

Differential Equations for Engineers
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Lecture No 29
Frobenius solutions for Bessel Equation

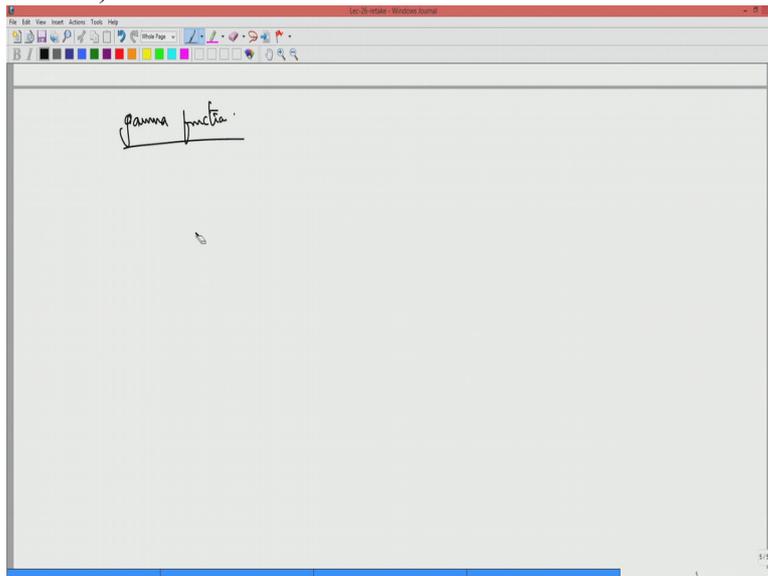
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Last video we were looking at the Frobenius solutions for the Bessel equation and we looked at the case 1, that is the difference between the roots of the indicial equation is a non-integer. This video we will see a difference is integer case, Ok. So we have seen in the last video when the roots difference is a non-integer and you found two solutions, one is y_1 and y_2 and you multiplied certain constant to y_1 and you defined what is called as Bessel function of first kind that is J_α of x . So in that what is the constant involved, gamma function, so that you might not be knowing so let's discuss about what is a gamma function.

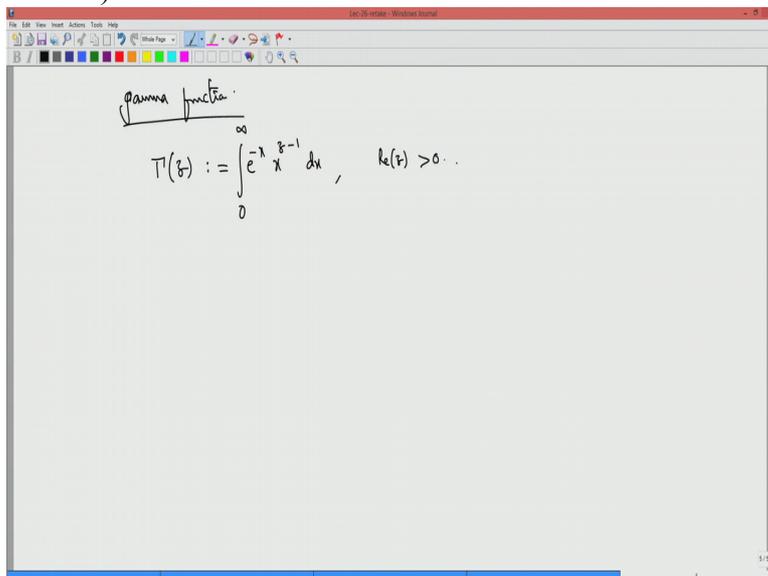
So we have a gamma function, so gamma function,

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so its definition is, so definition of gamma function, it is defined for all complex numbers as integral zero to infinity power minus x, x power minus 1 d x. So this is what you have and real part of that is positive, Ok. So

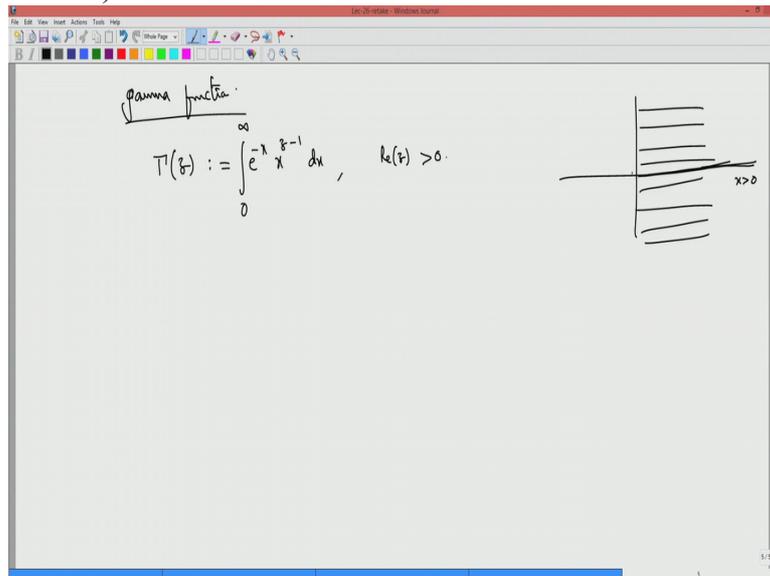
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this is what it is, so if you actually, so pictorially if you look at it, if you see this is defined everywhere whenever real part observed is positive.

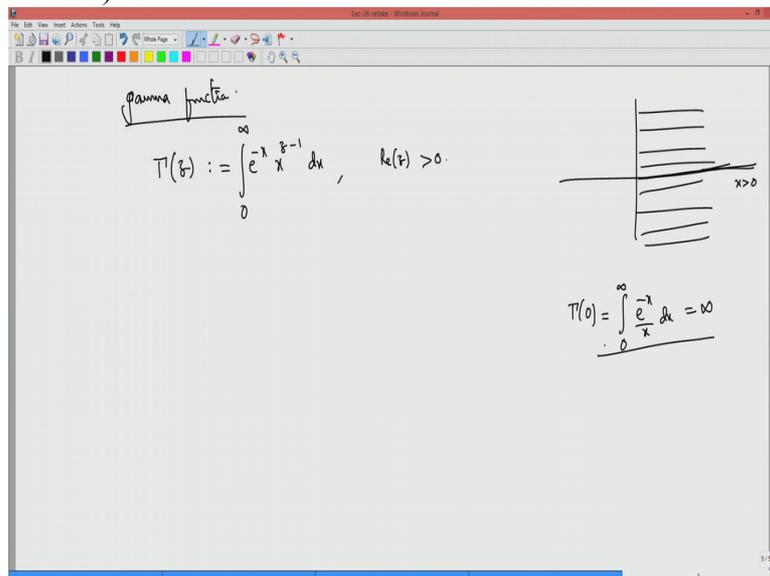
So everywhere so that means including, we are interested only for, we are interested only for the gamma function for real values. So the way it is defined, it is defined for all x positive, Ok.

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So when x is equal to zero if you put, then this integral what happens, gamma at zero, x equal to zero, this is the integral, so e power minus x by x d x, so this is actually certainly infinity. So

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because of this, this point is not defined, Ok.

Now we look at the property of the gamma function that is, if you calculate this at z plus 1, then what you get is e power minus x, x power z d x. Do the integration by parts and you see that e power minus x e power z, x power z so the limits zero to infinity minus minus plus, zero to infinity e power minus x, z into, z comes out because this constant, z into x power z minus 1 d x. So

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Gamma function:

$$\Gamma(z) := \int_0^{\infty} e^{-x} x^{z-1} dx, \quad \operatorname{Re}(z) > 0.$$

Diagram of the complex plane showing a branch cut on the positive real axis for $x > 0$.

$$\Gamma(z+1) = \int_0^{\infty} e^{-x} x^z dx = -e^{-x} \frac{x^z}{z} + \frac{1}{z} \int_0^{\infty} e^{-x} x^{z-1} dx$$

$$\Gamma(z+1) = z \Gamma(z)$$

$\Gamma(0) = \int_0^{\infty} \frac{e^{-x}}{x} dx = \infty$

this is nothing but, so you can see at infinity, e power minus x is close to zero, so zero, x power z is zero. So we have these terms, doesn't contribute, so we have z times, what you are left with is simply a gamma function.

So this is your property of gamma function. So if you use this

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$$\Gamma(z+1) = z \Gamma(z)$$

$\Gamma(0) = \int_0^{\infty} \frac{e^{-x}}{x} dx = \infty$

what we see is the, we are interested to see what are all the values of real, negative real numbers for which gamma function is defined, Ok. So you see that is zero, zero positive 1, it is defined, Ok. It is of course including 1, everywhere it is defined,

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Gamma function

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$$\Gamma(z+1) = z \Gamma(z)$$

Diagram: Complex plane with a branch cut on the positive real axis for $x > 0$.

$$\Gamma(0) = \int_0^{\infty} \frac{e^{-x}}{x} dx = \infty$$

~~$\Gamma(z) = \frac{\Gamma(z+1)}{z}$~~

this side, so zero to 1, so you look, you take gamma z I am defining here in the negative side, at zero is not defined, zero to minus 1, so gamma z equal to, so you take from here gamma z is gamma z plus 1 by z.

So 1 by z times gamma z plus 1,

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Gamma function

$$\Gamma(z) := \int_0^{\infty} e^{-x} x^{z-1} dx, \quad \operatorname{Re}(z) > 0.$$

$$\Gamma(z+1) = \int_0^{\infty} e^{-x} x^z dx = -e^{-x} \frac{x^z}{z} + \frac{1}{z} \int_0^{\infty} e^{-x} x^{z-1} dx$$

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~~$\Gamma(z) = \frac{\Gamma(z+1)}{z}$~~

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

when z is here, z plus 1 will be here Ok which is well-defined. So z plus 1 is well-defined, z you take as negative numbers between minus 1 to zero, so gamma z plus 1 by z is well-defined, well-defined in minus 1 to zero, Ok. Again once you know this gamma z in minus 1 to zero; now you know everything here.

