Differential Equations for Engineers
Professor Doctor Srinivasa Rao Manam
Department of Mathematics
Indian Institute of Technology Madras
Lecture No 28
Bessel differential equation

(Refer Slide Time 00:19)

So in the last few videos, we have seen how to, how to find the two linearly independent solutions for second order homogenous equations with variable coefficients by the Frobenius method when x equal to zero is a regular singular point. So when zero is the only singular point and there is no other singular point when you have x greater than or equal to zero or like less than or equal to zero, the solution will be valid. So y 1 x and y 2 x when you calculated as a Frobenius series, the series is absolutely and uniformly convergent on any close interval in x positive or x negative, Ok.

That's what we have seen. We have seen all the three cases, when the indicial equation roots, when the difference is integer, when roots of indicial equation difference is integer which is non-zero and if it is they are equal roots, so difference is zero and the difference is non-integer, Ok. So all the three cases we have demonstrated with the example.

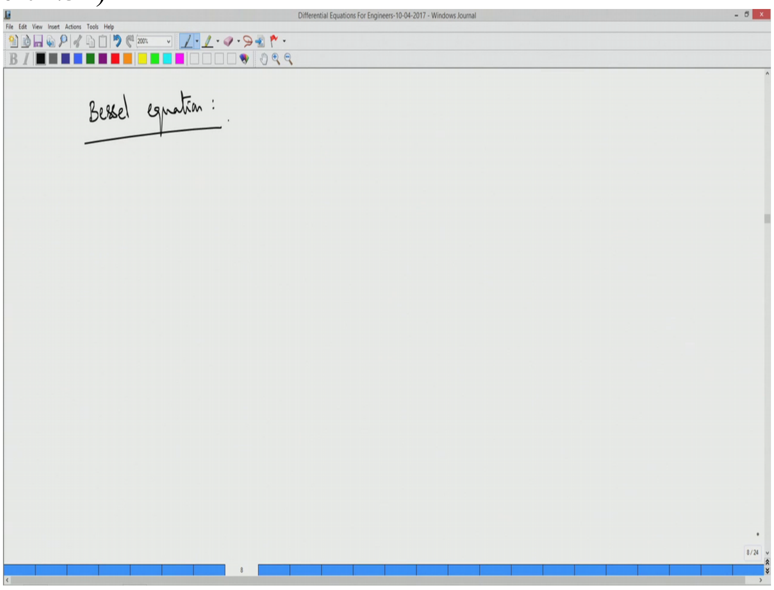
With the example we have shown that; find the general solution of second order homogenous equation with variable coefficients. So we will be applying this technique, Frobenius method to find the, to find the general solution of an important equation in physical sciences. It is called Bessel equation for which zero is a regular singular point and there is no other singular point in it, Ok.

So we will consider such equation, Bessel equation in this video. So we will try to see, it has a parameter, some alpha, so the parameter, depending on the parameter, you will be falling in one of those cases in the method. So either when you calculate the indicial equation, so indicial equation when you calculate the roots, the root difference will be depending on this parameter.

So depending on these parameter values, so the difference between the indicial roots is either zero or a non-zero integer or non-zero non integer, Ok, so we will see all the three cases. So we will start with this Bessel equation.

Bessel equation, we try to solve,

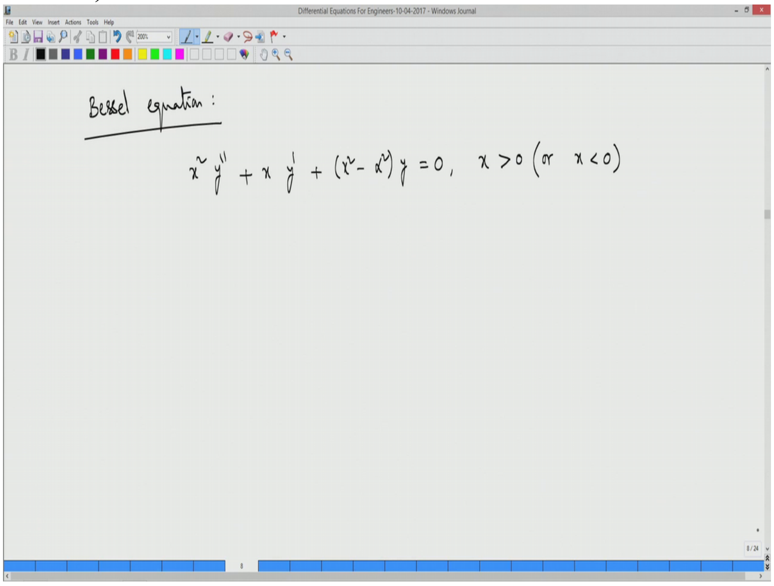
(Refer Slide Time 02:34)



find the general solution of this equation. So equation looks like this. We have x square y double dash plus x y dash plus x square minus alpha square. alpha is the parameter, y equal to zero.

So where is this valid, you see that zero is x square, a naught of x is x square. So it is zero at x equal to zero. So zero is a singular point and there is no other singular point so x greater than zero or x negative, x less than zero. So these are the two

(Refer Slide Time 03:11)



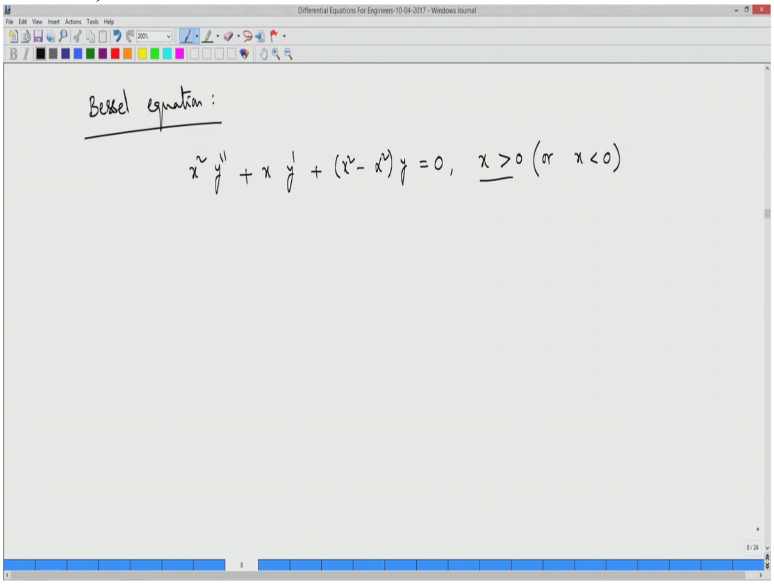
domains, each of the domain, you can consider the equations. So we will do only for x positive.

So x negative when you do, whatever the equation, whatever the solution you find when x is negative, wherever x is there

(Refer Slide Time 03:23)

you just put mod x. Wherever log x, wherever log x comes in the second solution, you have to use log mod x, Ok. Or when you have the powers of rational, real numbers, x power something k, k is the root of the indicial equation so there because you don't know, because x is negative. Negative real number power you don't know, real power of negative number, so you have to use positive, because the positive powers are defined. So mod x power k, so that is where you choose mod x, instead of x, you can write mod x so that it is the case for x negative. So mod x when x is less than zero is going to be

(Refer Slide Time 04:07)

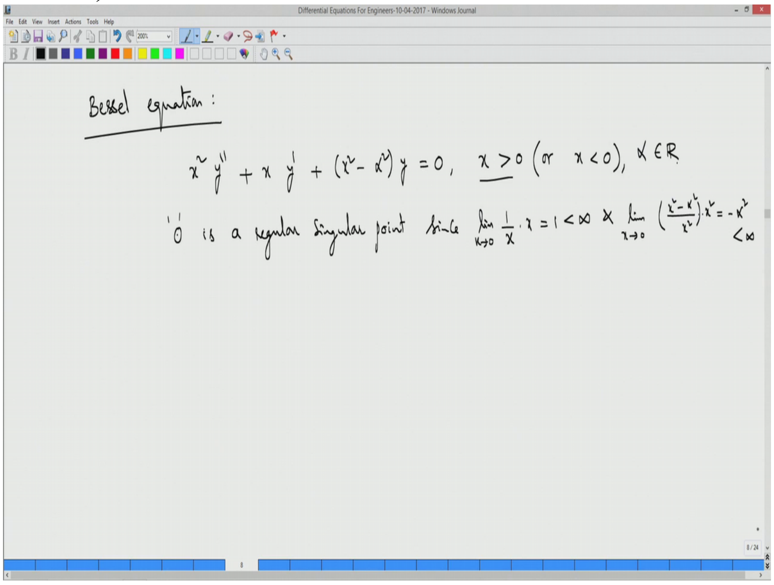


minus x that is positive, Ok, minus x power of k.

So this is the equation so you see that zero is a regular singular point as usual since, as the reason is a 1 divided by a naught that is 1 by x, x divided by x square is 1 by x into x limit x goes to zero, this is 1 which is finite and limit x goes to zero, a 2 of x that is x square minus alpha square by x square, so this is also, into x square. So this into x square is also, is actually minus alpha square.

So depending on the value you give, alpha, so alpha is given so alpha is a parameter so we can choose as a real number, Ok.

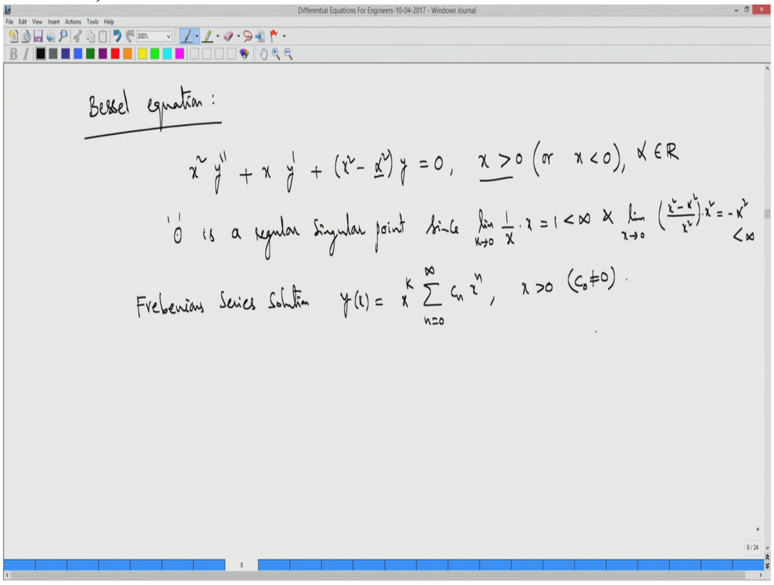
(Refer Slide Time 05:06)



So alpha square is always positive because alpha is a real number, so alpha square, square of any real number is positive number. So we have non-negative number you have alpha square. alpha can be alpha square, so without loss of generality you can choose alpha is always greater than or equal to zero, Ok.

So this is what, this is a regular singular point. So we will just, because it is a regular singular point, if you want solutions for x greater than zero, so you look for solutions by Frobenius series method, Frobenius, Frobenius series solution, solution if you look for this form x power k sigma n is from zero to infinity c n x power n for x positive and c zero without loss of generality we assume is non-zero.

