Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky
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Lecture No 19
The Invariance Under Inversion of the Spherical Metric on the Extended Complex Plane

Alright so let us continue with whatever we were doing, see our aim as a told you is to prove the Picard theorem for which we need to look at families of Meromorphic functions, so we need to do topology on a space of Meromorphic functions okay and so I told you in the previous lectures that you know if you want to do topology on a space of functions and you go and try to draw inspiration from usual topology than for example if you look at real valued or complex valued bounded functions continuous functions on a topological space then the topology is done by looking at uniform convergence okay.

So we define convergence in the space by uniform convergence but now if you try to it inspiration from this and come to complex analysis and then you are working of course with holomorphic functions that is analytic functions and then otherwise we want to be a little bit more general and we want to work with Meromorphic functions and Meromorphic functions you know are analytic or holomorphic functions accept for an isolated set of points where they have only poles singularities okay, so you want to work with such functions at the point is that even if you take analytic functions okay and you look at sequence of analytic functions converging okay.

Normally what happens is that you do not get uniform convergence okay but you get only normal convergence, so when you come from a topology to complex analysis you must remember that you should not work with uniform convergence but you should work with uniform convergence restricted to compact sets and that is called normal convergence okay that is the 1st point then the 2nd point is this pathology that you know you can have a domain in which you can have a decent sequence of analytic functions which goes to infinity everywhere okay. Now so for example I was discussing about this domain which is the exterior of the unit disk and I said you take the you know the sequence of functions given by Z power N, so the 1st function is Z the 2nd function is Z square and so on and then this sequence you know it converges point wise to the function which is infinity okay.

So and the point is that is convergence and normally if you are if you are only thinking of a 1st course in complex analysis we will say that this sequence diverges okay because if you take any complex number Z with modulus greater than 1 and you look at the sequence Z power n that is not going to go to limit it is going to go to infinity okay you will simply say this is the sequence diverges but the fact is that we want to think of this as the convergence sequence and there are 2 reasons for this the 1st thing is that it converges to the constant function which is infinity at every point outside the unit disk and we have to allow the value infinity because we want to think of Meromorphic options because say if you are thinking of Meromorphic functions then you can define the value at a pole to be infinity because that is the limit that you get as you approach the poles okay.

So we have to allow the value infinity and once you allow the value infinity then you must also allow the constant function which is infinity and if you go by that reasoning then the sequence Z power n actually tends to the constant function infinity and that is the reason that we say that Z power n tends to the constant function infinity it tends to the constant function infinity outside the unit disk okay and in the exterior of the unit disk alright and of course unit circle is not include we are looking at the exterior of the unit circle and then the point is that if you want to talk about this convergence in terms of a metric okay then you will have to worry about giving a metric on the extended complex plane and I told you that that metric that you can use is the spherical metric there is a so-called spherical metric and what is the spherical metric?

You take 2 points in the extended complex plane take their images in on the Riemann sphere under the stereographic projection and then use spherical metric on the Riemann sphere, the spherical metric on the Riemann sphere is just it is the length of the minor arc of the big circle passing through those 2 points on the Riemann sphere okay, so in this way you get spherical metric on the extended complex plane and then with respect to this spherical metric I can actually defined the distance between any 2 points in the extended plane, so for example I can have one point which is a point in the complex plane I have I can have the other point to be a point at infinity and I can still have this notion of distance okay and it is with respect to this metric the spherical metric at the convergence is actually normal okay.

So here this is the point that we have to understand if you take the sequence of function at power and in the domain mod Z greater than 1 then the sequence converges normally to the constant function infinity okay and this is the viewpoint it is very important okay and what I

am trying to tell you today is that you know this is the only pathology that can occur. If you take a domain in the complex plane and suppose you have sequence of analytic functions which converge normally to a function on the domain okay and assume that you allow this function the limit function to take the value infinity suppose you assume you allow that then is very beautiful that either the limit function is completely analytic or it is completely infinity you do not get anything in between.