Now you can define minus 2 to minus 1, so again, again in minus 2 to minus 1, if we take z, z plus 1 will be here which you have seen already,

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Gamma function

$$\Gamma(z) := \int_0^{\infty} e^{-x} x^{z-1} dx, \quad \operatorname{Re}(z) > 0.$$

$$\Gamma(z+1) = \int_0^{\infty} e^{-x} x^z dx = -\frac{e^{-x}}{x} \Big|_0^{\infty} + z \int_0^{\infty} e^{-x} x^{z-1} dx$$

$$\Gamma(z+1) = z \Gamma(z)$$

Diagram of the complex plane showing the region of convergence for the integral definition ($\operatorname{Re}(z) > 0$) and the region where the function is well-defined ($z \notin (-1, 0), (-2, -1), \dots$).

$\Gamma(0) = \int_0^{\infty} \frac{e^{-x}}{x} dx = \infty$

$\Gamma(z) = \frac{\Gamma(z+1)}{z}$ is well defined in $(-1, 0)$
in $(-2, -1)$

Ok, iterative, like this iteratively you can go on defining everywhere, every single interval, interval of length 1 except these negative integers,

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Gamma function

$$\Gamma(z) := \int_0^{\infty} e^{-x} x^{z-1} dx, \quad \operatorname{Re}(z) > 0.$$

$$\Gamma(z+1) = \int_0^{\infty} e^{-x} x^z dx = -\frac{e^{-x}}{x} \Big|_0^{\infty} + z \int_0^{\infty} e^{-x} x^{z-1} dx$$

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$\Gamma(0) = \int_0^{\infty} \frac{e^{-x}}{x} dx = \infty$

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in $(-2, -1)$

Ok, except these negative integers including zero, you don't have gamma function defined there, Ok. So everywhere else it is defined, that is what is important, Ok.

So let us see what is the last video, we have defined what is a gamma function, gamma function what we, Bessel function, Bessel function of first kind we defined in the last video that we will see. So $J_\alpha(x)$ is x^α by 2^α power α , gamma of $\alpha + 1$ times 1

plus sigma m is from 1 to infinity, minus 1 power m x power 2 m by 2 power 2 m, m factorial, now we have here, alpha plus 1, alpha plus 2 up to alpha plus m. That is what you have.

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Gamma function:

$$\Gamma(\alpha) := \int_0^{\infty} e^{-x} x^{\alpha-1} dx, \quad \operatorname{Re}(\alpha) > 0.$$

$$\Gamma(\alpha+1) = \int_0^{\infty} e^{-x} x^{\alpha} dx = -e^{-x} \frac{x^{\alpha}}{\alpha} \Big|_0^{\infty} + \int_0^{\infty} e^{-x} x^{\alpha-1} dx$$

$$\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$$

$$\frac{\Gamma(x)}{x} = \frac{x^x}{x \Gamma(x+1)} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! (x+1)(x+2)\dots(x+m)} \right]$$

Graphs showing the Gamma function for $x > 0$ and its poles at $x = 0, -1, -2, \dots$.

$\Gamma(0) = \int_0^{\infty} \frac{e^{-x}}{x} dx = \infty$

$\Gamma(x) = \frac{\Gamma(x+1)}{x}$ is well defined in $(-1, 0)$ in $(-2, -1)$

Now you can see that, gamma, so use this property. If you use this property, use this property here, so if you use this, alpha plus 1 into gamma alpha plus 1 is gamma alpha plus 2. Again alpha plus 2 into gamma alpha plus 2 is gamma alpha plus 3. So finally you take with alpha plus m into gamma alpha plus m will 1 be gamma alpha plus m plus 1. So if you do that x power alpha by 2 power alpha, that is common, so you have, take it inside here, 1 divided by gamma of alpha plus 1 and here you can club this gamma with this, with this so what you end up is minus 1 power m, 2 power x power 2 m by 2 power 2 m m factorial, here gamma of alpha plus m plus 1, Ok. So this is what is the Bessel function of first kind, Ok.

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$$\Gamma(z+1) = \int_0^{\infty} e^{-x} x^z dx = -e^{-x} \frac{x^z}{z} + \int_0^{\infty} e^{-x} x^{z-1} dx$$

$$\Gamma(z+1) = z \Gamma(z)$$

$$\Gamma(x) = \frac{x^x}{z^x \Gamma(z+1)} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! (x+1)(x+2)\dots(x+m)} \right]$$

$$\Gamma(k) = \frac{x^k}{z^k} \left[\frac{1}{\Gamma(k+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(k+m+1)} \right]$$

$$\Gamma(0) = \int_0^{\infty} \frac{e^{-x}}{x} dx = \infty$$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$
 is well defined in $(-1, \infty)$ and $(-2, -1)$

So we will get the case of, second case when the difference between two integers, two alpha is, $k_1 - k_2$ which is 2α , α minus of minus α , so which is integer, Ok which is an integer. So 2α is always positive, so which is a positive integer, non-integer case we have seen in the last video. We will see how, what is

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$$\Gamma(x) = \frac{x^x}{z^x \Gamma(z+1)} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! (x+1)(x+2)\dots(x+m)} \right]$$

$$\Gamma(k) = \frac{x^k}{z^k} \left[\frac{1}{\Gamma(k+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(k+m+1)} \right]$$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$
 is well defined in $(-1, \infty)$ and $(-2, -1)$

$k_1 - k_2 = 2\alpha$ is a +ve integer.

now, for integer case.

So, and 2α is integer means 2α is either; zero case we will deal separately in the next video. So 2α is 1, 2, 3, 4 and so on, Ok, 4, 5, 6, 7 and so on. So you look at

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$$J_{\alpha}(x) = \frac{x^{\alpha}}{\Gamma(\alpha)} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! (\alpha+1)(\alpha+2)\dots(\alpha+m)} \right]$$

$$J_{\alpha}(k) = \frac{x^{\alpha}}{\Gamma(\alpha)} \left[\frac{1}{\Gamma(\alpha)} + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(\alpha+m+1)} \right]$$

$\Gamma(\beta) = \frac{\Gamma(\beta+1)}{\beta}$ is well defined
 in $(-1, \infty)$
 in $(-2, -1)$

$k_1 - k_2 = 2k$ is a +ve integer.
 $k = 1, 2, 3, 4, 5, 6, 7, \dots$

this case when 1,3,5, 7 all odd integers, alpha is 1 by 2, 3 by 2, 5 by 2, 7 by 2 and so on, Ok. So you can look

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$$J_{\alpha}(x) = \frac{x^{\alpha}}{\Gamma(\alpha)} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! (\alpha+1)(\alpha+2)\dots(\alpha+m)} \right]$$

$$J_{\alpha}(k) = \frac{x^{\alpha}}{\Gamma(\alpha)} \left[\frac{1}{\Gamma(\alpha)} + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(\alpha+m+1)} \right]$$

$\Gamma(\beta) = \frac{\Gamma(\beta+1)}{\beta}$ is well defined
 in $(-1, \infty)$
 in $(-2, -1)$

$k_1 - k_2 = 2k$ is a +ve integer.
 $k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

at this one. So this gamma function, gamma of, alpha is half or 3 by 2 or anything. So half of odd integer plus m plus 1 is still, never be negative integer Ok whatever may be the case. This will always be positive, still it is well-defined, right even if I choose alpha is negative, that is, even if, what I mean to say is let's take J minus alpha, just replace alpha by minus alpha. What you have is J alpha by 2 power minus alpha is 1 by gamma of minus alpha plus 1 plus sigma m is from 1 to infinity x by 2 power 2 power n divided by m factorial gamma of minus alpha plus m plus 1, Ok.

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$$J_{\alpha}(x) = \frac{x^{\alpha}}{2^{\alpha} \Gamma(\alpha+1)} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(\alpha+1)(\alpha+1)\dots(\alpha+m)} \right]$$

$$J_{\alpha}(x) = \frac{x^{\alpha}}{2^{\alpha}} \left[\frac{1}{\Gamma(\alpha+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(\alpha+m+1)} \right]$$

$\alpha - k_x = 2x$ is a +ve integer.
 $k_x = 1, 2, 3, 4, 5, 6, 7, \dots$
 $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

$$J_{-\alpha}(x) = \frac{x^{-\alpha}}{2^{-\alpha}} \left[\frac{1}{\Gamma(-\alpha+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m}}{m! \Gamma(-\alpha+m+1)} \right]$$

$\Gamma(\beta) = \frac{\Gamma(\beta+1)}{\beta}$ is well defined in $(-1, 0)$ and $(-2, -1)$

So when alpha is one of these values, this gamma of minus alpha plus m plus 1, so you can see that gamma of, let us say take 5 by 2, minus 5 by 2 plus m plus 1. This can be

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$$J_{\alpha}(x) = \frac{x^{\alpha}}{2^{\alpha} \Gamma(\alpha+1)} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(\alpha+1)(\alpha+1)\dots(\alpha+m)} \right]$$

$$J_{\alpha}(x) = \frac{x^{\alpha}}{2^{\alpha}} \left[\frac{1}{\Gamma(\alpha+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(\alpha+m+1)} \right]$$

$\alpha - k_x = 2x$ is a +ve integer.
 $k_x = 1, 2, 3, 4, 5, 6, 7, \dots$
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$\Gamma(\beta) = \frac{\Gamma(\beta+1)}{\beta}$ is well defined in $(-1, 0)$ and $(-2, -1)$

negative number when m is 1, Ok but this never be a negative integer, Ok. So this is never be one of these values especially when alpha is this one.