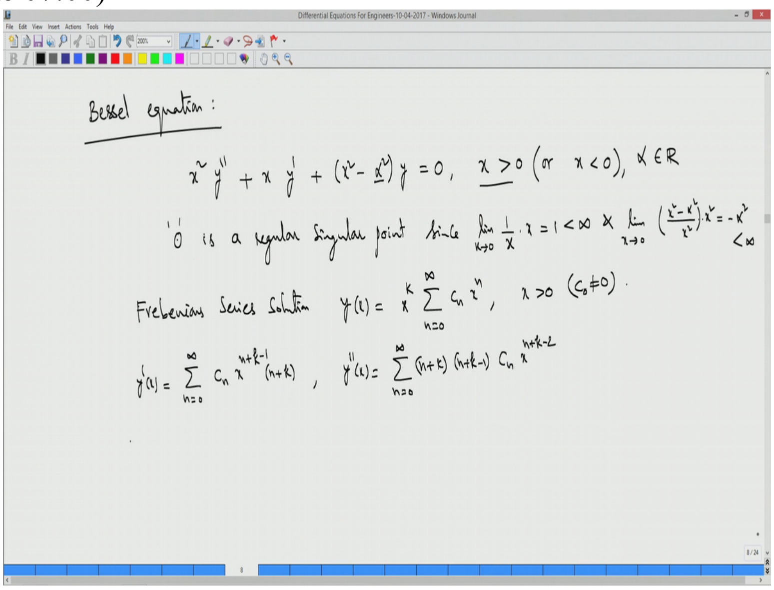
(Refer Slide Time 06:06)



So this is how you have chosen as a solution. Now you simply substitute into the equation, get all the unknowns k and c ns. So substitute, so you have y dash of x which is sigma n is from zero to infinity c n x power n plus k into n plus k. So n plus k comes out, so you have minus 1. n plus k, x power k minus 1, this is 1, y double dash of x will be n is from zero to infinity, n plus k, n plus k minus 1 into x power c n, c n constant, x power n plus k minus 2.

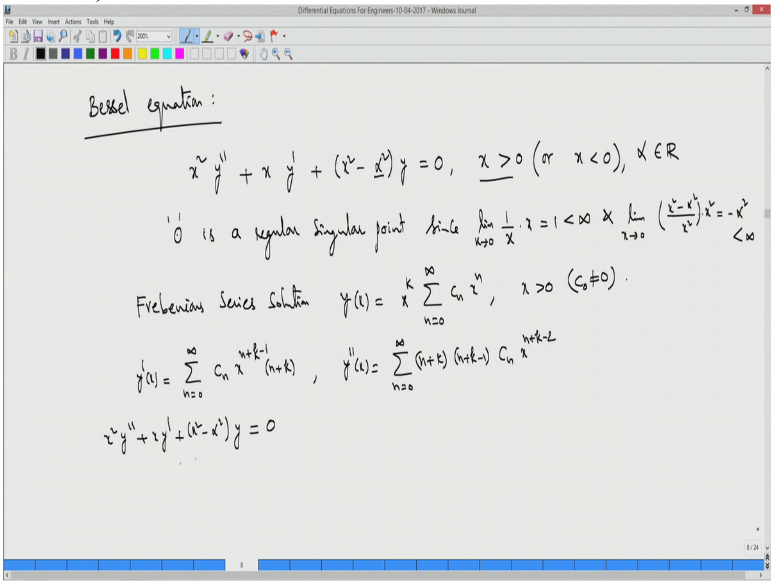
So this is, these three, you substitute into the equation, given equation, so you will have

(Refer Slide Time 07:00)



x square since x square y double dash plus x y dash plus x square minus alpha square y equal to zero,

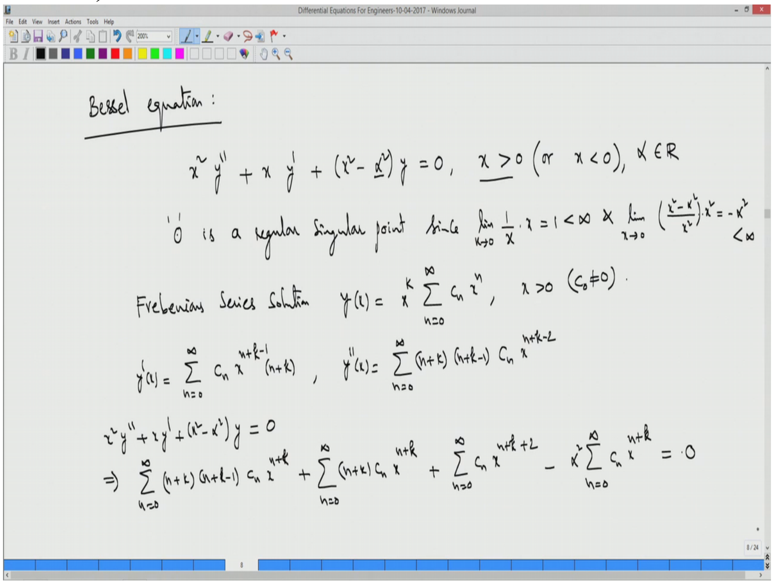
(Refer Slide Time 07:12)



Ok, so this is the equation. So you substitute all this y and y dash, so you have x square y double dash will be, n is from zero to infinity, n plus k, n plus k minus 1 c n x power n plus k. I have minus 2 and multiply with x square, it is going to be x plus, x power n plus k.

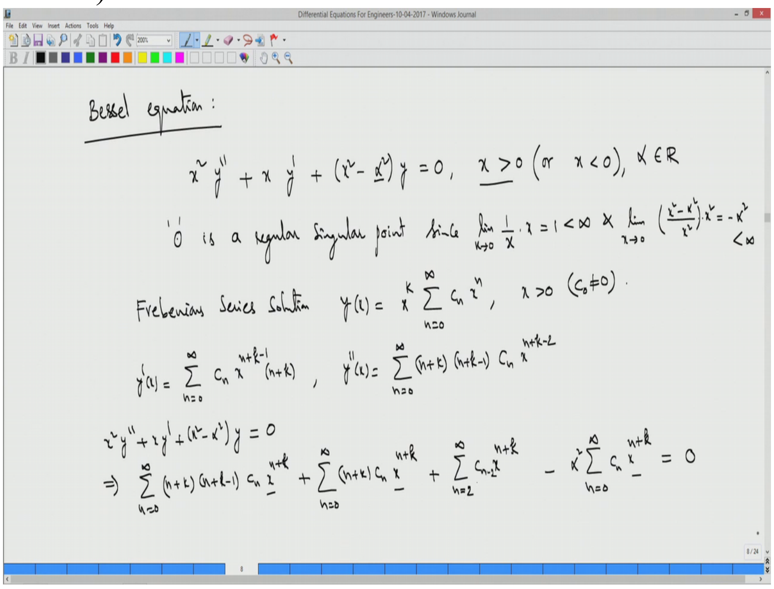
Similarly this will be from n is from zero to infinity, n plus k c n x power n plus k, minus 1 goes because you are multiplying with x, plus x square y you can write it like n is from zero to infinity, c n x power n plus k plus 2. So this, you see this, Ok and then we will see, we will change the index later, so right now you write minus alpha square y. So that is alpha square, write y, c n x power n plus k which is equal to zero.

(Refer Slide Time 08:18)



So you see that all the powers are x power n plus k in these three, except this one, so you change the index as usual, so n is, if you replace n is n minus 2, then this x powers will become x power n plus k. So n equal to n minus 2 make it n plus k, Ok. So n equal to n minus 2. And then when you put n equal to n minus 2, it is running from n is from 2 to infinity,

(Refer Slide Time 08:48)



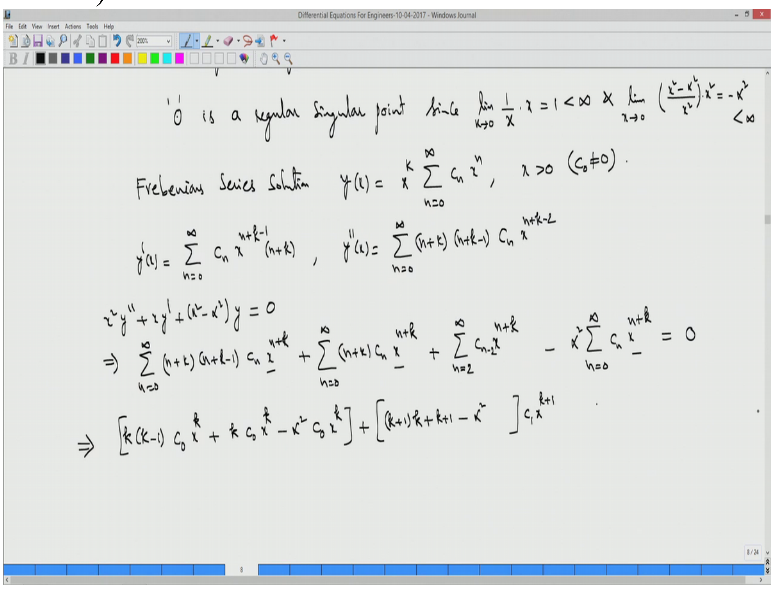
so this is what equation...

Now all x powers are n plus k. So what is the isolated terms, you write. n equal to zero and n equal to 1, you write separately and all the series are running from 2 to infinity that you can write together. So this gives me n equal to zero, you have k into k minus 1 into c zero x power n, x power k Ok plus, n equal to zero so you have k times c zero x power k and then n is equal to zero here so I have minus alpha square c zero x power k,

So this is what you have for n equal to zero. n equal to 1, you have k plus 1 into n equal to 1, so k into k plus 1 into c 1, Ok so you can also, you can take c 1 is always common here in these three series, so you have c 1 you can take it out, c 1 times x power k plus 1. Use n equal to 1.

So this, this is what you have. The first series you have k into k plus 1, next series plus you have n equal to 1, you have k plus 1 and then minus alpha square, simply alpha square. This is what you have, so plus,

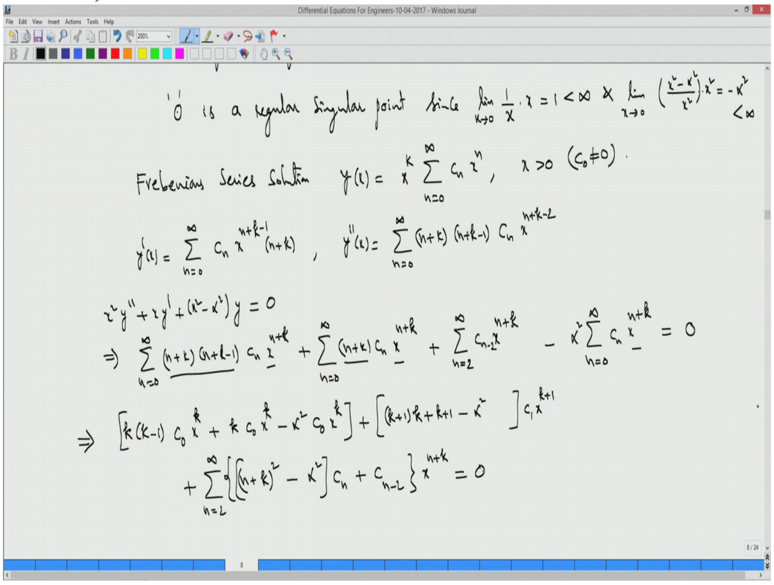
(Refer Slide Time 10:26)



now everything is running from 2 to infinity. So you can combine together.

