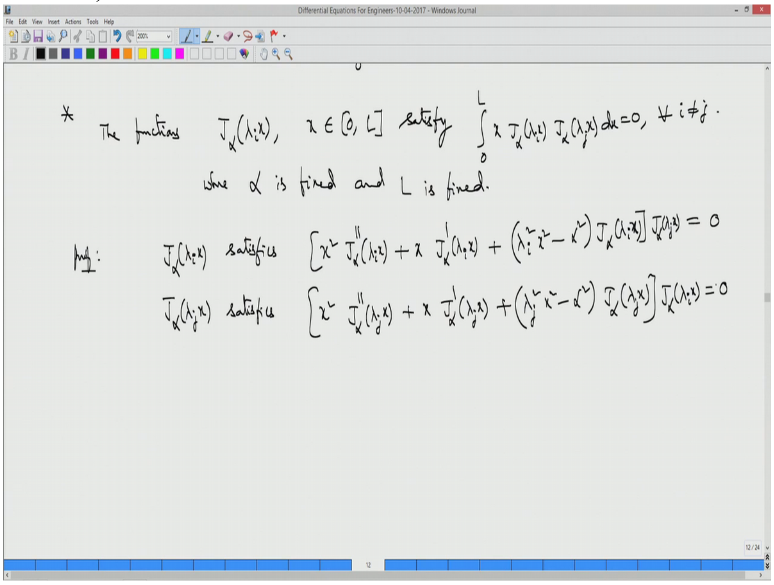

equal to zero. So both sides if you multiply, there is no change, Ok. This satisfies, satisfies equation implies this.

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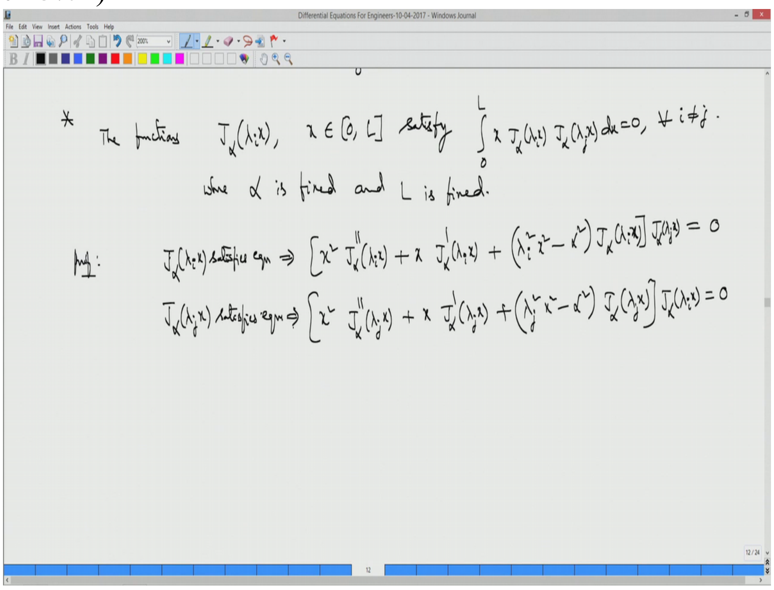

This satisfies equation implies this. For this I multiply both, if j, J alpha of lambda j of x satisfies this, that you multiply to that, that equation and whatever this satisfied for this equation, here, this equation, J alpha of lambda i x, you multiply to this. And then you take the difference. Ok so this is equation number 1