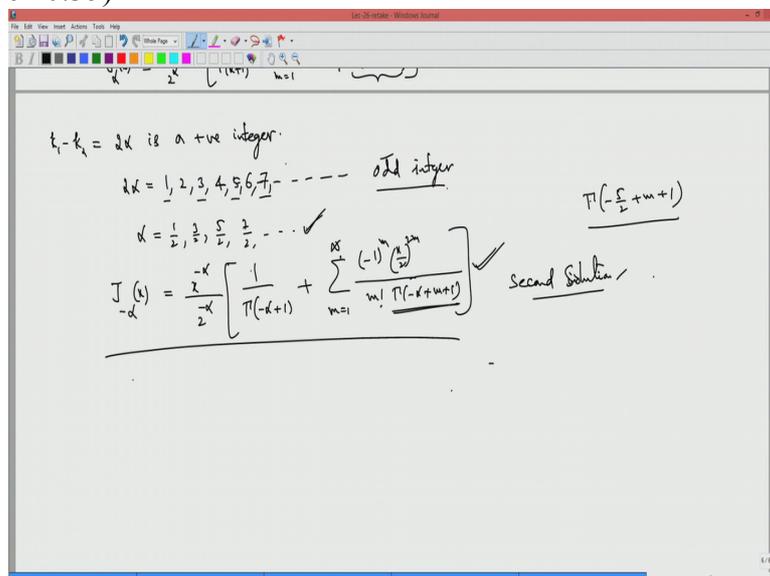
So that means from the recurrence relation of the Bessel equation, if you simply replace k 1, k by k 2, that is minus alpha, you don't have issues to find, you don't have problem to find the constants. Those constants you will get it without a problem because this gamma function is well-defined. So what you get is the same number with alpha is replaced with minus alpha. And replace alpha with minus alpha this becomes zero but this cannot be, this is not a

problem because this is not even defined at these values. And when alpha is a negative integer, 2α is a negative integer, so 2α is odd integer, sorry, Ok when 2α is odd integer that is alpha is half, 3 by 2, 5 by 2, 7 by 2 and so on, this gamma function when alpha is replaced with negative, minus, so this one is never be negative integer and is well-defined. This is well-defined.

So you will be able to find all the coefficients so that finally when you end up a solution like this with some constant, constant now I replace in the gamma function as x power alpha by 2 power alpha into gamma alpha plus 1 with alpha being minus alpha. x power minus alpha divided by 2 power minus alpha gamma of minus alpha plus 1, with that constant you multiply, what you end up is this same Bessel function of second, first kind but for minus alpha. So this is well-defined.

So what you have, this is another solution, this is the second solution. That means you don't have problem when it is odd integer. Odd integer, odd integer case you don't have problems. You can get the second solution directly from the recurrence relation

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because I don't have problem with this, 1 by zero, or zero by zero when you try to calculate those constants in the power series. Only trouble with this, so, only alpha, 2α is 2,4,6,8, and so on. So that is, we are left with only one case. alpha is 1, 2, 3 and so on, Ok, so actual integer. So, or you can

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$$J_{\alpha}(x) = \frac{x^{\alpha}}{2^{\alpha}} \left[\Gamma(\alpha+1) + \sum_{m=1}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \Gamma(\alpha+m+1)} \right]$$

$k_1 - k_m = 2k$ is an integer.

$n = 1, 2, 3, 4, 5, 6, 7, \dots$ --- odd integer

$\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$ ✓

$$J_{-\alpha}(x) = \frac{x^{-\alpha}}{2^{-\alpha}} \left[\frac{1}{\Gamma(-\alpha+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \Gamma(-\alpha+m+1)} \right]$$

Second Solution ✓

$\alpha = 1, 2, 3, \dots$

write alpha equal to n, n is from 1, 2, 3 onwards. So this is what is we are interested. So this is actually case

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$$J_{\alpha}(x) = \frac{x^{\alpha}}{2^{\alpha}} \left[\Gamma(\alpha+1) + \sum_{m=1}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \Gamma(\alpha+m+1)} \right]$$

$k_1 - k_m = 2k$ is an integer.

$n = 1, 2, 3, 4, 5, 6, 7, \dots$ --- odd integer

$\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$ ✓

$$J_{-\alpha}(x) = \frac{x^{-\alpha}}{2^{-\alpha}} \left[\frac{1}{\Gamma(-\alpha+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \Gamma(-\alpha+m+1)} \right]$$

Second Solution ✓

$\alpha = n, n = 1, 2, 3, \dots$

2. Case 2 is actually only about this, Ok.

So we have seen J_n , J_n of x is one solution. So you have x power n divided by 2 power n , 1 by gamma when alpha is n , it's gamma n plus 1 is n factorial plus sigma m is from 1 to infinity minus 1 power m x by 2 power $2m$ divided by m factorial. Here gamma of n plus m plus 1 , that is n plus m factorial. So this is what is your J_n , Bessel function of first kind. This is one solution. One solution in this case which you always get when

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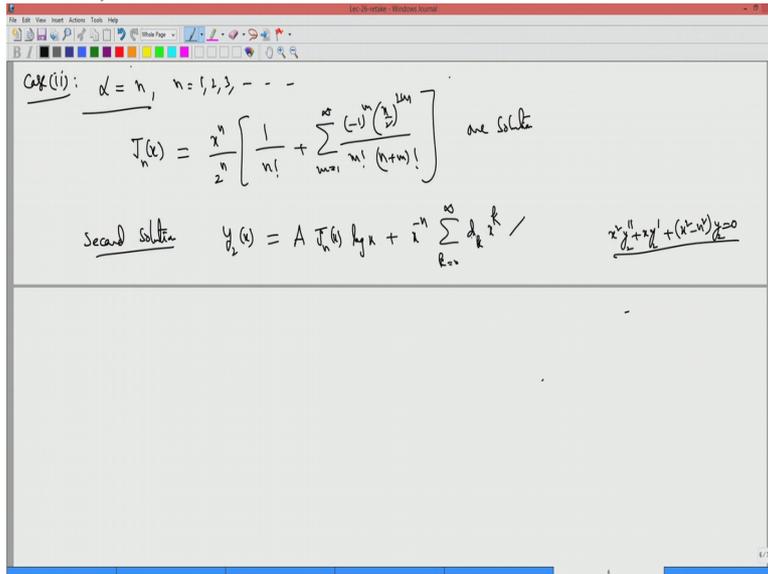
$\alpha - k_x = 2x$ is a +ve integer.
 $\alpha = 1, 2, 3, 4, 5, 6, 7, \dots$ odd integer
 $\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$
 $J_{-\alpha}(x) = \frac{x^{-\alpha}}{2^{-\alpha}} \left[\frac{1}{\Gamma(-\alpha+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m \binom{-\alpha}{m}}{m! \Gamma(-\alpha+m+1)} \right]$ Second Solution
 $\Gamma(-\frac{\alpha}{2} + m + 1)$
 Case (ii): $\alpha = n, n = 1, 2, 3, \dots$
 $J_n(x) = \frac{x^n}{2^n} \left[\frac{1}{n!} + \sum_{m=1}^{\infty} \frac{(-1)^m \binom{-n}{m}}{m! (n+m)!} \right]$ one solution

for the bigger root in the Frobenius method. Of course you multiply certain constant, conventional constant to define a, this solution as a Bessel function of first kind.

So the second solution, the second solution in the Frobenius method, you should look for y_2 of x . y_2 of x I define it as this way. So you look for solution in this form, $A y_1$ is my $J_n(x)$, I take it as $J_n(x) \log x$ plus x power minus n k is from zero to infinity, $i d k x$ power k . So this is simply, k is a dummy variable. I look for solution in this form. For this I need, I need to substitute into the equation $x^2 y'' + x y' - (x^2 - n^2) y = 0$. This is the Bessel equation plus x^2 minus α is n , x^2 minus n^2 y equal to zero. So in this I need to substitute y_2 , Ok. So this is actually satisfying y_2 .

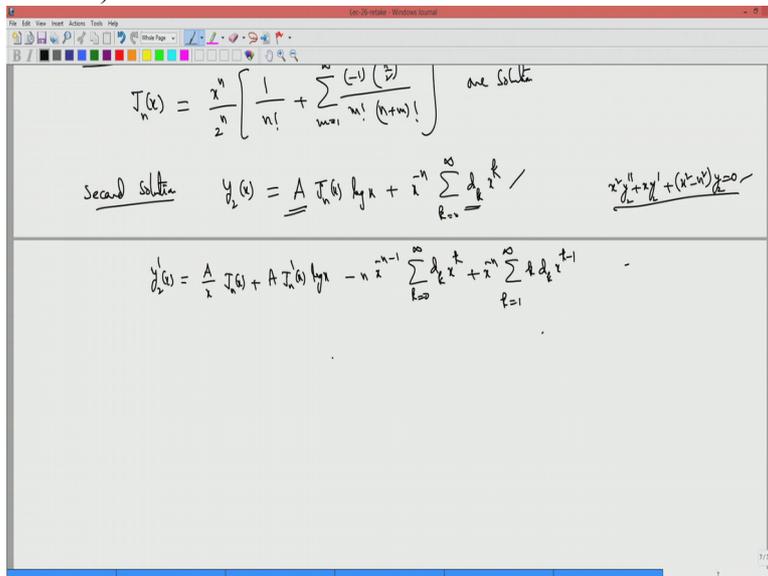
So if I look for solution in this form,

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substitute the Bessel equation and try to get your unknown constants A and d_k, Ok. That's the idea of the Frobenius method. So you need y₂. So y₂ of x is A by x, so you are differentiating 2 functions, product of 2 functions, so you do that. So A by x j_n of x plus A J_n dash of x log x plus minus n minus n x power minus n minus 1, this series as it is, x power k plus x power minus n k is from zero to, 1 to infinity, so k d_k x power k minus 1, Ok, this one is your y₂.