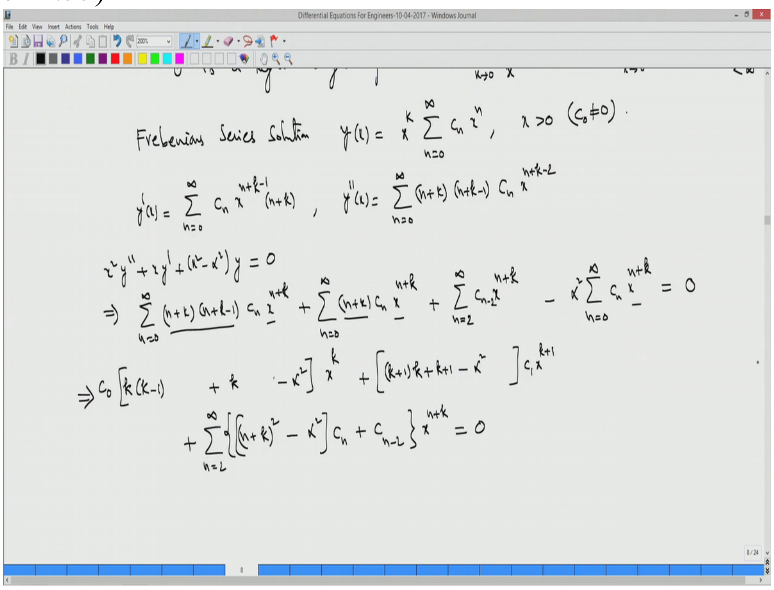
So you have n plus k, so n plus k into n plus k minus 1 and here n plus k, so you are adding them together, so that will give n plus k square. So n plus k is out, common, you have n plus k minus 1 and there is a plus 1, so there is going to be n plus k. So you have together it is a square, and you have here minus alpha square, so this is what with c n. And then, and here simply you have c n minus 2, Ok, so this into x power n plus k, n is running from 2 to infinity. So this is zero. So it is what,

(Refer Slide Time 11:15)



So you see that c naught is common so I can write it outside, c zero, so I remove this c naught out here. And x power k also common, so I can take it out, x power k. So

(Refer Slide Time 11:33)

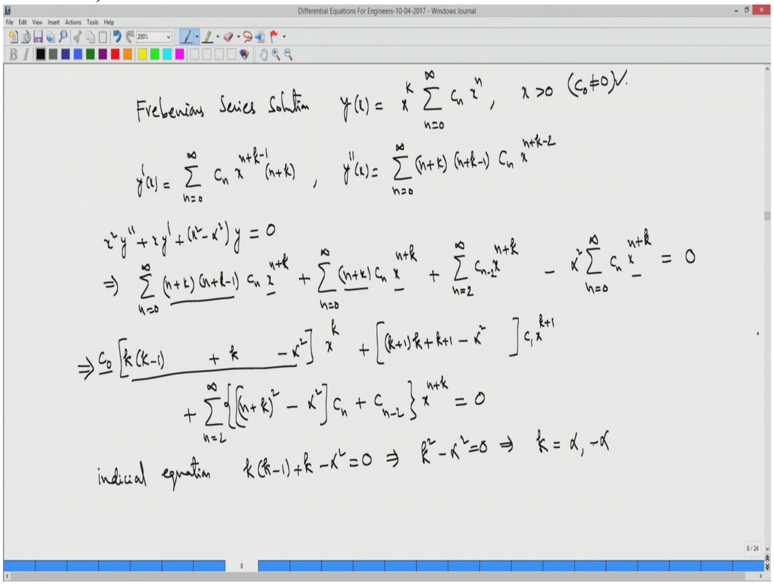


you see this is a power series so you can equate because c zero not zero, that is the assumption, you have chosen that. c zero is non-zero so this has to be zero. That means this has to be zero. This is the indicial equation.

So x power coefficient, x power k coefficient is zero will give me indicial equation. k into k plus 1, k minus 1 plus k minus alpha square is equal to zero. So this is actually equal to k square minus alpha square equal to zero. So this will give me k equal to plus or minus alpha, alpha and minus alpha.

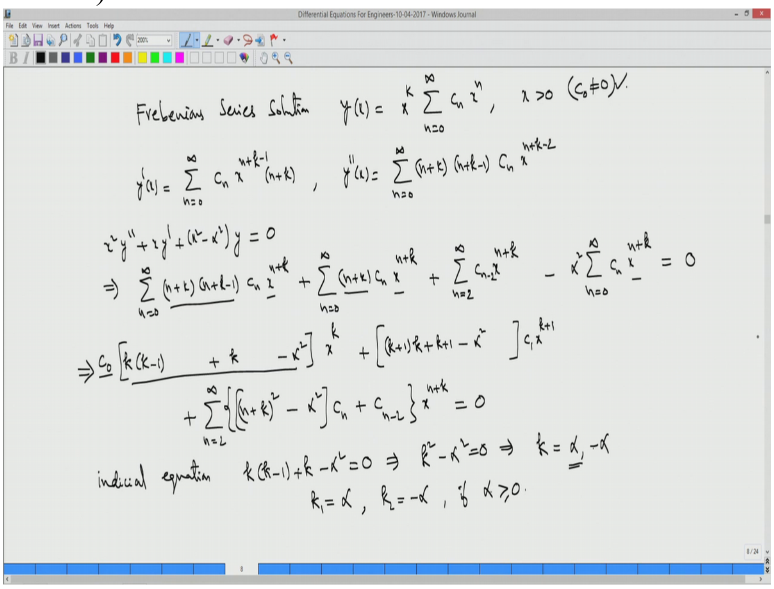
So alpha if you assume

(Refer Slide Time 12:21)



without loss of generality, alpha is positive, greater than or equal to zero. So this root k 1 will be alpha, Ok. So you can fix what is your bigger root, k is k 1, k 1, what is your k 1, bigger root? Call this alpha and k 2 is minus alpha if alpha is greater than or equal to zero.

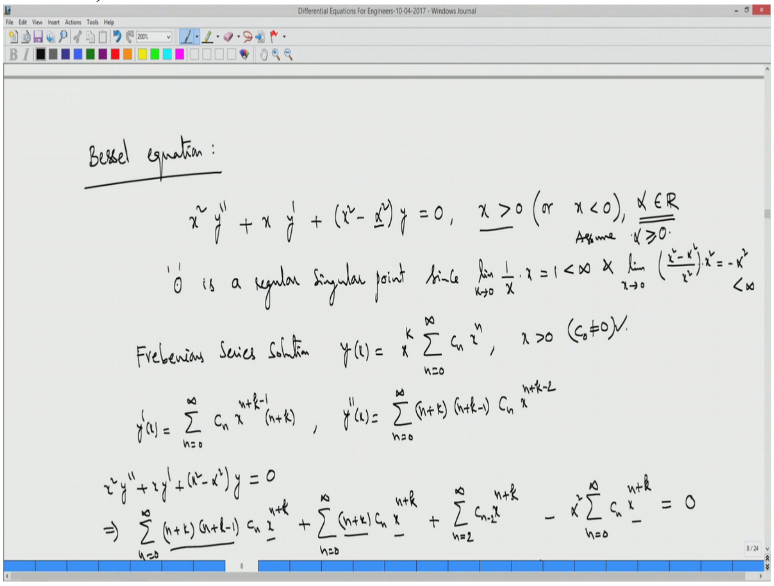
(Refer Slide Time 12:44)



If alpha is negative, less than equal to zero so you can choose k 1 as minus alpha, Ok? Bigger root will be, when alpha is negative, bigger root will be minus alpha, Ok. So assume that alpha is always, so you can say without, alpha is actually full R but without loss of generality now we can choose that this is whether or not equal to zero, Ok.

Assume,

(Refer Slide Time 13:15)

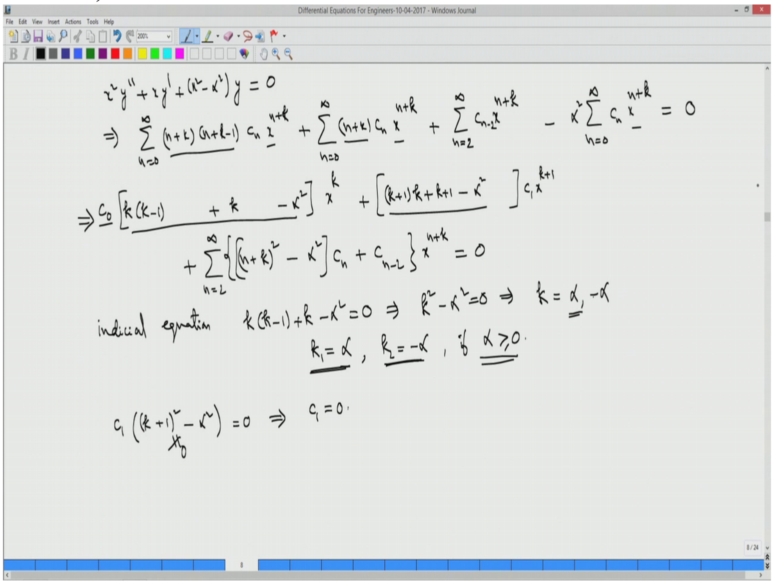


assume that alpha is greater than equal to zero, so even if alpha is negative you can simply change the roles of k 1 and k 2. So bigger root will be minus alpha if alpha is negative. So you have the bigger root, bigger root alpha and this is the smaller root, this one. So what happens?

So because now what happens to the other part? So now if we equate, equate coefficient of x power k plus 1 c 1 into k square, k plus 1 square rather, Ok, k plus 1 square minus alpha square, Ok into c 1 has to be zero. If we equate x power k plus 1 coefficient. So this implies, because k, k is alpha or minus alpha, right. So these are your roots k equal to alpha or minus alpha.

So k plus 1 square will never be alpha square, Ok. So minus alpha plus 1 square, alpha plus 1 square can never be alpha square. So that means this is non-zero. So that means this cannot be zero, so that means c 1 has to be zero. So this is what you found,

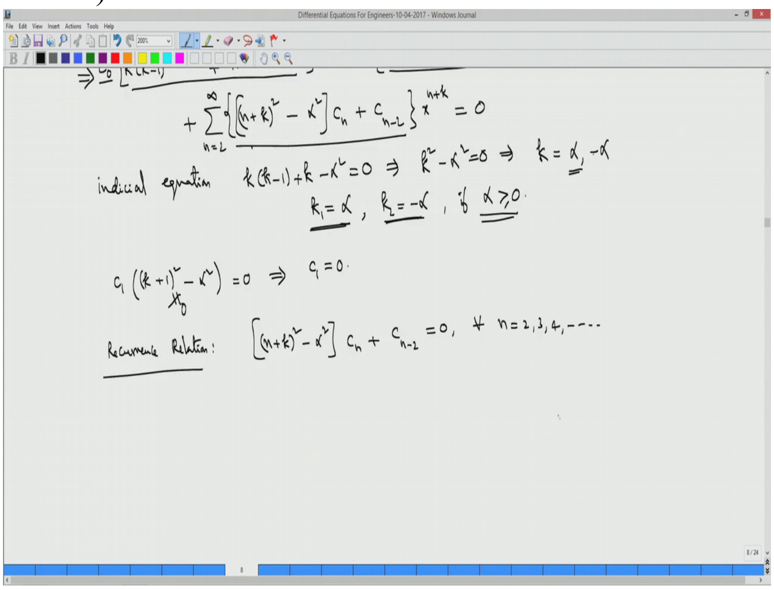
(Refer Slide Time 14:32)



what is the first coefficient c 1.