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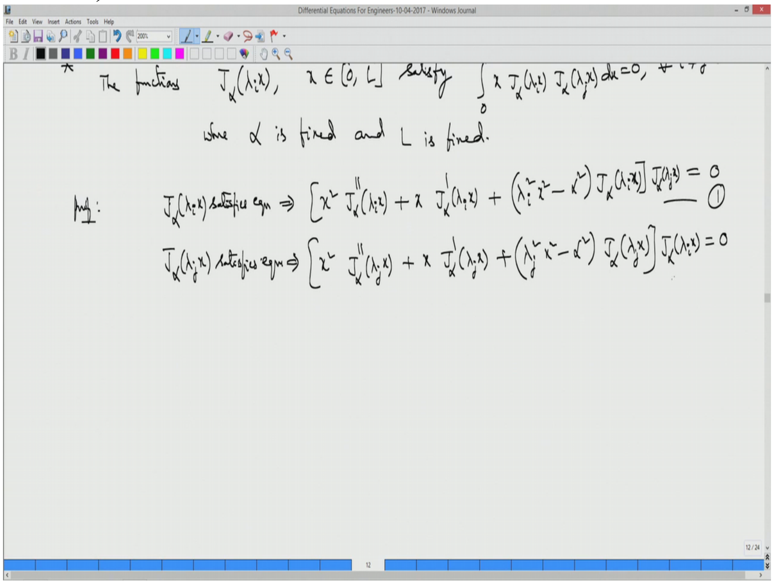

and 2,

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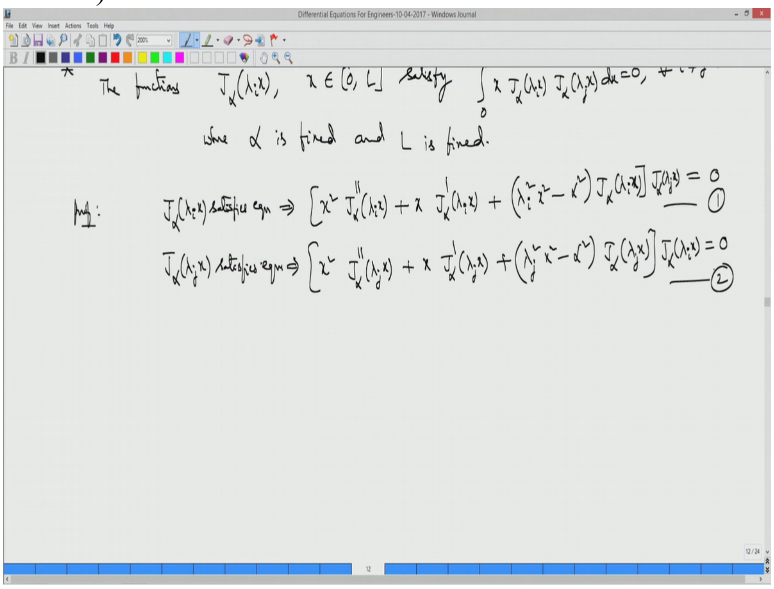

1 minus 2 will give me, what happens 1 minus 2,

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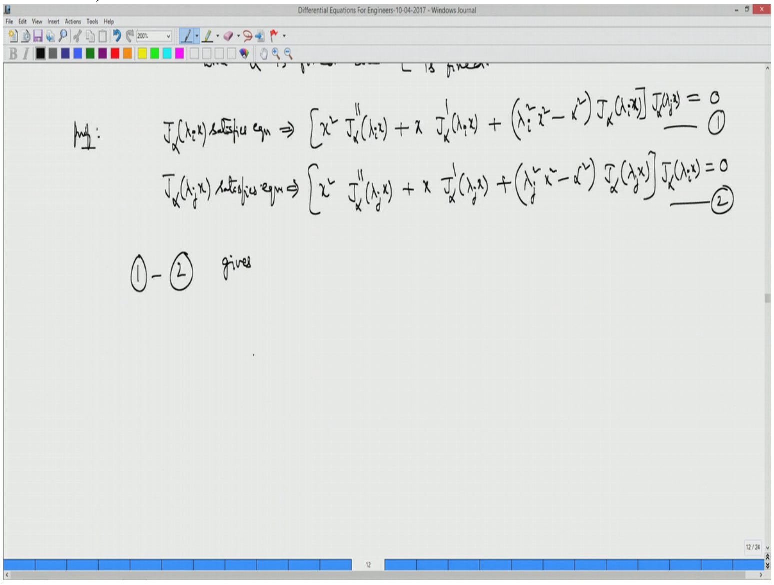

it is simply x square J alpha double dash of lambda i x, J alpha of lambda j x minus J alpha dash of lambda j x J alpha of lambda i x. This is what you get plus x times, again here so J alpha dash of lambda i x, J alpha of lambda j x minus J alpha dash of lambda j x into J alpha dash of lambda, J alpha of lambda i x. That is what is