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y₂' double dash, we calculate again, so what you see is minus A by x square J_n x plus k A by x J_n dash of x, so here when you do, when you differentiate log x you get the same term, so you have 2 plus A times J_n double dash of x log x, Ok.

And this one and what you have is minus n into into minus n minus 1, so it will be plus, n into n plus 1, x power minus n minus 2, sigma k is from 0 to infinity d k x power k minus n x power minus n minus 1. And this is running from 1 to infinity, k d k x power k minus 1. Now you differentiate

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The whiteboard shows the following steps:

$$J_n(x) = \frac{x^n}{2^n} \left[\frac{1}{n!} + \sum_{m=0}^{\infty} \frac{(-1)^m \binom{n}{m}}{m!(n+m)!} \right] \text{ one soln}$$

$$\text{Second Soln } y_2(x) = A J_n(x) h_2(x) + x^{-n} \sum_{k=0}^{\infty} d_k x^k \quad \text{with } x^2 y'' + xy' + (x^2 - n^2)y = 0$$

$$y_2'(x) = \frac{A}{x} J_n(x) + A J_n'(x) h_2(x) - n x^{-n-1} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$y_2''(x) = -\frac{A}{x^2} J_n(x) + \frac{2A}{x} J_n'(x) + A J_n''(x) h_2(x) + n(n+1) x^{-n-2} \sum_{k=0}^{\infty} d_k x^k - n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

the last term, it will give me minus n x power minus n minus 1, k is from 1 to infinity k d k x power k minus 1 plus x power minus n, this is running from 2 to infinity, k into k minus 1 d k x power k minus 2. So this is what is

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The whiteboard shows the following steps:

$$\text{Second Soln } y_2(x) = A J_n(x) h_2(x) + x^{-n} \sum_{k=0}^{\infty} d_k x^k \quad \text{with } x^2 y'' + xy' + (x^2 - n^2)y = 0$$

$$y_2'(x) = \frac{A}{x} J_n(x) + A J_n'(x) h_2(x) - n x^{-n-1} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$y_2''(x) = -\frac{A}{x^2} J_n(x) + \frac{2A}{x} J_n'(x) + A J_n''(x) h_2(x) + n(n+1) x^{-n-2} \sum_{k=0}^{\infty} d_k x^k - n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$-n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1} + x^{-n} \sum_{k=2}^{\infty} k(k-1) d_k x^{k-2}$$

2 derivatives.

Now put this into this Bessel equation to get finally, so see that what you get is substitute into the, substitute, substitute into Bessel equation y double dash plus x y dash plus x square minus n square y equal to zero to get,

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The whiteboard shows the following steps:

$$y_1'(x) = \frac{A}{x} J_n(x) + A J_n'(x) \log x - n x^{-n-1} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$y_1''(x) = -\frac{A}{x^2} J_n(x) + \frac{2A}{x} J_n'(x) + A J_n''(x) \log x + n(n+1) x^{-n-2} \sum_{k=0}^{\infty} d_k x^k - n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$- n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1} + x^{-n} \sum_{k=2}^{\infty} k(k-1) d_k x^{k-2}$$

Substitute into Bessel equation $x^2 y'' + x y' + (x^2 - n^2) y = 0$, to get

what you get is minus a, so you are multiplying x square for y 2 double dash, so minus A J n 2 A x J n dash of x plus A x square J n double dash of x log x plus n into n plus 1, x power minus n k is from zero to infinity d k x power k minus n into x power minus n sigma k is from 1 to infinity k d k x power k. So I have taken my one x here, one x here. So,

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The whiteboard shows the following steps:

$$y_1'(x) = \frac{A}{x} J_n(x) + A J_n'(x) \log x - n x^{-n-1} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$y_1''(x) = -\frac{A}{x^2} J_n(x) + \frac{2A}{x} J_n'(x) + A J_n''(x) \log x + n(n+1) x^{-n-2} \sum_{k=0}^{\infty} d_k x^k - n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$- n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1} + x^{-n} \sum_{k=2}^{\infty} k(k-1) d_k x^{k-2}$$

Substitute into Bessel equation $x^2 y'' + x y' + (x^2 - n^2) y = 0$, to get

$$-A J_n(x) + 2A x J_n'(x) + A x^2 J_n''(x) \log x + n(n+1) x^{-n} \sum_{k=0}^{\infty} d_k x^k - n x^{-n} \sum_{k=1}^{\infty} k d_k x^k$$

minus n x power minus n, again k is from 1 to infinity k d k, x power k plus x power minus n sigma k is from 2 to infinity, k into k minus 1 d k x power k.

Now this 1 plus x times y 2 dash so that is A J n of x plus A x J n dash of x log x minus x x power minus n k is from zero to infinity d k x power k plus x power minus n k is from 1 to infinity k d k x power k. So that's my second term and here,

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$$y_1(x) = \frac{A}{x} J_n(x) + A J_n'(x) \log x - n x^{-n-1} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$y_2(x) = -\frac{A}{x^2} J_n(x) + \frac{2A}{x} J_n'(x) + A J_n''(x) \log x + n(n+1) x^{-n-2} \sum_{k=0}^{\infty} d_k x^k - n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$- n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1} + x^{-n} \sum_{k=2}^{\infty} k(k-1) d_k x^{k-2}$$
 Substitute into Bessel equation $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$, we get

$$-A J_n''(x) + 2A x J_n'(x) + A x^2 J_n''(x) \log x + n(n+1) x^{-n} \sum_{k=0}^{\infty} d_k x^k - n x^2 \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$- n x^{-n} \sum_{k=1}^{\infty} k d_k x^k + x^{-n} \sum_{k=2}^{\infty} k(k-1) d_k x^k + A J_n''(x) + A x J_n'(x) \log x - n x^{-n} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

plus x square minus n square and you have this, that is the term A n, J n of k log x plus x square minus n square sigma x power minus n right so you have x power minus n sigma n k is from zero from infinity d k x power k equal to zero. So this is what I substitute into this. So from this relation

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$$y_1(x) = \frac{A}{x} J_n(x) + A J_n'(x) \log x - n x^{-n-1} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$y_2(x) = -\frac{A}{x^2} J_n(x) + \frac{2A}{x} J_n'(x) + A J_n''(x) \log x + n(n+1) x^{-n-2} \sum_{k=0}^{\infty} d_k x^k - n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1}$$

$$- n x^{-n-1} \sum_{k=1}^{\infty} k d_k x^{k-1} + x^{-n} \sum_{k=2}^{\infty} k(k-1) d_k x^{k-2}$$
 Substitute into Bessel equation $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$, we get

$$\left\{ \begin{aligned} &-A J_n''(x) + 2A x J_n'(x) + A x^2 J_n''(x) \log x + n(n+1) x^{-n} \sum_{k=0}^{\infty} d_k x^k - n x^2 \sum_{k=1}^{\infty} k d_k x^{k-1} \\ &- n x^{-n} \sum_{k=1}^{\infty} k d_k x^k + x^{-n} \sum_{k=2}^{\infty} k(k-1) d_k x^k + A J_n''(x) + A x J_n'(x) \log x - n x^{-n} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^{k-1} \\ &+ (x^2 - \nu^2) A J_n''(x) \log x + (x^2 - \nu^2) x^{-n} \sum_{k=0}^{\infty} d_k x^k = 0 \end{aligned} \right.$$

I should get all my constants A and d ks. That is from the Frobenius method. That is what we will see now, Ok. So if you simplify, further so you first simplification we can see this one. You add this one. Look at these 3 terms, this one, this one. This is exactly Bessel equation,

Bessel equation, left hand side of the Bessel equation. So because A is common, J and, J and of x is satisfying the Bessel equation, so these, sum of these 3 terms is zero, Ok. So they won't contribute here.

So what you are left with is, so you will see, also you can cancel these 2 terms and you are left with this term, Ok, so

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$$\begin{aligned}
 & -A J_n''(x) + 2A x^{-1} J_n'(x) + A x^{-2} J_n''(x) y x + n(n+1) x^{-n} \sum_{k=0}^{\infty} d_k x^k - n x^{-n} \sum_{k=1}^{\infty} k d_k x^k \\
 & -n x^{-n} \sum_{k=1}^{\infty} k d_k x^k + x^{-n} \sum_{k=0}^{\infty} k(k-1) d_k x^k + A x^{-2} J_n''(x) + A x^{-1} J_n'(x) y x - n x^{-n} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=0}^{\infty} k d_k x^k \\
 & + (x^{-n}) A J_n''(x) y x + (x^{-n}) x^{-n} \sum_{k=0}^{\infty} d_k x^k = 0
 \end{aligned}$$

you write this, $2 A x^{-1} J_n'(x)$. Now, now each of these series, you just pick up common terms, common index that is 2 to infinity, Ok. So if I write this from k is from 2 to infinity, so then what you, I can take x^n inside so what you have is this is what you have. So you have $d_k x^{k-n}$.