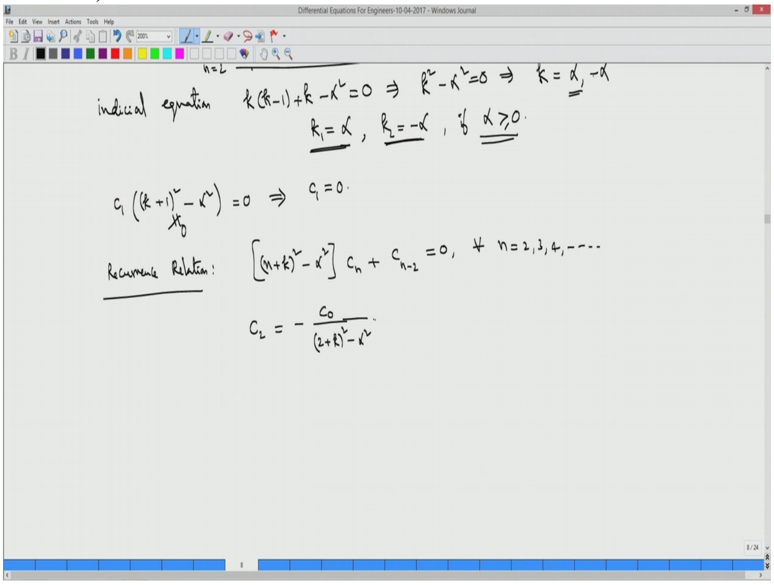
And now equate the general coefficient for n equal to 2, if you make it zero, you get the recurrence relation which is equation, coefficient of x power n plus k, n is from 2 to onwards, zero. So you have n plus k whole square minus alpha square into c n plus c n minus 2 equal to zero for every n, 2, 3 onwards.

(Refer Slide Time 15:10)



So now you can put n equal to 2, so you get c 2. So you calculated c 1. You find c 2. c 2 will be equal to minus c naught divided by 2 plus k, k plus 2 square minus alpha square, Ok.

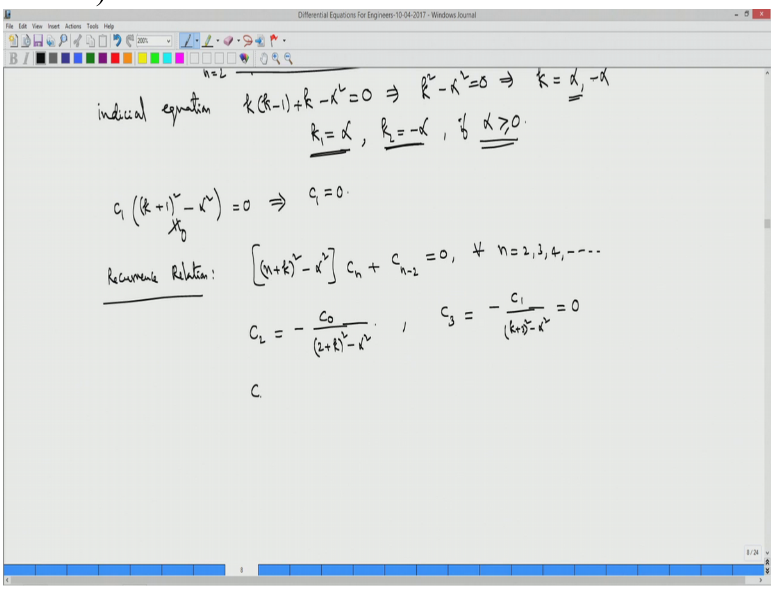
(Refer Slide Time 15:27)



And then what is c 3? What is c 3? If I calculate c 3, minus c 1 divided by, and c 3, c 3 will be k plus 3 whole square minus alpha square. k value, either alpha or minus alpha, so this will never be zero, the denominator will never be zero but I know that c 1 is zero, so this is, that means zero.

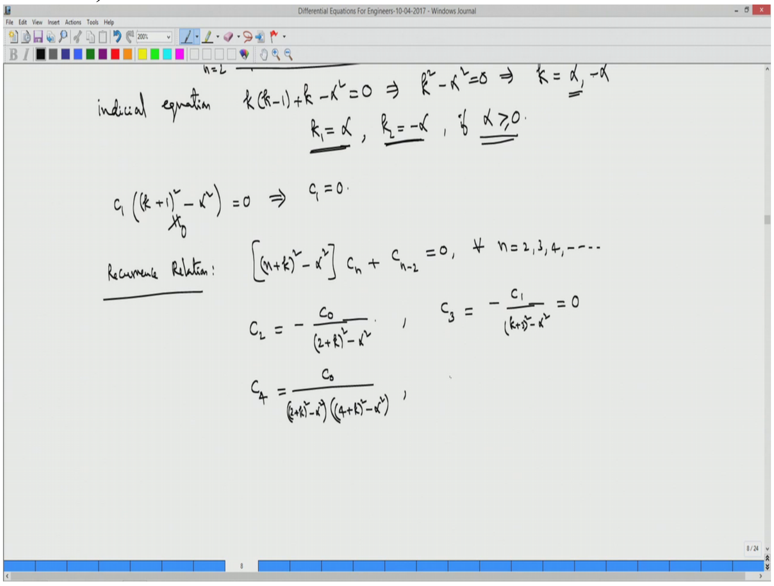
So you get

(Refer Slide Time 15:54)



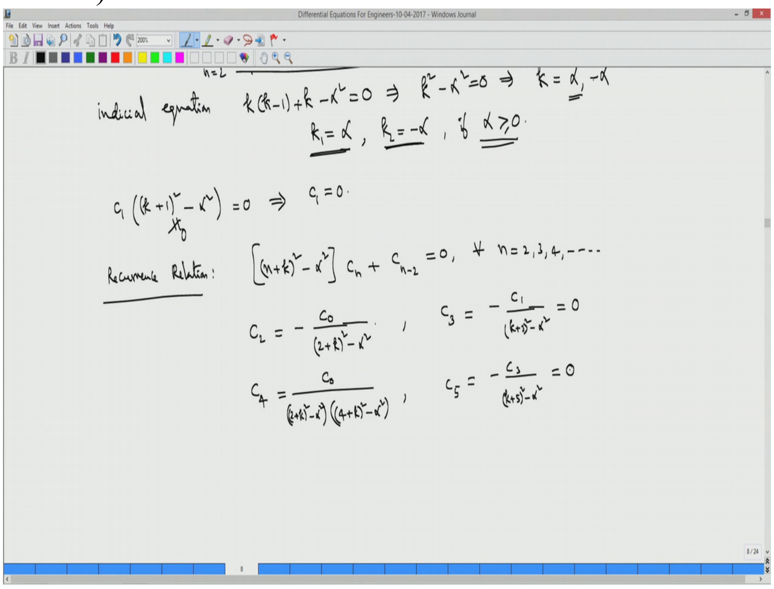
c 4, c 4 will be minus c 2 divided by this one, so you have plus c zero divided by 4 plus k whole square minus alpha square, this is one, and you already for c 2, c zero divided by this, this you have 2 plus k whole square minus alpha square. So this is what you have. And then

(Refer Slide Time 16:19)



c 5 which is again, minus c 3 divided by denominator which is k plus 5 whole square minus alpha square which is non-zero, but you already know that c 3 is zero, so this is zero.

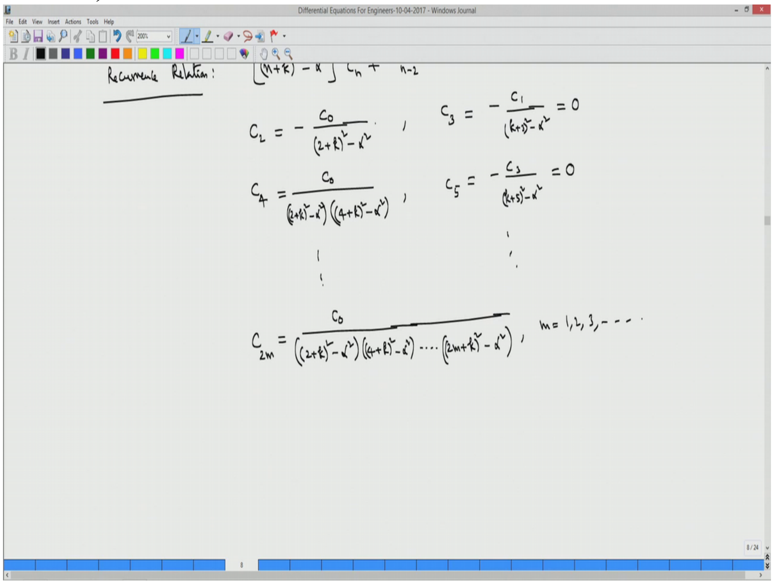
(Refer Slide Time 16:32)



So like that you can go on, Ok. All the odd coefficients will be zero, all the even coefficients you can write generally, you can expect something like this, c 2 m which is c 0 divided by, and you can write, now you can fix it, the bigger root. If you fix your bigger root, where k is, that where k is. So you can write it if you want, so k is actually, or you can simply write c zero divided by, so you have 2 plus k whole square, so you can fix it later also, minus alpha square 4 plus k whole square minus alpha square up to, what is the next part, so you have 2 m plus k whole square minus alpha square. This is what you get.

So whenever you have even coefficients, 2 m, m is from 0,1, 2 onwards. m is actually from 1 onwards, so 2 onwards. Ok that means m is from 1, 2, 3 and so on. That means c 2, c 4

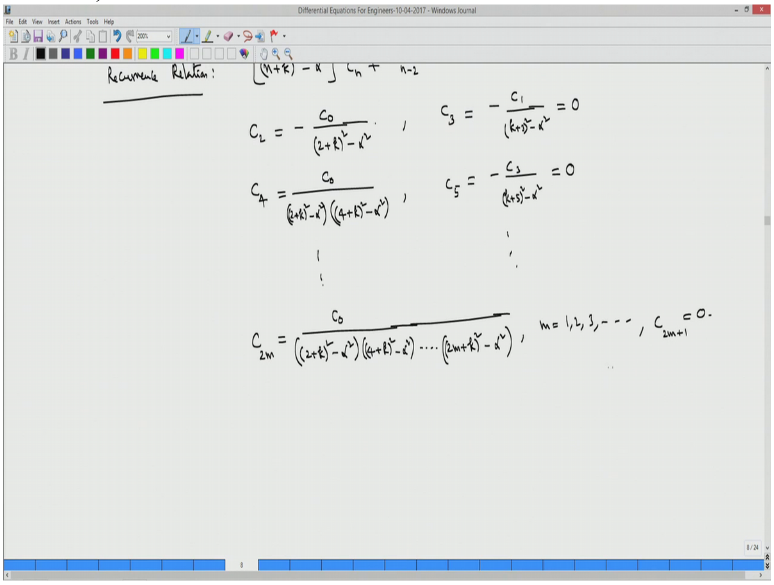
(Refer Slide Time 17:48)



and so on. So c naught, you don't know still arbitrary. So c naught you cannot calculate. So what you get if you put m equal to, that is actually valid. So it is only from m is equal to 1 to infinity.

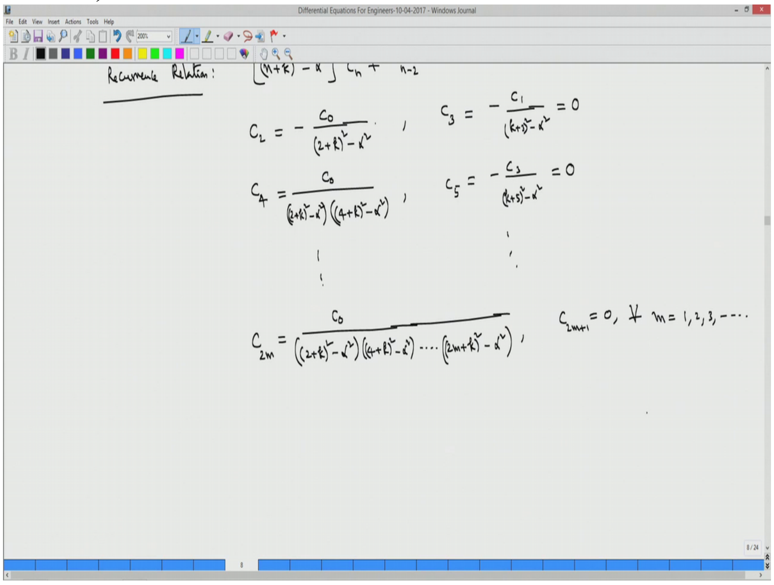
So if we put m equal to zero, you have, that is like you have c c zero by zero, so that is not defined. So m is from 1, so that is where you know, c 2, c 4 you have these and all others c 2 m plus 1 will be zero.