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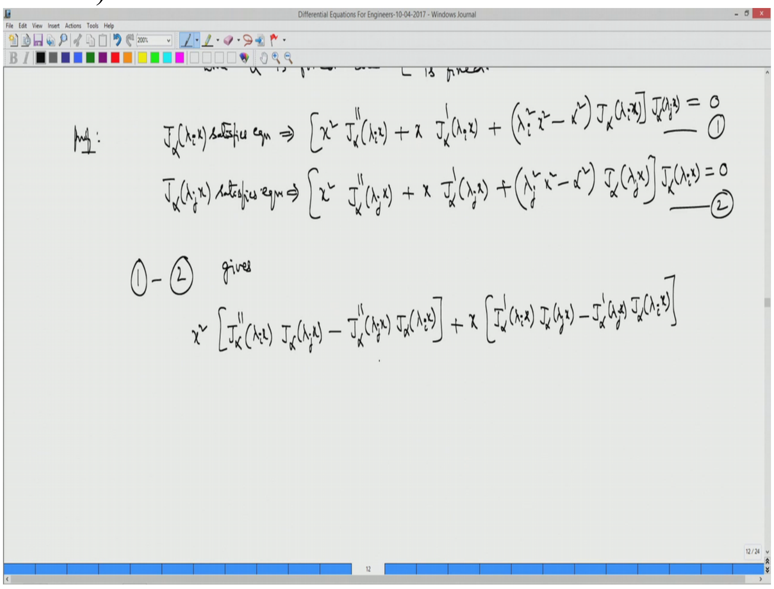

this difference. Plus see alpha square, alpha square, this will be zero because both are same. So J alpha, J alpha lambda j, J alpha lambda i. Here also you have this same. So we have lambda i square minus lambda j square, other thing is common, that is x square J alpha of lambda i x into J alpha of lambda j x equal to zero.

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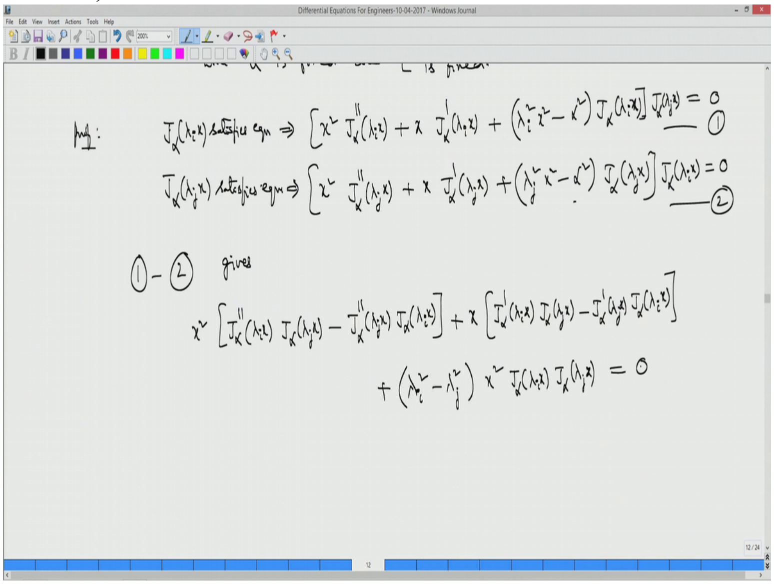

Ok, so here you cancel one x, one x you cancel so that means and you can see that, x times Ok, x is, what you have is, if you do that x times J alpha dash of lambda i x J alpha of lambda j x minus J alpha dash of lambda j x into J alpha of lambda i x, you consider this function. You differentiate this with x. If you differentiate this function, this is a function of, 2 functions, multiplication of 2 functions, if you differentiate, derivative of this is, if you differentiate, so first you differentiate x that will be simply this part. That you already have here, Ok. If you, if you differentiate this one, what you get is this one. You can easily see that, Ok. So if you differentiate this part, what you get is J alpha double dash of lambda i x, Ok into J alpha of lambda x minus this one. If you differentiate this, this is what you get. So these two terms I can put it like this, plus lambda i square minus lambda j square into x into J alpha of lambda i x, J alpha of lambda j x equal to zero. You simply integrate both sides from zero to l d x, integral zero to l d x, Ok. Now what happens?