So wherever you have picked up these things, so n into $n+1$, Ok first from here and then you have $-n k$, $-n k$ and $+n k$, so you have $-2 n k$. So you have $-n k$ and another one here, Ok and then here, $+k$ into $k-1$ and then here, $-n$ and $+k$, Ok. And then what happens to this term? So this term, again you have $-n$, in this second term, so you have a $-n^2$, Ok so and then what you are left with is only the terms that involve zero and 1. So you have a zero here. So before I see this I will take x^2 , x^2 and so you write this, x^2 , k is from zero to infinity, $d_k x^{k+2}$. So if I put this term, with x^2 multiplication, this you can rewrite by replacing k equal to $k-2$. So it is actually running from 2 to infinity, k equal to $k-2$ so that this becomes this, Ok and I have, what I am left with

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$$\begin{aligned}
 & -A J'_n(x) + 2Ax J'_n(x) + Ax^2 J''_n(x) + n(n+1)x^{-n} \sum_{k=0}^{\infty} d_k x^k - nx^{-n} \sum_{k=1}^{\infty} k d_k x^k \\
 & -nx^{-n} \sum_{k=1}^{\infty} k d_k x^k + x^{-n} \sum_{k=2}^{\infty} k(k-1) d_k x^k + A J'_n(x) + Ax J'_n(x) + nx^{-n} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^k \\
 & + (x^{-n}) A J'_n(x) + (x^{-n}) x^{-n} \sum_{k=0}^{\infty} d_k x^k = 0
 \end{aligned}$$

$$2Ax J'_n(x) + \sum_{k=2}^{\infty} [n(n+1) - 2nk + k(k-1) - n + k - n^2] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^k$$

is only the terms with k equal to zero and k equal to, k equal to zero, if you choose, k equal to zero here, so this is the term n into n plus 1 and then k equal to zero where else, you have here, minus n and again here, so you have minus n square Ok, so this is simply zero, that is 1. So k equal to zero terms are not there

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Substitute into Bessel equation $x^2 y'' + xy' + (x^2 - n^2)y = 0$, let $y = \sum_{k=0}^{\infty} d_k x^k$

$$\begin{aligned}
 & -A J'_n(x) + 2Ax J'_n(x) + Ax^2 J''_n(x) + n(n+1)x^{-n} \sum_{k=0}^{\infty} d_k x^k - nx^{-n} \sum_{k=1}^{\infty} k d_k x^k \\
 & -nx^{-n} \sum_{k=1}^{\infty} k d_k x^k + x^{-n} \sum_{k=2}^{\infty} k(k-1) d_k x^k + A J'_n(x) + Ax J'_n(x) + nx^{-n} \sum_{k=0}^{\infty} d_k x^k + x^{-n} \sum_{k=1}^{\infty} k d_k x^k \\
 & + (x^{-n}) A J'_n(x) + (x^{-n}) x^{-n} \sum_{k=0}^{\infty} d_k x^k = 0 \quad \underline{k=0 \quad n(n+1) - n^2 = 0}
 \end{aligned}$$

$$2Ax J'_n(x) + \sum_{k=2}^{\infty} [n(n+1) - 2nk + k(k-1) - n + k - n^2] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^k$$

and now k equal to 1, k equal to 1 terms now you calculate. So n into n plus 1 here and then minus n k equal to 1, and so minus 2 n actually here, Ok and here again minus n Ok and then k equal to 1 here so plus 1, and here. So you have finally minus n square that is for d 1. So this is corresponds to into d 1 x power minus n plus 1.

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Substitute into Bessel equation $x^2 y'' + x y' + (x^2 - n^2) y = 0$ to get

$$\left\{ \begin{aligned} & -A \cancel{I_n'(x)} + 2Ax I_n'(x) + Ax^2 I_n''(x) + n(n+1) x^{-n} \sum_{k=0}^{\infty} d_k x^k - n^2 \sum_{k=1}^{\infty} k d_k x^k \\ & -n x^n \sum_{k=1}^{\infty} k d_k x^k + x^n \sum_{k=2}^{\infty} k(k-1) d_k x^k + A \cancel{I_n'(x)} + Ax^2 I_n''(x) - n x^n \sum_{k=0}^{\infty} d_k x^k + x^n \sum_{k=1}^{\infty} k d_k x^k \\ & + (x^2 - n^2) A I_n'(x) + (x^2 - n^2) x^n \sum_{k=0}^{\infty} d_k x^k = 0 \end{aligned} \right.$$

$k=0: n(n+1) - n - n^2 = 0$
 $k=1: (n(n+1) - n - n^2 + 1 - n^2)$

$$2Ax I_n'(x) + \sum_{k=2}^{\infty} [n(n+1) - 2nk + k(k-1) - n + k - n^2] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^k$$

\therefore
 \therefore

So you have, what you are left with is n^2 , n^2 goes, plus n minus n goes, so what you are left with is $1 - 2n$. So you finally

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Substitute into Bessel equation $x^2 y'' + x y' + (x^2 - n^2) y = 0$ to get

$$\left\{ \begin{aligned} & -A \cancel{I_n'(x)} + 2Ax I_n'(x) + Ax^2 I_n''(x) + n(n+1) x^{-n} \sum_{k=0}^{\infty} d_k x^k - n^2 \sum_{k=1}^{\infty} k d_k x^k \\ & -n x^n \sum_{k=1}^{\infty} k d_k x^k + x^n \sum_{k=2}^{\infty} k(k-1) d_k x^k + A \cancel{I_n'(x)} + Ax^2 I_n''(x) - n x^n \sum_{k=0}^{\infty} d_k x^k + x^n \sum_{k=1}^{\infty} k d_k x^k \\ & + (x^2 - n^2) A I_n'(x) + (x^2 - n^2) x^n \sum_{k=0}^{\infty} d_k x^k = 0 \end{aligned} \right.$$

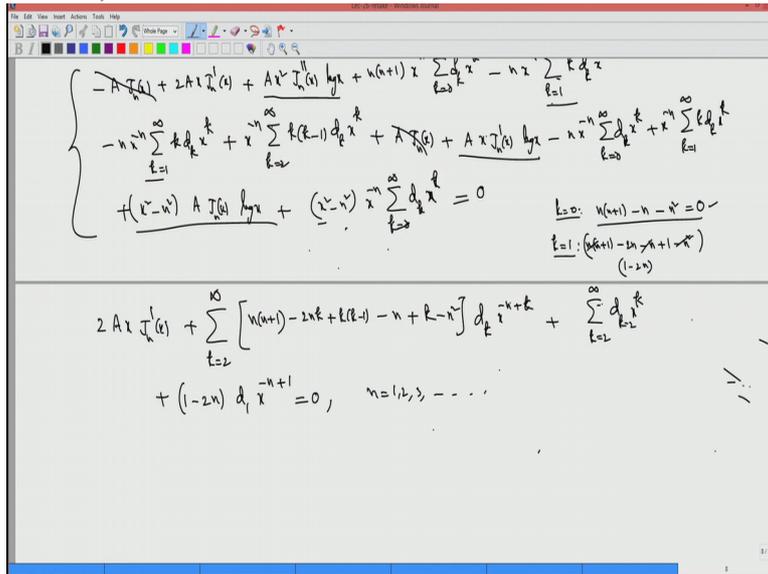
$k=0: n(n+1) - n - n^2 = 0$
 $k=1: (n(n+1) - n - n^2 + 1 - n^2)$
 $(1 - 2n)$

$$2Ax I_n'(x) + \sum_{k=2}^{\infty} [n(n+1) - 2nk + k(k-1) - n + k - n^2] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^k$$

\therefore
 \therefore

end up with $1 - 2n$ and $1 \times \text{power} - n$ plus 1 equal to zero. and equal to zero. So this is, what is n ? n so far is running from 1, 2, 3 onwards. So n is actually 1, 2, 3 onwards.

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$$\begin{aligned}
 & -A J_n''(x) + 2Ax J_n'(x) + Ax^2 J_n''(x) + n(n+1)x \sum_{k=0}^{\infty} d_k x^k - nx \sum_{k=1}^{\infty} k d_k x^k \\
 & -nx^n \sum_{k=1}^{\infty} k d_k x^{k-1} + x^n \sum_{k=2}^{\infty} k(k-1) d_k x^{k-2} + A J_n''(x) + Ax^2 J_n''(x) + n(n+1)x \sum_{k=0}^{\infty} d_k x^k - nx \sum_{k=1}^{\infty} k d_k x^k \\
 & + (x^n - n) A J_n''(x) + (x^n - n) \sum_{k=2}^{\infty} d_k x^k = 0
 \end{aligned}$$

$$\begin{aligned}
 & k=0: n(n+1) - n - n^2 = 0 \\
 & k=1: (k^2 + 1) - 2n - n + 1 - n^2 \\
 & \quad \quad \quad (1 - 2n)
 \end{aligned}$$

$$\begin{aligned}
 & 2Ax J_n'(x) + \sum_{k=2}^{\infty} [n(n+1) - 2nk + k(k-1) - n + k - n^2] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^k \\
 & + (1-2n) d_1 x^{-n+1} = 0, \quad n=1,2,3, \dots
 \end{aligned}$$

So this is the equation.

So now whole equation is this. Now because, now you can substitute what is your J_n of x , Ok, J_n dash. So for this you need J_n . You know J_n , you calculate J_n dash. J_n dash if you calculate, and you can get, what you get J_n dash, so you can, you can, you can just get this term, Ok. So from this, you calculate J_n . I am writing directly. So this is your J_n . You differentiate this J_n and then try to substitute, so you will get $2Ax J_n$ dash of x will be $A n x$ power n divide by 2 power n minus 1 by n factorial plus A times sigma n is from 1 to infinity minus 1 power n . $2m$ plus n , x power $2m$ plus n divided by 2 power $2m$ plus n minus 1 , m factorial m plus n factorial. So can directly see. It is not difficult. You simply calculate J_n dash and multiply by $2Ax$. This is what is the result. Then plus, plus you have this k is from 2 to infinity. Now if you calculate this one, if you just see this, n square n square goes. Minus n and plus n goes and what you are left with is, minus k goes. And k square so you have a k into k square minus $2nk$. So you have again k minus $2n$. So k square and k $2n$ is this one, so into $d_k x$ power minus n plus k .