(Refer Slide Time 18:16)



Ok so you can write here. So if you want you can write c 2 m plus 1 is zero for every m equal to 1,2,3, and so on,

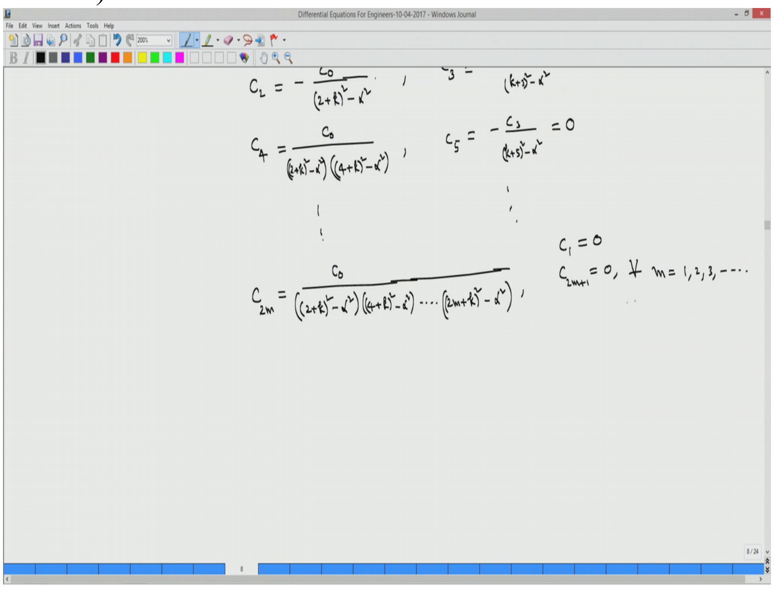
(Refer Slide Time 18:31)



Ok with c 1 zero.

So that will not have zero

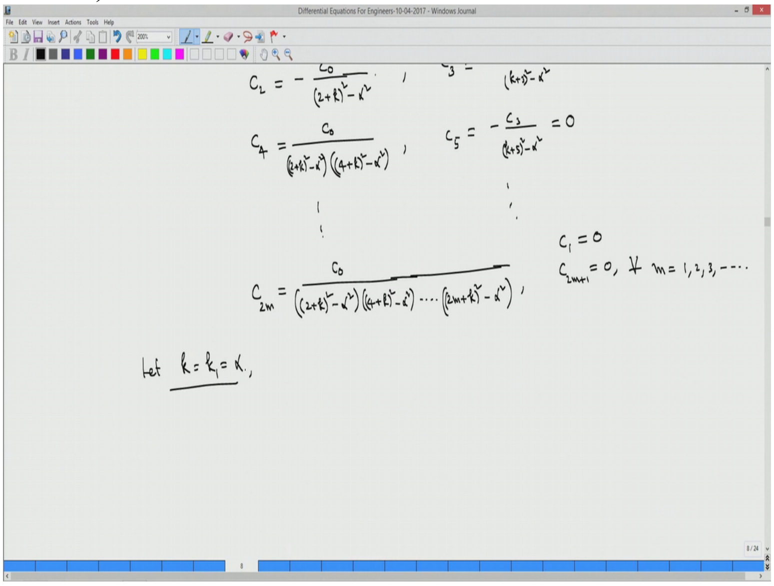
(Refer Slide Time 18:39)



m equal to 1, c 3 onwards. c 3 onwards you wrote, Ok. So c naught is arbitrary. So this is what you have in general. So I didn't fix my k. k is actually either alpha or minus alpha. I know that whatever may be the case, whatever may be the value of alpha, always bigger root, when alpha is greater than equal to zero, bigger root is always k 1. So put k equal to k 1 which is alpha so you can get all your coefficients in terms of c naught, Ok.

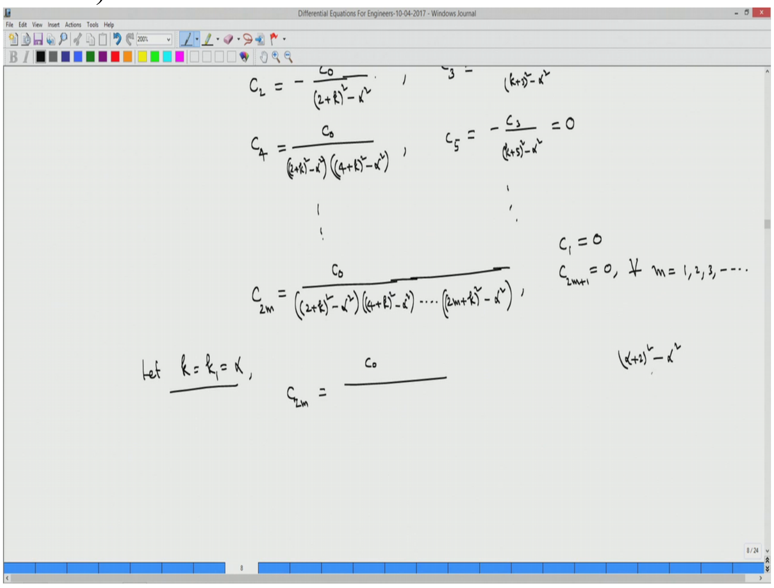
So let k equal to k 1 which is alpha which is positive, Ok. If you put this,

(Refer Slide Time 19:22)



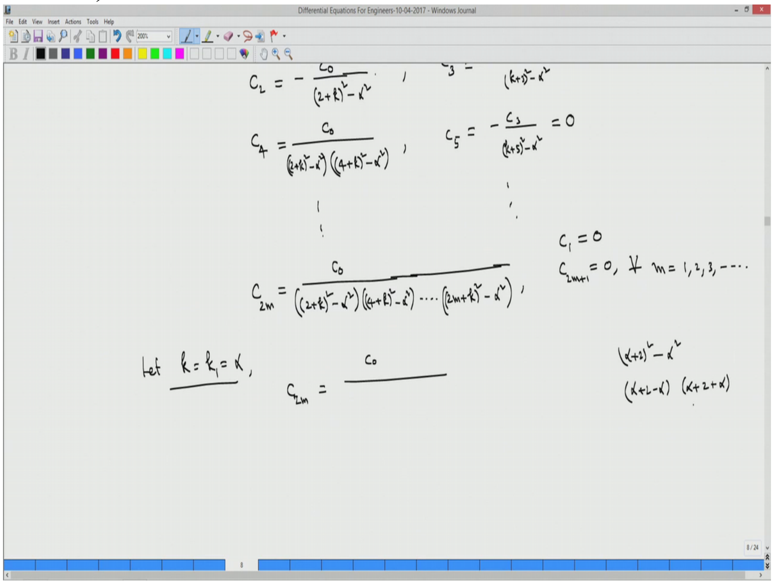
what I get is c 2 n equal to c zero divided by, k equal to alpha, so you have alpha plus 2, so what is the first term, alpha plus 2 whole square minus alpha, alpha plus 2 whole square minus alpha square, that is

(Refer Slide Time 19:39)



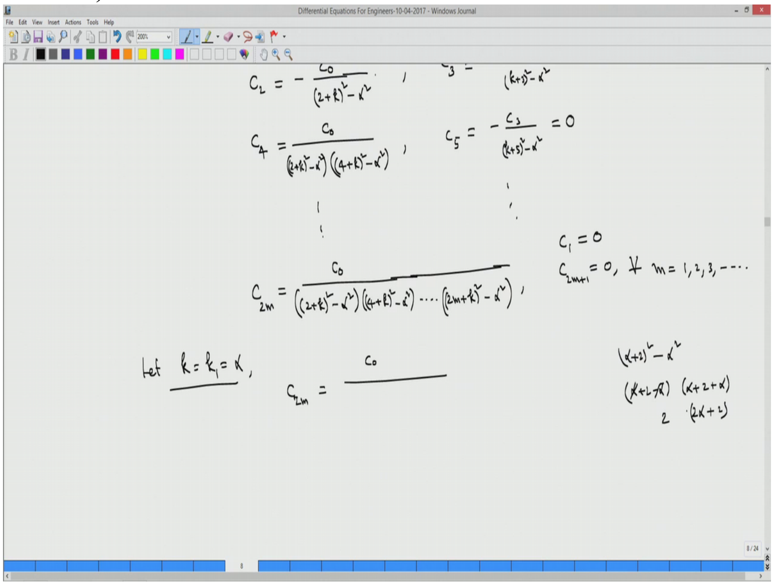
actually alpha plus 2 minus alpha into alpha plus 2 plus alpha.

(Refer Slide Time 19:46)



So I have 2 alpha plus 2. Here alpha alpha goes, 2,

(Refer Slide Time 19:53)

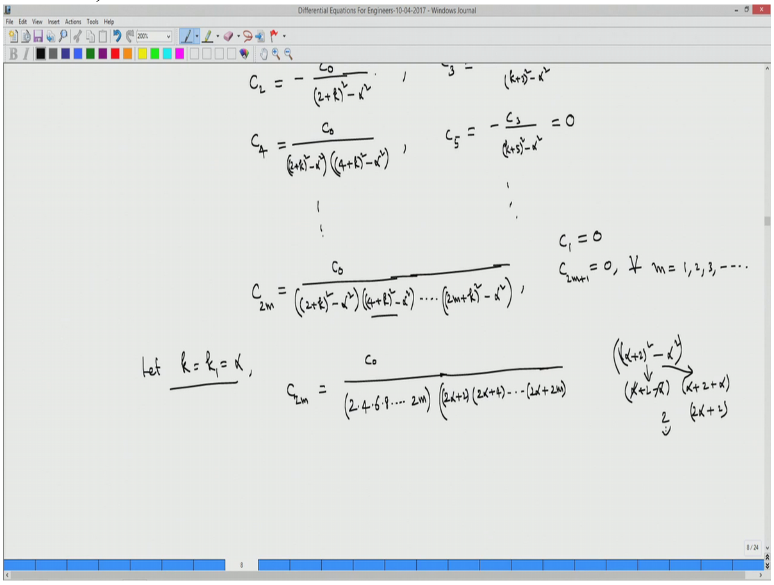


this is what you will get. Ok.

So you get 2, similarly here 4, and then 6, 8 up to 2 n, Ok. So this is what you get from first term. Each of these if you expand like this, Ok so this if you write, so 2, next here you get 4, like that you get it here 2 m.