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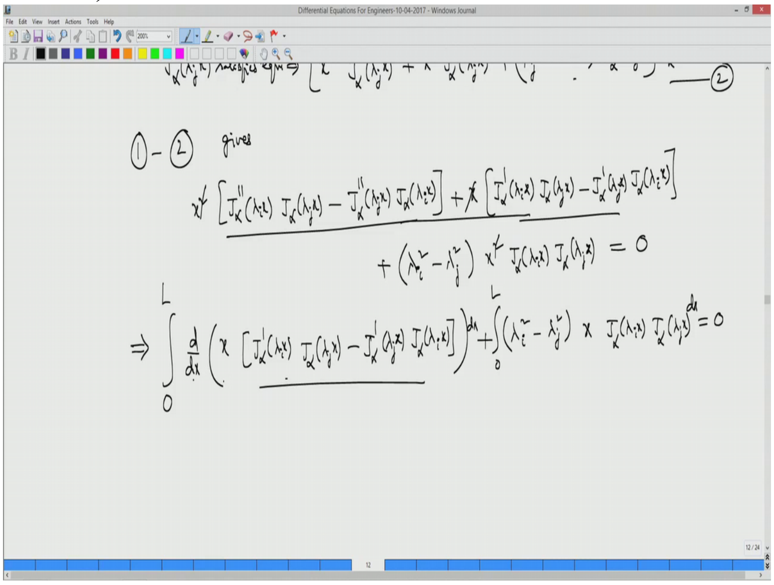

Because J alpha of lambda j l, these are the roots right, lambda j l, that is how we have chosen your J alpha of lambda i l for every l, so j, lambda, when you put x equal to l, J alpha of lambda j or J alpha of lambda i l will be zero. Ok. So at l, this part will be zero. When you put x equal to zero, x into whatever may be the quantity, its derivatives, at zero, it may be zero, Ok. So overall this quantity will be zero. What is left is the other part, that is lambda i minus lambda j whole square, lambda i square minus lambda j square into this integral. There is a constant, I take it out. x times J alpha of lambda i x, J alpha of lambda j x d x equal to zero. Because i is not equal to j, this cannot be zero, Ok. If i is not equal to j, lambda i is certainly not equal to lambda j, because lambda i is nothing but x i by l. So what is your lambda i? x i by l, so x i by l is not, cannot be same as x j by l, because these are distinct roots. That is how you have chosen. For x 1, x 2, x 3 these are all distinct roots of J alpha of x. Ok so because they are distinct, x is are distinct, lambda is are distinct. That is so if

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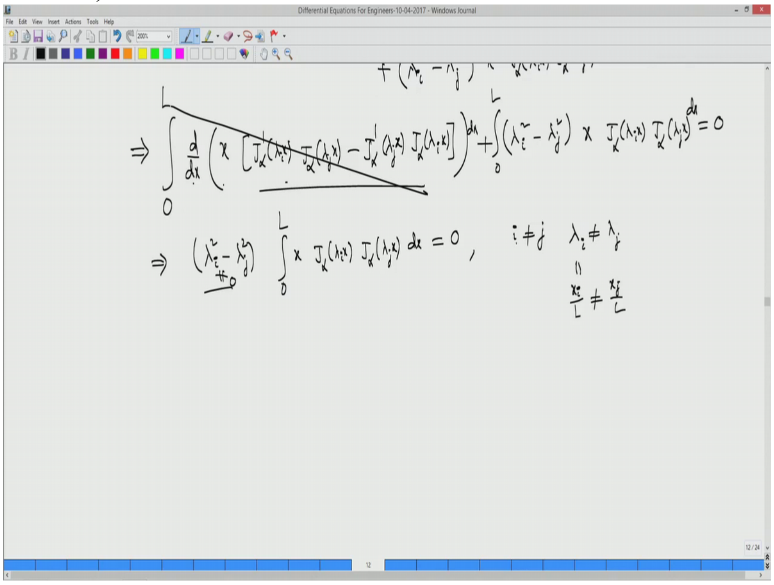