So you can put it together. I think I, I missed here this x square into this one, I have missed minus n . So you can put it together, these two, so this will become, this will become d_k minus 2 times, now you multiply x power minus n plus k . So this is what,

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$$(x-n) A J_n'(x) + (x-n)^{-n} \sum_{k=2}^{\infty} d_k x^k = 0$$

$$k=0: n(n+1)-n-x=0$$

$$k=1: \frac{(n(n+1)-2n-n+1-x)}{(1-2n)}$$

$$2Ax J_n'(x) + \sum_{k=2}^{\infty} [n(n+1)-2nk+k(k-1)-n+k-x] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^{k-n} + (1-2n) d_1 x^{-n+1} = 0, \quad n=1,2,3, \dots$$

$$\frac{Ax^n}{2^{n-1} n!} + A \sum_{h=1}^{\infty} \frac{(-1)^h (2n+h) x^{2n+h}}{2^{2n+h-1} h! (n+h)!} + \sum_{k=2}^{\infty} [k(k-2n) d_k + d_{k-1}] x^{-n+k}$$

so this together, this and this is this, plus you have 1 minus 2 n d 1 x power minus n plus 1 equal to zero, n is from 1,2,3 onwards. Now what you do is you simply multiply

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$$k=1: \frac{(n(n+1)-2n-n+1-x)}{(1-2n)}$$

$$2Ax J_n'(x) + \sum_{k=2}^{\infty} [n(n+1)-2nk+k(k-1)-n+k-x] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^{k-n} + (1-2n) d_1 x^{-n+1} = 0, \quad n=1,2,3, \dots$$

$$\frac{Ax^n}{2^{n-1} n!} + A \sum_{h=1}^{\infty} \frac{(-1)^h (2n+h) x^{2n+h}}{2^{2n+h-1} h! (n+h)!} + \sum_{k=2}^{\infty} [k(k-2n) d_k + d_{k-1}] x^{-n+k}$$

$$+ (-2n) d_1 x^{-n+1} = 0, \quad n=1,2,3, \dots$$

both sides, left hand side equal to zero, so this is what you have. You simply multiply x power, multiply x power n both sides. If you do this what you get the same. So

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$$2Ax^j(x) + \sum_{k=2}^{\infty} [n(n-1) - 2nk + k(k-1) - n + k - 2k] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^{k-n}$$

$$+ (1-2n)d_1 x^{-n+1} = 0, \quad n=1,2,3, \dots$$

$$\frac{A n x^n}{2(n-1)n!} + A \sum_{h=1}^{\infty} \frac{(-1)^h (2n+h) x^{2n+h}}{2^{2n+h-1} h! (n+h)!} + \sum_{k=2}^{\infty} [k(k-2n)d_k + d_{k-2}] x^{-n+k}$$

$$+ (-2n)d_1 x^{-n+1} = 0, \quad n=1,2,3, \dots$$

multiply x^n both sides

I multiply, if I multiply $2n$, so what you have is $2n$ here, and here this goes, this simply x power k and here is simply x , Ok.

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$$2Ax^j(x) + \sum_{k=2}^{\infty} [n(n-1) - 2nk + k(k-1) - n + k - 2k] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^{k-n}$$

$$+ (1-2n)d_1 x^{-n+1} = 0, \quad n=1,2,3, \dots$$

$$\frac{A n x^n}{2(n-1)n!} + A \sum_{h=1}^{\infty} \frac{(-1)^h (2n+h) x^{2n+h}}{2^{2n+h-1} h! (n+h)!} + \sum_{k=2}^{\infty} [k(k-2n)d_k + d_{k-2}] x^{-n+k}$$

$$+ (-2n)d_1 x^{-n+1} = 0, \quad n=1,2,3, \dots$$

multiply x^n both sides

If you simply multiply this is what you get.

So now this is the equation now. So from this you have to calculate all your d_k and A , the unknown constants. So how do I do this? So if you look at the series, it is starting from x^2 , x^3 and so on, Ok. So it is starting from x^2 and so on, so you start with, and here and this one is starting from $2n+2$, $2n+4$, $2n+4$ x power, this is the coefficient, x powers are $2n+2$, n is fixed. Once you fix n , anything here, what you have is x power $2n+2$ and x power $2n+4$ and so on, Ok, something bigger. This is x power 2

n. This is the coefficient of x power 2 n here. So lowest powers are x, first of all x, you have a x term, so equate the coefficients. If you equate the coefficients, coefficient of x power 1 equal to zero, this will give me d 1 equal to zero, first form, Ok.

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The whiteboard shows the following steps:

$$2Ax \frac{d}{dx} f(x) + \sum_{k=2}^{\infty} \left[n(n-1) - 2nk + k(k-1) - n + k - n \right] d_k x^{-n+k} + \sum_{k=2}^{\infty} d_k x^{k-n}$$

$$+ (-2n) d_1 x^{-n+1} = 0, \quad n=1, 2, 3, \dots$$

$$\frac{An \frac{d}{dx} f(x)}{2} + A \sum_{k=1}^{\infty} \frac{(-1)^k (2n+k) x^{2n+2k}}{2^{2n+k-1} n! (n+k)!} + \sum_{k=2}^{\infty} \left[k(k-2n) d_k + d_{k-2} \right] x^k$$

$$+ (-2n) d_1 x = 0, \quad n=1, 2, 3, \dots \quad \checkmark$$

multiply x^n both sides

coeff of $x^1 = 0$: $d_1 = 0$

What are the other terms? x square, so 2, 3 up to, you can go 2 n up to minus 1, because the next, next terms are 2 n minus 1. So 2 n is here, 2 n plus 2 is here. So 2 n minus, up to 2 n minus 1, I have to take only from here, Ok. So coefficient of x square equal to zero will give me, what you get, so d 2 so x 2, so k equal to 2, 2 is d 2 equal to, this is zero, minus d zero divided by k equal to 2 so you get 2 divided by 2 minus 2 n. So 2 square, 1 minus n, if you can make it plus, n minus 1 you can write, Ok.

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The whiteboard shows the following steps:

$$+ (-2n) d_1 x = 0, \quad n=1, 2, 3, \dots \quad \checkmark$$

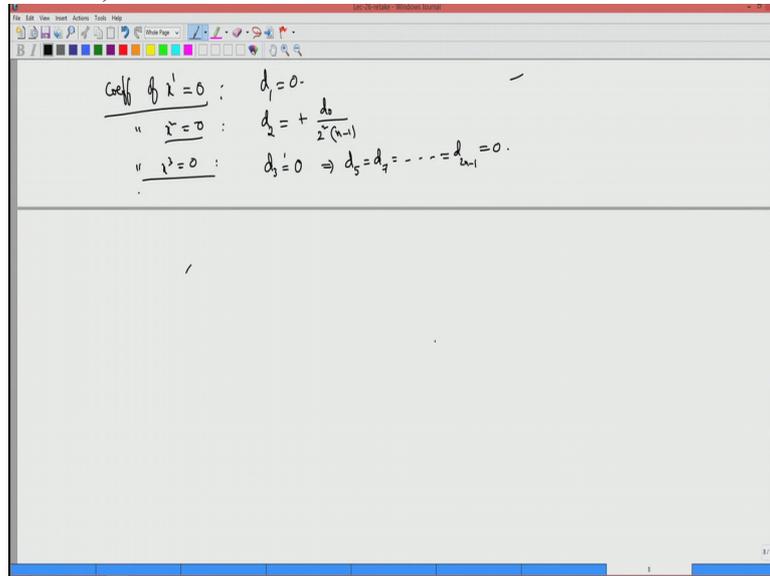
multiply x^n both sides

coeff of $x^1 = 0$: $d_1 = 0$

" $x^2 = 0$: $d_2 = + \frac{d_0}{2^2(n-1)}$

Again coefficient of x cube is equal to zero will give me; d 3 will be in terms of d 1. So d 1 is zero, so it will become zero. So implies all d 5, d 7 up to d 2 n minus 1 equal to zero, Ok. Now what happens

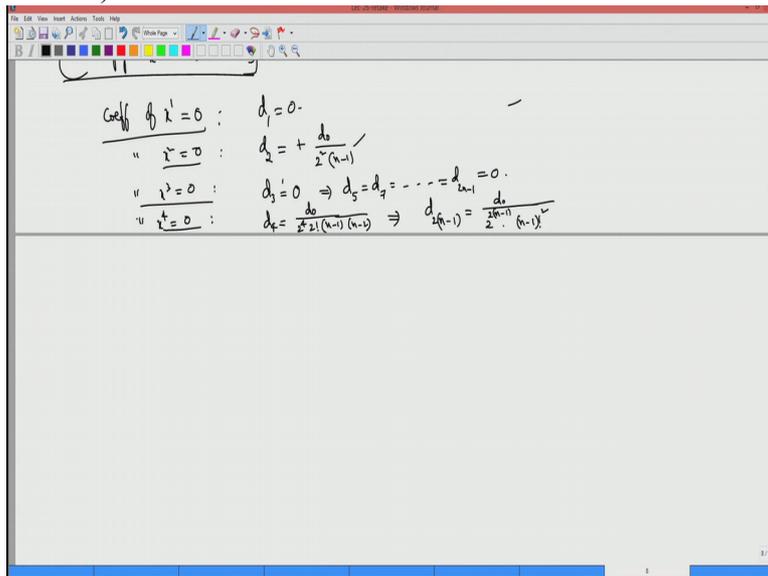
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to d 4? So you can calculate, say x power 4 equal to zero, x power 4, coefficient of x power 4 equal to zero, if you make it equal so you get d 4 as d zero, you just calculate. So d 2 divided by something, d 2 already you know from here, if you substitute, you will see that d zero divided by 2 power 4 into 2 factorial n minus 1, n minus 2. So this is what you will get.