What is other part? Other part will be 2 alpha plus 2 and here 4 plus alpha plus also 4, so 2 alpha plus 4 equal up to 2 alpha plus 2 m. This is what you have as your

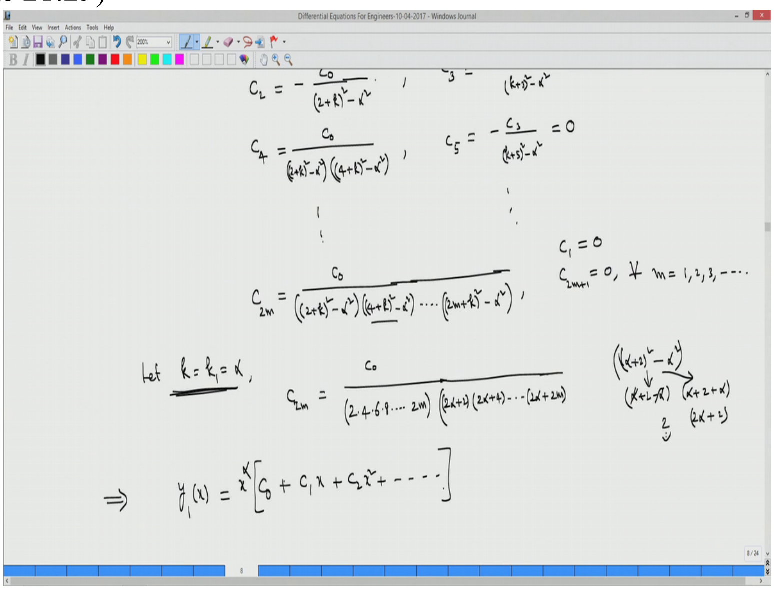
(Refer Slide Time 20:51)



c zero. So this implies I always have, for bigger root I always have a solution. That is the power series solution. Without (()) I do not any problem the method of Frobenius for the bigger root, soyou will always get all the coefficients which I got, only I could get only even coefficients.

So what is the solution y 1? c zero into x power k, k is alpha so you have x power alpha, x power k is alpha and the series is c 0 plus c 1 x plus c 2 x square this is what if you expand. This is the

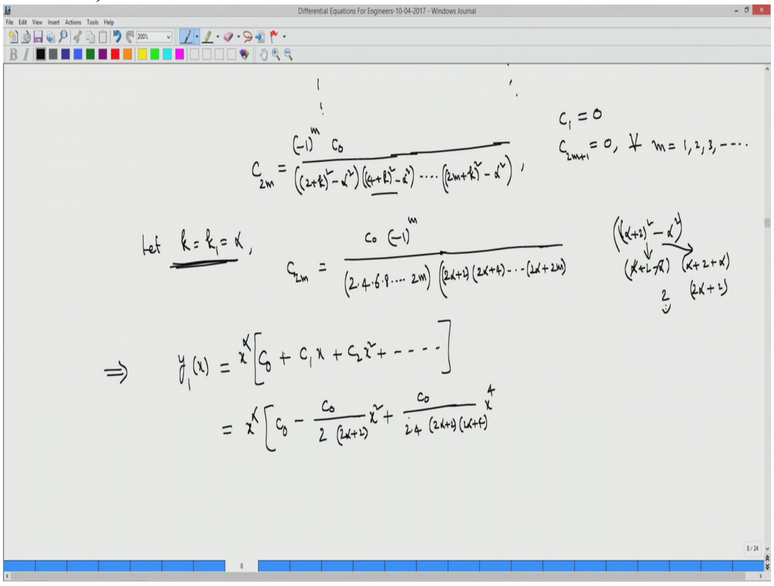
(Refer Slide Time 21:29)



series you look for, right? So this is equal to x 4 alpha, c zero I could not find, c 1 is zero, c 2 is, what is c 2? c 2 is minus c naught by, so minus c naught by what you get, if you do this, I will put k is equal to alpha so you have 2 into 2 alpha plus 2. So this is your x square.

And next term will be c 4. c 4 when I do this is plus, Ok, so now it will be plus, so you have c 0 divided by 2 4 2 alpha plus 2 and then 2 alpha plus 4 into x power 4. So because of that alternative positive and negative sign you will see you have that that power n. So that I missed, so you have, actually what you have is minus 1 power n. So you have minus here, plus here when m equal to

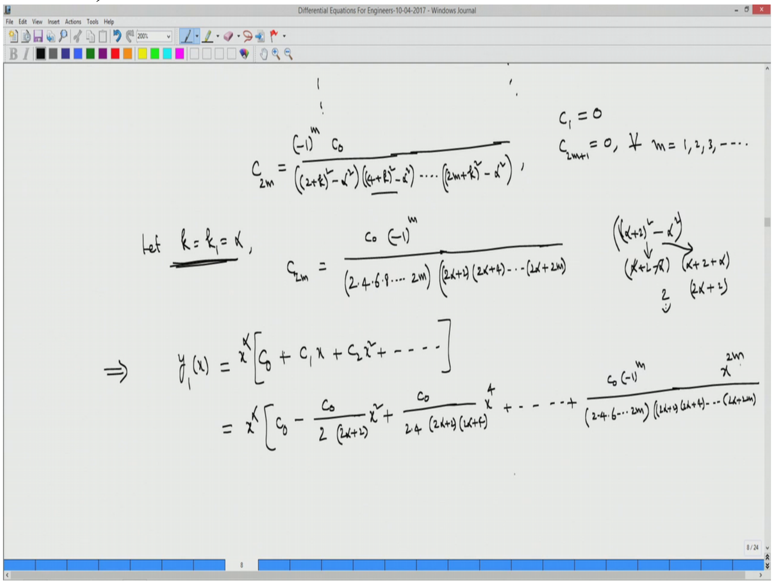
(Refer Slide Time 22:44)



1, it is minus, m equal to 2 you have plus and so on.

So you can have the mth term, so you can write even term which is c zero minus 1 power m divide by whatever you have here 4 2 4 6 up to 2 m, is 1 and 2 alpha plus 2, 2 alpha plus 4, 2 alpha plus 2 m. So this is your nth term. Once you have nth term you have

(Refer Slide Time 23:21)

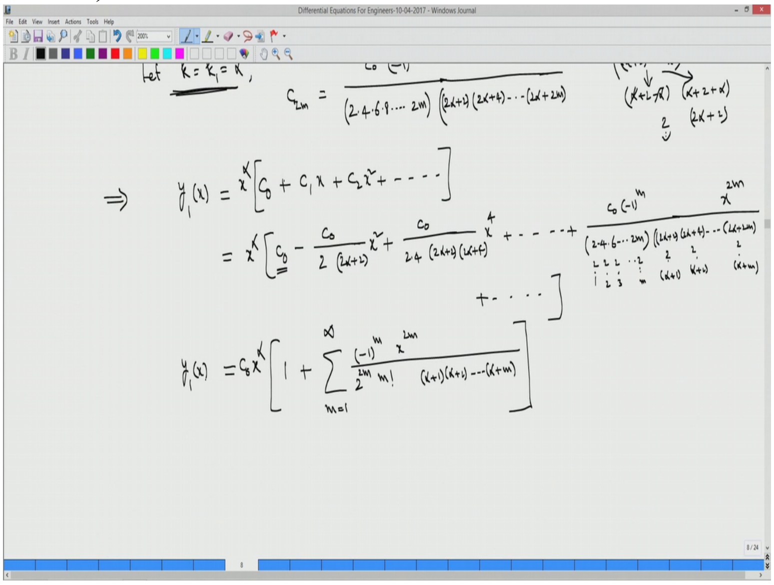


x 4 2 m and so on. So this is what is your series, solution. So this is actually equal to x power alpha, c zero you can take it out and you can sum it up now.

If you sum it up now, so what you have, m is running from, if m is running from 1, you write it as it is. So you have 1, Ok and then plus sigma, this term I am not writing, I am not putting in the sum. So we have m is from 1 to infinity. So all, for m is from 1 to infinity I have my c 2 m, Ok, 1 to m, you can simply, now c naught you took it out and what you have is minus 1 power m x power 2 m divided by, so you can write this, 2 power so this will be 2, 2 here, 2 here, Ok so what do you have, 2 here, 2 here up to 2 you remove. What is left, 2 I can write, 2 into 1, 4 you can write 2 into 2, 6 I can write 2 into 3 and so on, n. So these are the product of all these. So I have m 2 ns, so you have 2 power m and then I have m factorial, Ok and what you have here is again, you have 2 here, 2 take it out like that. What is left here, 2 into alpha plus 1, alpha plus 2, into alpha plus n.

So that also if you add, you have 2 power m into alpha plus 1, alpha plus 2 and so on up to alpha plus m, Ok. So this is what you have, so put it together, 2 power m 2 times, I have 2 power 2 m. So this is what is your series solution, first non-zero solution,

(Refer Slide Time 25:31)



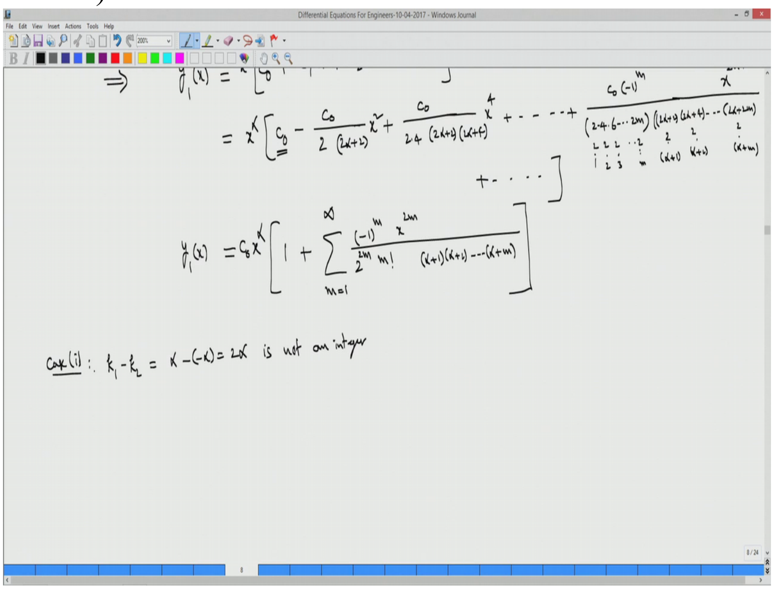
Ok for the bigger root in the Frobenius method, Ok.

Now I have not chosen any, I have not fixed any alpha so alpha is, a bigger root is always, when alpha is positive, bigger root is always alpha. So I have this, one series solution I will always get. That is your y 1.

Now you will see the different cases. Different cases, so what happens? The simplest case 1, what is the indicial roots, the roots are k 1 minus k 2 which is alpha minus minus alpha, that is 2 alpha. If 2 alpha is not an integer, not an integer then this is what is the case 1.

Case 1

(Refer Slide Time 26:24)

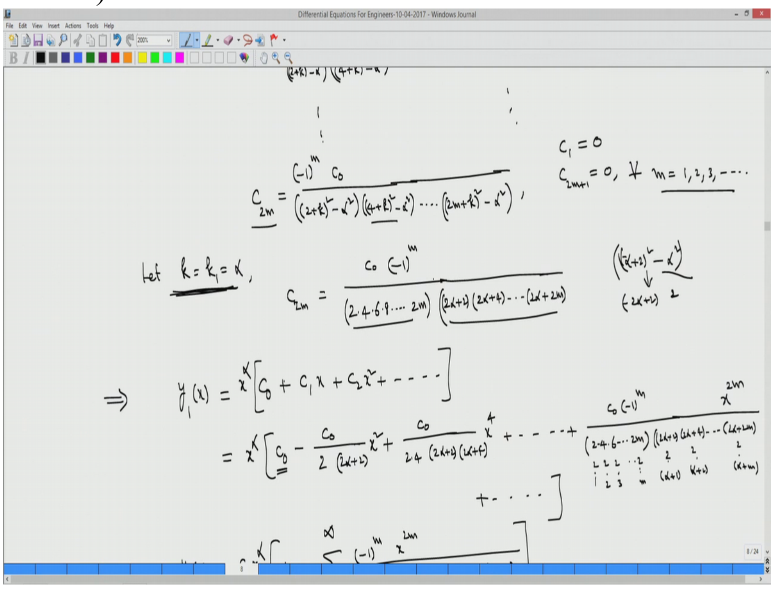


in the Frobenius method, in this case actually even if you do for k equal to k 2, Ok, so whatever you have done here, I put k equal to k 1, I got one solution. When the difference of the roots is non-integer, you see that when you put the other root which is k equal to minus alpha, you get the same solution. So we canget all the coefficients.