lambda is, i is not equal to j, this is, because of this this cannot be zero so this whole thing is zero. So this implies this integral has to be zero. That means these functions when i is not equal to j, they have to be orthogonal. J alpha of lambda j x d x equal to zero if i is not equal to j. This is called orthogonal relation,

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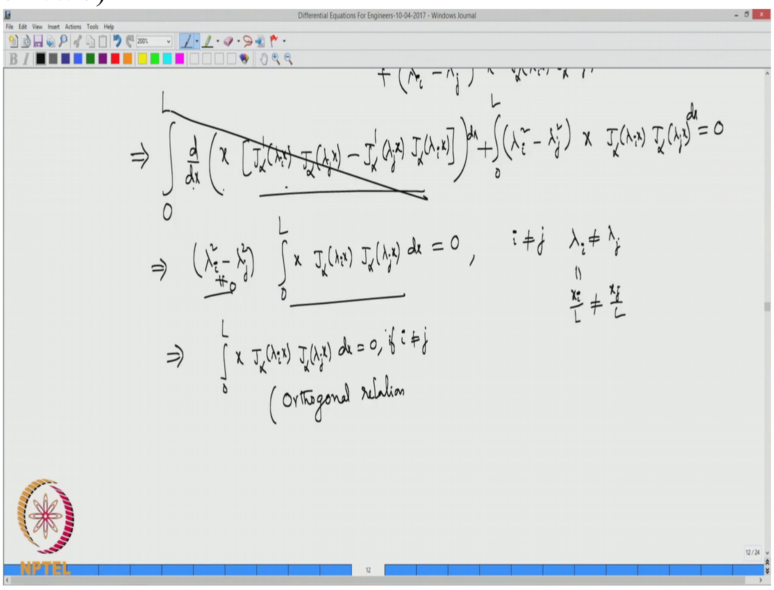

Ok,

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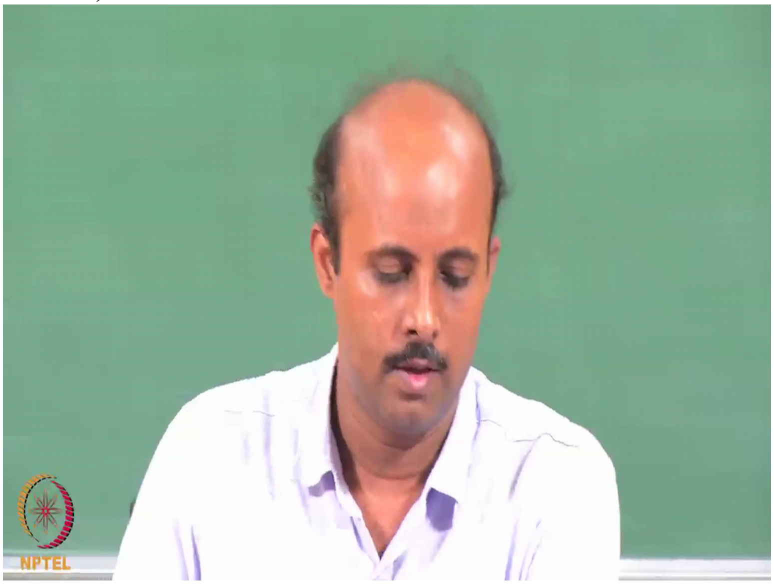

so what happens if i equal to j, we can show that that integral value will be some constant. That we can find out, so that I will do in the next video.