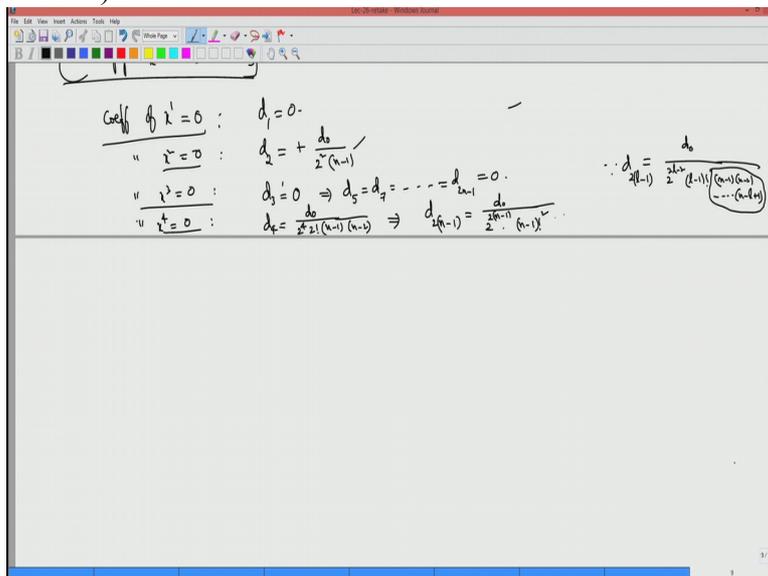
So from this inductively you can write, Ok, so from this and you may be, you may have to calculate d 6. You try to calculate d 6 and put it in this form. From that you can see the trend and you can write d 2 n, so how long you can go, you can only go 2 n minus 1. So 2 n minus 2, so let's write 2 n minus 2, that means 2 into n minus 1, Ok. 2 into 2, 2 into n minus 1, this am writing directly, directly I am writing so d zero divided by 2 power 2 into n minus 1 into n minus 1 factorial square is what you will see, Ok.

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This is because, you see that this is because d_{2l-1} for any l , if you write it, what you see is d_0 divided by 2^{2l-2} , $(l-1)!$ and $n-1$, $n-2$ up to $n-l$, that is $l+1$, so this is what. So this sum when l equal to $n-1$,

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it becomes $n-1$ factorial, So you get along with this, l equal to n , you see that this will be square of this factorial. So this is what you see. So l , if you want, you can write like this, l equal to $2, 3, 4$, l equal to 2 , if you put, d_2 , so you have d_2 . d_0 is in arbitrary, we didn't find. So $2, 3, 4$ and so on, $n-1$, Ok. So because of this

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$$k=0 : d_0 = d_0$$

$$k=1 : d_1 = 0$$

$$k=2 : d_2 = \frac{d_0}{2 \cdot (n-1)}$$

$$k=3 : d_3 = 0$$

$$k=4 : d_4 = \frac{d_0}{2^2 \cdot 1 \cdot (n-1) \cdot (n-2)}$$

$$k=5 : d_5 = 0$$

$$k=6 : d_6 = \frac{d_0}{2^3 \cdot 1 \cdot 2 \cdot (n-1) \cdot (n-2) \cdot (n-3)}$$

$$k=7 : d_7 = 0$$

$$k=8 : d_8 = \frac{d_0}{2^4 \cdot 1 \cdot 2 \cdot 1 \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)}$$

$$\dots = d_{2n-1} = 0$$

$$d_{2k} = \frac{d_0}{2^k \cdot k! \cdot (n-1)!}$$

we can see this one.

So I found all sum coefficients in terms of d_0 . Sum of odd coefficients here zero, up to $2n-1$, only from here I picked up. Now for $2n$, what is the value of $2n$, so $2n$ you have to pick up from here also and here, so coefficient equation of coefficient of, coefficient of x^{2n} equal to zero. If you do this one, what you see is A_n divided by 2^{n-1} into $n!$, that is from here. That is the coefficient. Plus now you can pick up from x^{2n} . x^{2n} is $k=2n$, so this will not contribute. What you get is d_{2n-2} equal to zero.

So this will actually give me A equal to minus d_0 , I know that my d_{2n-2} from here, so if you substitute and directly if you write you see that is going to be d_0 divided by 2^{n-1} and then $n!$ factorial square, that is my d_0 minus of this and you have into 2^{n-1} into $n!$. So this will give me, a n is there. So you have a n here, that comes down. So you have minus, finally you get, you see that this n , n goes, $n-1$ factorial, that 1 factorial goes, you have 2^{n-1} Ok, so 2^{n-2} , so $n-1$, one $n-1$ goes, so you have $n-1$ and you have finally $n-1$ factorial. So this is what you will see as your A , Ok.

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$$\begin{aligned}
 \text{if } x^0 = 0 : & \quad d_2 = + \frac{d_0}{2^{(n-1)}} \\
 \text{if } x^2 = 0 : & \quad d_3 = 0 \Rightarrow d_5 = d_7 = \dots = d_{2n-1} = 0. \\
 \text{if } x^4 = 0 : & \quad d_4 = \frac{d_0}{2^{2!} (n-1)(n-2)} \Rightarrow d_{2(n-1)} = \frac{d_0}{2^{2(n-1)} \cdot (n-1)!}
 \end{aligned}$$

$$\therefore d_k = \frac{d_0}{2^{k!} \cdot 2^{2k} \cdot (n-1)(n-2) \dots (n-k)} \quad k=2,3,\dots,n-1$$

$$\text{Coeff of } x^{2n} = 0 : \quad \frac{A_n}{2^{n-1} n!} + d_{2n-2} = 0 \Rightarrow A = - \frac{d_0 \cdot 2^{n-1} \cdot n!}{2^{2n-2} \cdot (n-1)! \cdot n} = - \frac{d_0}{2^{n-1} (n-1)!}$$

Now what else you can do, so this coefficient is gone, Ok. This will not contribute, this we have taken care. Now you have to go with this, I am picking from here. So when m equal to 1, coefficient of x power 2 n plus 2, Ok, 2 n plus 2, coefficient of x power 2 n plus 2 equal to zero. If you do this one, what you see, well you can also show, but here, from here, but 2 n plus 1 may be, you can pick up from here, Ok, so that is simple. So coefficient of x power 2 n plus 1 equal to zero, first form. From this, from the last this, this series if you do, this is basically x equal to 2 n minus 1, Ok so if you write that, if you write that actually, so I will write that, so 2 n plus 1 plus d 2 n minus 1 equal to zero. But you know that d 2 n minus 1 is zero so it makes it d 2 n plus 1 equal to zero. So, implies d 2 n plus 3 and so on, everything is zero, d 2 n plus 5 all that will be zero,

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$$\text{multiply } x^m \text{ both sides}$$

$$\begin{aligned}
 \text{Coeff of } x^0 = 0 : & \quad d_1 = 0. \\
 \text{if } x^2 = 0 : & \quad d_2 = + \frac{d_0}{2^{(n-1)}} \\
 \text{if } x^4 = 0 : & \quad d_3 = 0 \Rightarrow d_5 = d_7 = \dots = d_{2n-1} = 0. \\
 \text{if } x^6 = 0 : & \quad d_4 = \frac{d_0}{2^{2!} (n-1)(n-2)} \Rightarrow d_{2(n-1)} = \frac{d_0}{2^{2(n-1)} \cdot (n-1)!}
 \end{aligned}$$

$$\therefore d_k = \frac{d_0}{2^{k!} \cdot 2^{2k} \cdot (n-1)(n-2) \dots (n-k)} \quad k=2,3,\dots,n-1$$

$$\text{Coeff of } x^{2n} = 0 : \quad \frac{A_n}{2^{n-1} n!} + d_{2n-2} = 0 \Rightarrow A = - \frac{d_0 \cdot 2^{n-1} \cdot n!}{2^{2n-2} \cdot (n-1)! \cdot n} = - \frac{d_0}{2^{n-1} (n-1)!}$$

$$\text{Coeff of } x^{2n+2} = 0 : \quad (2n+1) d_{2n+1} + d_{2n-1} = 0 \Rightarrow d_{2n+1} = 0 \Rightarrow d_{2n+3} = d_{2n+5} = \dots = 0.$$

$$\text{Coeff of } x^{2n+4} = 0 :$$

Ok.

That is why you will only look at x^{2n+2} , x^{2n+4} and so on. So if you do this, now x^{2n+2} , what you see is $-n+2$, so that means I picked up now from here, here 1 term and here 1 term so together if you write you see that $(2n+1)n+1$ factorial plus 2 into $2n+2$ d $2n+2$ plus d $2n$ equal to zero.

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Handwritten notes on a whiteboard:

$$\begin{aligned}
 & \text{For } x^{2n+2}: d_{2n+2} + d_{2n} = 0 \Rightarrow d_{2n+2} = -d_{2n} \\
 & \text{For } x^{2n+1}: (2n+1)d_{2n+1} + d_{2n-1} = 0 \Rightarrow d_{2n+1} = -\frac{d_{2n-1}}{2n+1} \\
 & \text{For } x^{2n}: -\frac{(n+2)}{2(n+1)!} + 2(2n+1)d_{2n+2} + d_{2n} = 0 \\
 & \Rightarrow d_{2n+2} = -\frac{d_{2n}}{2(n+1)} + \frac{(n+2)}{2(n+1)!}
 \end{aligned}$$

So this will actually give me d_{2n+2} in terms of $-d_{2n}$ into $\frac{n+2}{2(n+1)!}$ plus d_{2n} over $2(n+1)$. Ok so this is over.