What happens when I put instead of k if I put k equal to minus alpha, where you have k you have minus alpha. So, so instead of alpha plus 2, you have minus alpha plus 2, minus alpha square, so that becomes, so instead of...you will only have to interchange, Ok. So what happens the product now? So minus alpha 2 plus 2 minus alpha that is 2 minus 2 alpha or minus 2 alpha plus 2, Ok and then other one is minus alpha plus 2, plus alpha will get 2. So you will only change the, you will interchange. Instead of you are writing 2 into 2 alpha plus 2, you will get minus 2 alpha plus 2 into 2, Ok?

So this whole thing will be as it is, with alpha is minus alpha and this will be as it is,

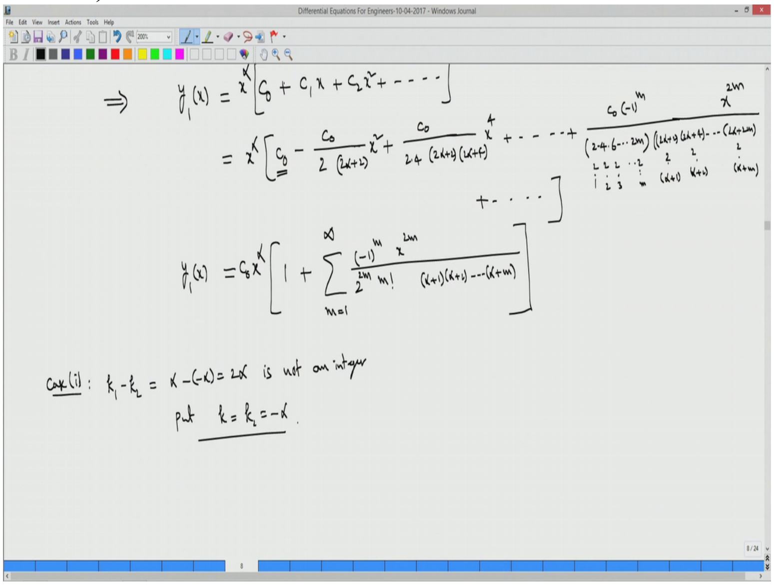
(Refer Slide Time 27:40)



Ok. So you will get the same coefficients but alpha is minus alpha, Ok. So that is what is the, that is what you will get if the difference is non-integer. Non-integer you should not have problem, Ok because the denominator will never be zero if the difference is non-zero, non-integer. You see that I have got 2 alpha plus 2, right? The denominator I have minus 2 alpha plus 2, minus 2 alpha plus 4, only if 2 alpha equal to 2 that is where is the problem, Ok. 2 alpha is equal to 4, that is where the problem, because alpha, 2 alpha it is not an integer, it will never be 2, 4 and so on. So I don't have issue.

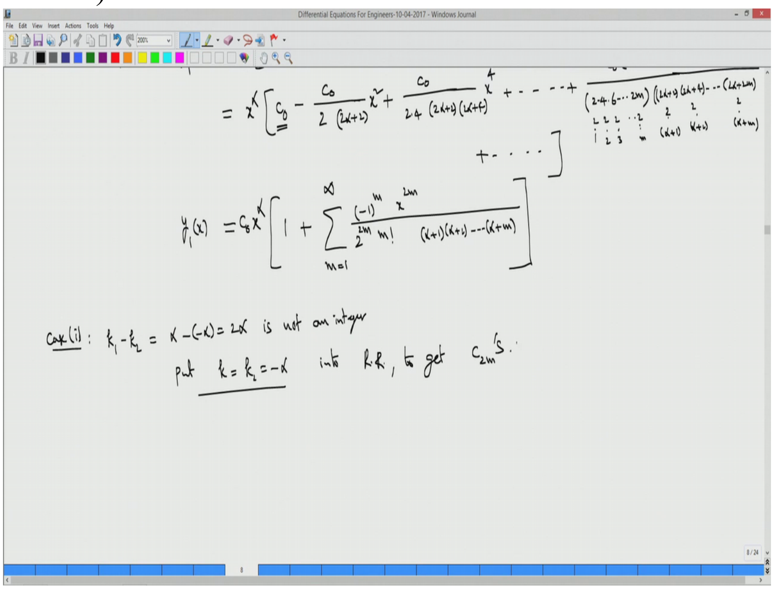
You get simply by putting, put k equal to k 2 which is minus alpha will give me in to the,

(Refer Slide Time 28:36)



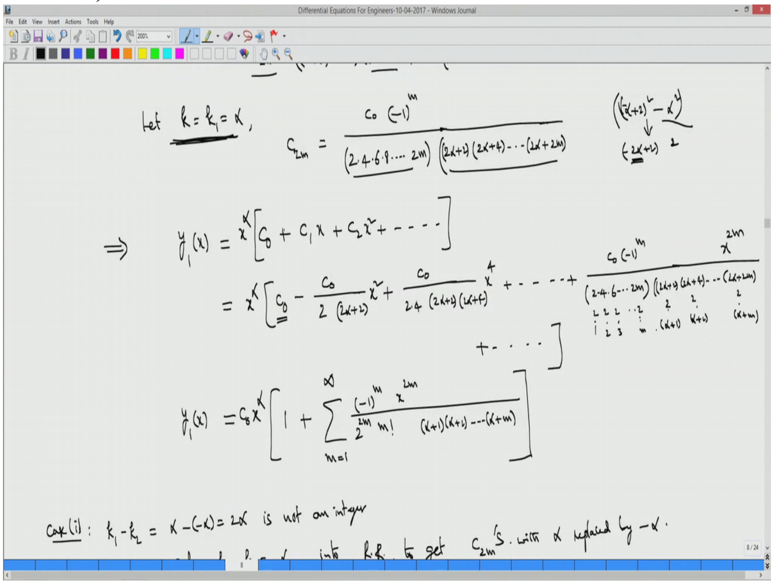
into where, into the recurrence relations, into the recurrence relation, into the recurrence relation, R R I may write it as R R, recurrence relation to get c 2 ms, c 2 ms Ok as it is, c 2 ms, you get the same c 2 ms

(Refer Slide Time 29:02)



with alpha replaced by minus alpha. This is what you will get, Ok,

(Refer Slide Time 29:10)

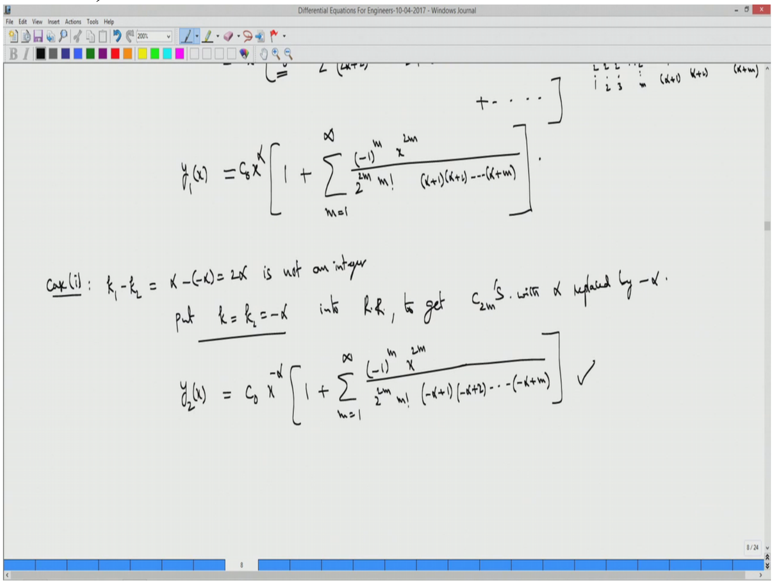


you can check it.

From this step onwards you try to put k equal to minus alpha. You will see that this is what you will get. Instead of alpha you have minus alpha but you know that 2 alpha is not an integer, this denominator never be zero so I could get what is my c 2ms. And all the odd coefficients are any way zero because, whether you put alpha equal to minus alpha, doesn't matter. When k is equal to minus alpha, still this is not zero. So because c 1 already we found zero, and you see that all the odd coefficients are zero.

So you have a second solution very easily as c zero x power minus alpha and here 1 plus sigma, m is from 1 to infinity, minus 1 power m x power 2 m divided by 2 power 2 m m factorial. Here we have minus alpha plus 1, minus alpha plus 2 and so on, minus alpha plus m. This is what the second solution, Ok.

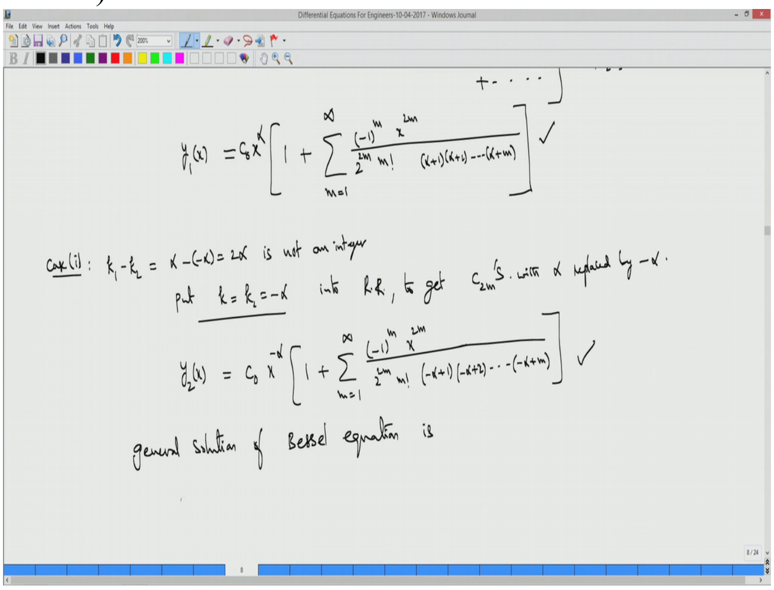
(Refer Slide Time 30:13)



So this is your first solution, this is your second solution in the case 1, when difference of, difference between these two roots, that is 2 alpha is not an integer case. Ok.

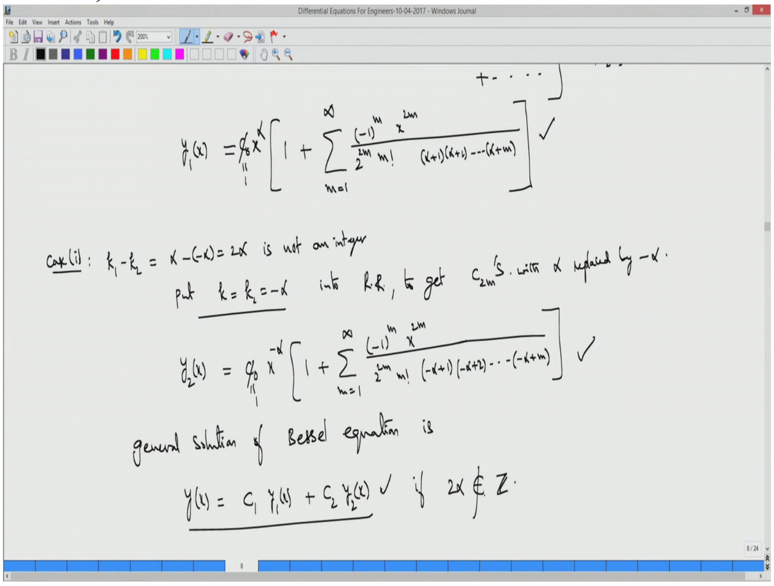
So we can write a general solution, general solution of Bessel equation, of Bessel equation is,

(Refer Slide Time 30:36)



y x if we can write y x, some arbitrary constant c 1 times Ok, c 1 times, c zero you can take it as 1 here Ok, c zero you can take it as 1, c zero you can take it as 1, so that you have 2 non-zero solutions. So c 1 is arbitrary constant. So you take with y 1and then c 2 is another arbitrary constant, you take another linearly independent solution. So this is your general solution of Bessel equation if 2 alpha does not belong to Z, non-zero integer, Ok, doesn't belong to.