But you see that so far

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Handwritten notes on a whiteboard:

$$\begin{aligned}
 & \text{For } x^{2n+2}: d_{2n+2} + d_{2n} = 0 \Rightarrow d_{2n+2} = -d_{2n} \\
 & \text{For } x^{2n+1}: (2n+1)d_{2n+1} + d_{2n-1} = 0 \Rightarrow d_{2n+1} = -\frac{d_{2n-1}}{2n+1} \\
 & \text{For } x^{2n}: -\frac{(n+2)}{2(n+1)!} + 2(2n+1)d_{2n+2} + d_{2n} = 0 \\
 & \Rightarrow d_{2n+2} = -\frac{d_{2n}}{2(n+1)} + \frac{(n+2)}{2(n+1)!}
 \end{aligned}$$

we see that we have all the coefficients are in terms of d naught. But this one is in terms of d 2 n, Ok. Similarly coefficients of x power $2 n$ plus, let us make it general terms $2 m$ equal to zero. So m equal to 1 we have seen here, Ok. If I choose, if I see this as the general term, m equal to 1 we have seen, this one, Ok this is for m equal to 1. So now for m equal to 2, so in general if you write minus 1 power m $2 m$ plus n divided by 2 power $2 m$ plus n minus 1 m factorial into m plus n factorial plus $2 m$, $2 m$ plus $2 n$, d $2 m$ plus $2 n$, d $2 m$ plus $2 n$ minus 2 equal to zero.

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The image shows a whiteboard with handwritten mathematical derivations. The derivations are as follows:

$$\text{Coeff of } x^{2n} = 0: \frac{A_n}{2^{n-1} n!} + d_{2n-2} = 0 \Rightarrow A_n = -\frac{d_0 2^{n-1} n!}{2^{2n-2} (n-1)! n} = -\frac{d_0}{2^{n-1} (n-1)!}$$

$$\text{Coeff of } x^{2n+1} = 0: (2n+1) d_{2n+1} + d_{2n-1} = 0 \Rightarrow d_{2n+1} = 0 \Rightarrow d_{2n+3} = d_{2n+5} = \dots = 0$$

$$\text{Coeff of } x^{2n+2} = 0: -\frac{(n+2)}{2^{n+1} (n+1)!} + 2(2n+1) d_{2n+2} + d_{2n} = 0$$

$$\Rightarrow d_{2n+2} = -\frac{d_{2n}}{2^{n+1} (n+1)} + \frac{(n+2)}{2^{n+1} (n+1)!}$$

$$\text{Coeff of } x^{2m+2n} = 0: \frac{(-1)^m (2m+n)}{2^{2m+n-1} m! (n+1)!} + 2m(2m+2n) d_{2m+2n} + d_{2m+2n-2} = 0 \dots$$

So this is what is the general coefficient, here. So we take the x power $2 m$ plus $2 n$ and here now put k equal to $2 m$ plus $2 n$, and collect these three terms, that is what we have here, Ok. Now m equal to 1, we have seen, this is what you result. m equal to 2, what you get is, I am writing directly, so if you will see, m equal to 2, so you will see that d $2 n$, so you can see that d $2 n$, m equal to 2, you will see that d $2 n$ plus 4, what you see is finally, this only coefficient m equal to 2 is 4 minus..so, $2 n$ plus 2. So you have d $2 n$ plus 2, you have from here. Make use of this here and then finally you see that d $2 n$ plus 4 will be d $2 n$ divided by 2 power 5 n plus 1, n plus 2 minus 1 divided by 2 power 4 plus n , n plus 1 factorial into 1 plus n plus 4 divided by 2 factorial 2 square n plus 2 whole square.

So don't worry about all this. So what you are seeing is d $2 n$ plus 2 is in terms of d $2 n$, d $2 n$ plus 4 will be in terms of d $2 n$.

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$$\text{Coeff of } x^{2n+2} = 0: \frac{(-1)^m (2m+n)}{2^{2m+n-1} m! (m+1)!} + 2m(Lm+2n) d_{2m+2n} + d_{2m+2n-2} = 0$$

$$\Rightarrow d_{2n+2} = -\frac{d_{2n}}{2^{2(n+1)}} + \frac{(n+2)}{2^{n+1} (n+1)!}$$

$$\text{Coeff of } x^{2n+2m} = 0: \frac{(-1)^m (2m+n)}{2^{2m+n-1} m! (m+1)!} + 2m(Lm+2n) d_{2m+2n} + d_{2m+2n-2} = 0$$

$$m=2: d_{2n+4} = \frac{d_{2n}}{2^{2(n+1)(n+2)}} - \frac{1}{2^{4+n} (n+1)!} \left(1 + \frac{n+4}{2^{2(n+2)}}\right)$$

Like that you will get. So d_{2n+6} will be in terms of d_{2n} , something else you will get, Ok. Conventionally if I choose my d_{2n} , so you know that, right, in the power series method, you can all, if you want to choose, if you have a c zero of some power series you get, this is a constant, so you can choose this as a constant number, but if you have, if you write everything in terms of some, say, in some, c_n into some power series, Ok, known power series into c_n . If this is your solution, you have to choose, you can choose your c_n depending on n , Ok. c_n can be like 1 by n plus 1 . So this is the function depending on n ,

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$$\text{Coeff of } x^{2n+2m} = 0: \frac{(-1)^m (2m+n)}{2^{2m+n-1} m! (m+1)!} + 2m(Lm+2n) d_{2m+2n} + d_{2m+2n-2} = 0$$

$$m=2: d_{2n+4} = \frac{d_{2n}}{2^{2(n+1)(n+2)}} - \frac{1}{2^{4+n} (n+1)!} \left(1 + \frac{n+4}{2^{2(n+2)}}\right)$$

$$y(x) = C_n \left(\frac{1}{n+1} \right)$$

$$C_n = \frac{1}{n+1}$$

like that you can choose.

So using that, if I choose, conventionally this is our, if d_2 is chosen this way, d_2 is conventionally chosen this way. So 2 times n plus 1 divided by 2 power n minus 1 into n minus 1 factorial, this is just I am writing, Ok. So 1 divided by 2 power n plus 2 , n plus 1 factorial into 1 plus half plus 1 plus n up to 1 by n . This is what, if I choose this, this is what is conventionally, convention, Ok. Conventionally chosen constant. That

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The image shows a whiteboard with handwritten mathematical derivations. At the top, there are some terms: 2^{2m+2} , 2^{m+1} , and 2^{m+1} . Below that, the derivation starts with the equation $\text{Coeff of } x^{2m+2} = 0$. This leads to the equation:
$$\frac{(-1)^m (2m+1)}{2^{2m+1} m! (m+1)!} + 2m (2m+2) d_{2m+2} + \frac{d_{2m+4}}{2^{2m+2} (m+1)!} = 0$$
 Then, it rearranges to solve for d_{2m+4} :
$$d_{2m+4} = \frac{d_{2m+2}}{2^{m+1} (m+1)!} - \frac{1}{2^{2m+1} (m+1)!} \left(1 + \frac{m+2}{2! 2^{m+1}} \right)$$
 Finally, it states:
$$\text{If } d_{2n} = \frac{2^{n+1}}{2^{n-1} (n-1)!} \frac{1}{2^{n+2} (n+1)!} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \quad (\text{conventionally chosen constant})$$

in terms, in as d_{naught} . You can always choose d_{naught} . Any constant times $((()))$ is also fine. That depends on n .

So this is how if you choose, Ok, then your d_{2n+2} will be in terms of d_{naught} . Again d_{2n+4} will be in terms of d_{naught} , everything in terms of d_{naught} . And earlier we have seen A also in terms of d_{naught} . And you have seen that d_{2n-1} also in terms of d_{naught} , everything in terms of d_{naught} . So this implies you have your y_2 , whatever you get y_2 as some power series times d_{naught} , Ok, so choose your

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$$\text{If } d_{2n} = \frac{2(n+1)}{2^{n-1}(n-1)!} \frac{d_0}{2^{n+2}(n+1)!} \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right), \quad \left(\text{conventionally chosen constant}\right)$$

$$\Rightarrow y_2(x) = d_0 \left[\text{power series} \right]$$

and naught as 1, finally because they are constant, this y_2 of x is what you get is your y_n of x , so this is what is the Bessel function of, Bessel function of second kind, Ok.

So this is how you get the second solution in the case of

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$$\text{If } d_{2n} = \frac{2(n+1)}{2^{n-1}(n-1)!} \frac{d_0}{2^{n+2}(n+1)!} \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right), \quad \left(\text{conventionally chosen constant}\right)$$

$$\Rightarrow y_2(x) = d_0 \left[\text{power series} \right]$$

$$\text{If } d_0 = 1, \quad \underline{y_n(x) := y_2(x)} \quad \text{Bessel function of second kind.}$$

when the root difference is integer, positive integer and in that we avoided the case 2α equal to zero. That we will see in the next video and what we have seen is only 2α equal to 1, 2, 3 and so on. So that is actually odd integer case is simpler and even integer case, that is when α equal to 1, 2, 3 onwards, that is α equal to n , n is from

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1,2,3 onwards we have seen second solution y_2 . Conventional constant if you multiply final solution, what you get is Bessel function of second kind, Ok.

So you just don't worry about these horrible calculations, just procedure you understand, we are not going to do any problems on this, just try to watch this video, just see, understand how the Frobenius method works for the Bessel equation in the second case. Ok we will see the third case that is when the difference between the roots is zero, same, zero that means roots are same. That is possible only when n equal to zero case, Ok, so that we will see in the next video. Thank you very much.