(Refer Slide Time 31:15)



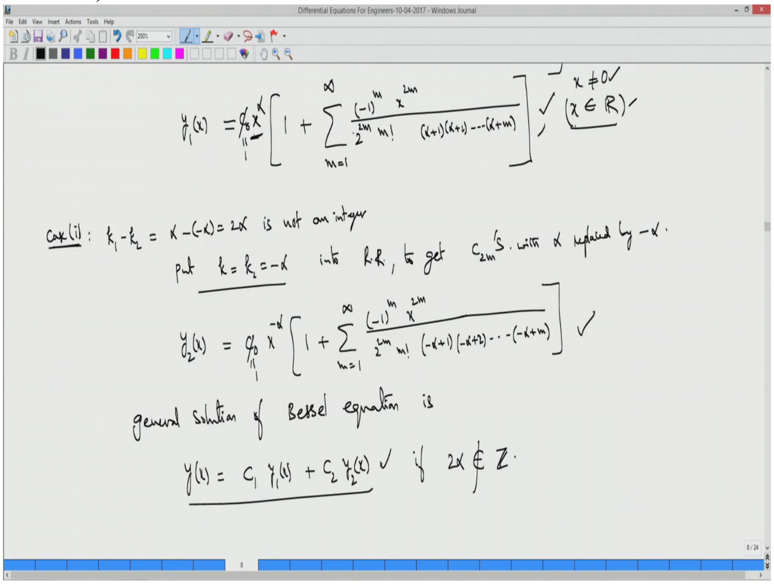
This is what is the case, Ok. As a straight forward case, case 1.

And it is also, so from this you can see that this is, this is x power alpha, alpha is positive. At zero, zero is ordinary; zero is a singular point but the solution as such it is defined even at x equal to zero, Ok. Actually it is defined x positive, Ok as a solution and as a function here, this solution, just like Legendre equation, Legendre function, Legendre solutions, Legendre polynomials though they are solutions

(Refer Slide Time 32:00)

of a differential equation in the open interval minus 1 to 1 but the polynomials are defined, so

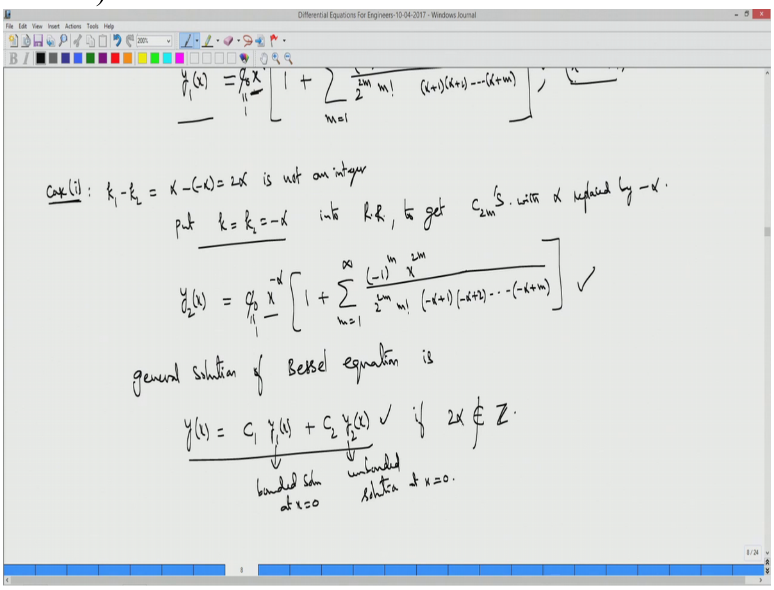
(Refer Slide Time 32:06)



y 1 of zero is actually x power alpha, Ok.

So you see that as a solution which is bounded, that is what you are seeing. But here, alpha is positive, x power minus alpha, then you put x equal to zero, so you have unbounded solution, Ok. So y 2 is unbounded. So this is a bounded solution at x equal to zero, unbounded solution, at x equal to zero, Ok.

(Refer Slide Time 32:41)



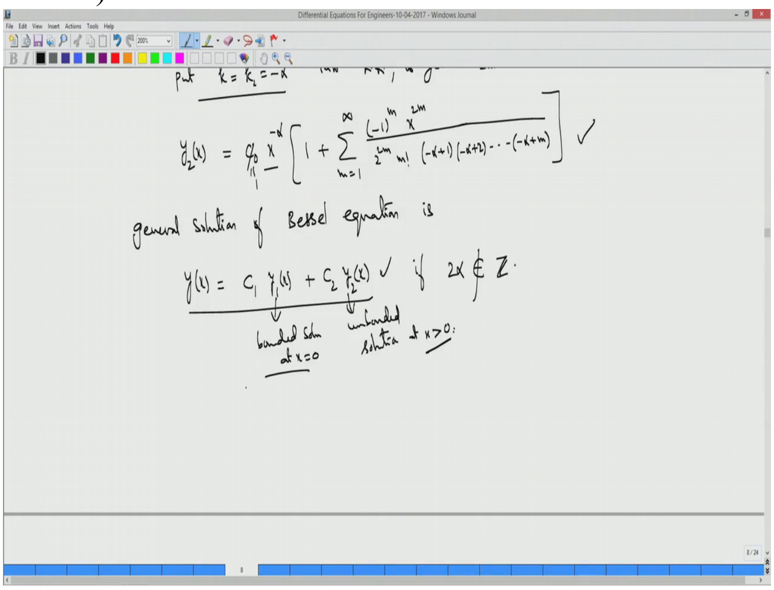
In the neighborhood also, in the unbounded, in the neighborhood of x positive because it is not defined at x equal to zero, Ok. When you closely, very, very small values of x is always, is going to be big value, Ok. So it is unbounded in the neighborhood of zero.

So you can say that this is also, as x varies close to zero, still bounded. So, Ok. As a limit it exists. As limit x close to zero, y 1 x is bounded, y 2 x, limit x close to zero is unbounded. This is what we see as a simple case

(Refer Slide Time 33:16)

what we learnt in the Frobenius method when you apply the case 1. So you get the nicely, truly linear independent solutions. So take linear combination of it, you get a general solution of the Bessel equation.

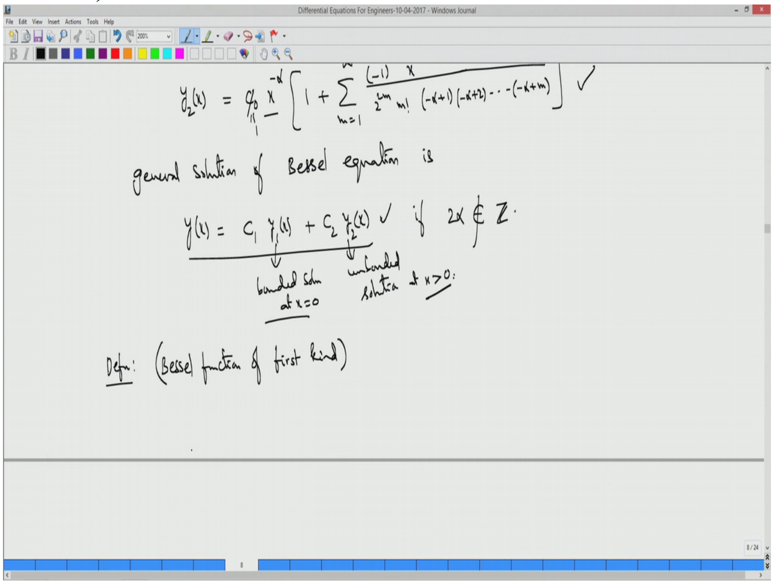
(Refer Slide Time 33:29)



And, so it is bounded solutions like in the Legendre equation, in the Bessel equation also this bounded solution y 1, by customerily by choosing this c naught, so you have this, so what we have chosen is c naught equal to 1, this is what is my y 1 as a non-zero solution but if you fix some conventionally some c zero so that, so that this whole thing is simplified, Ok that is how I can define as a Bessel function, Ok.

So I write what is the definition of Bessel function. Bessel function, Bessel function of first kind, so you get a solution as a Frobenius first solution for a bigger root, this is always called first kind Bessel functions.

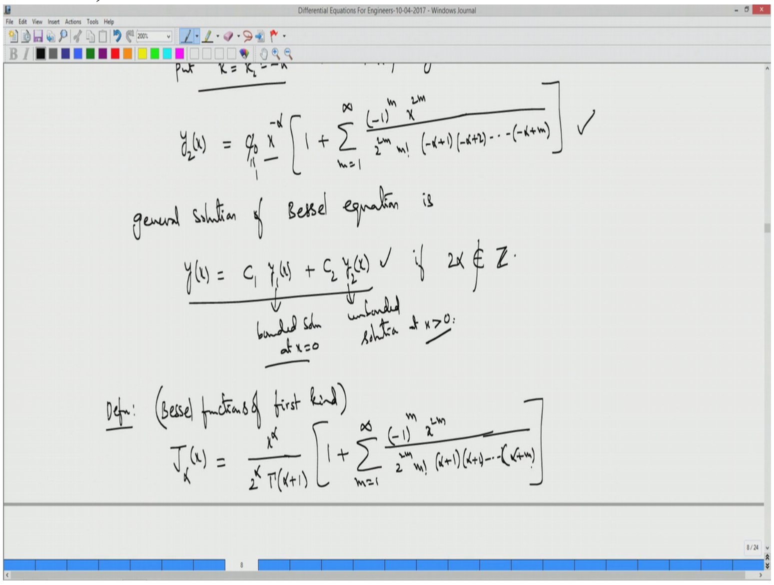
(Refer Slide Time 34:16)



Bessel functions of first kind we call this J alpha of x by choosing, simply take c zero as some fixed number. So conventionally we choose, c zero if we choose, 1 over 2 power alpha, gamma, gamma function this is a gamma function, we will see what exactly it is, gamma into alpha plus 1, so this you choose into x power minus alpha, x power alpha. So since, instead of c zero, you choose, instead of 1, you choose this number. Any constant multiple of y 1 is also a solution. So c zero if you choose like this, and then it will become x power alpha here, so you have x power alpha and you have sigma, n is from, rather, you have 1 plus sigma, m is from 1 to infinity minus 1 power m x power 2 m.

So whatever you have you write, x to power 2 m divided by 2 power 2 m into m factorial alpha plus 1, alpha plus 2 right, alpha plus 2, alpha plus 1, alpha plus 2, up to alpha plus m. So this is what, this is called

(Refer Slide Time 35:35)



Bessel function of first kind. This is what you get, so conventionally you have chosen this number, Ok for c naught. So that number will give me, so this is how we define. So this is the definition. So this is what we define. So we will see, we will again simplify, we will simplify, in the next video we will simplify exactly

(Refer Slide Time 35:55)

using, why we use this gamma function. Using the gamma function, we can simplify this nicely and then the definition of the Bessel function will be much simpler. So we will see that in the next video.