So you know you do not get this kind of horrible pathology that you have sequence of analytic function and they tend to for example a Meromorphic function so that will not happen, you cannot have a sequence of analytic function on a domain they converge normally okay that is they converge uniformly on compact subsets of the domain and in the limit you get low behold you suddenly get a Meromorphic function that will not happen, so this is very important thing it tells you that things are going on well as per usual intuition something does not you must always see whenever you work with some extraordinary cases you should always be careful, so for example when I am allowing a function to take the value infinity the function can very well be a Meromorphic function because Meromorphic function takes the value infinity at the poles.

So it can happen that I have a sequence of analytic functions on a domain it is converging normally what limit function, now if I allow the limit function take the value infinity okay and the limit function could very well be even Meromorphic okay but the fact is that does not happen, what happens is either the limit function is analytic that is the good thing and in the bad case the worst thing happens the limit function is always infinity okay and the reason or this is 2 important facts, one fact is the theorem of Hurwitz for analytic functions, the other thing is the so-called symmetry of the spherical metric with respect to inversion okay, so these are the 2 facts and this is what I wanted to concentrate today upon.

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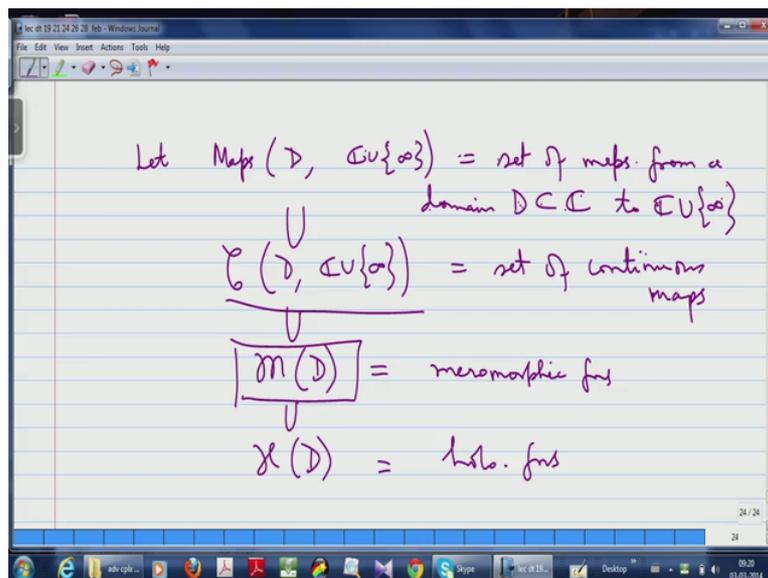
So I will start with the following thing, so let maps D, C union infinity so look at this set let this be the set of maps or just functions from D which is it is a domain D is a domain in the complex plane to the extended complex plane, so these are the important things I am allowing the value infinity alright and so what I am going to do is there are many subset of this which I am interested in, so let me write them down among this is inside this is I will put this C of D C union infinity and what is this C of C? This is the set of continuous maps, this is the set of continuous maps okay and you know when I say this is a set of continuous maps am looking at all those maps which are continuous this domain D to C union infinity and mind you C union infinity as a topology.

So C union infinity has 1 point compactification topology and with this topology it is holomorphic to the Riemann sphere under the stereographic projection and in fact it is in fact even a metric space you can take either the spherical metric or the cordial metric on the Riemann sphere and transport it to C union infinity via the stereographic projection and so C union infinity becomes even a metric space, it becomes a complete metric space it is compact it is very nice okay. So it does make sense to look at continuous maps into a topological space or a metric space so it discontinues in that sense okay.

So the point I want you to understand is that now if you look at the set of continuous maps into C union infinity then the constant map which is equal to infinity okay that is the map that takes every point to infinity that is also continuous because mind you always constant maps are always continuous okay, so the function which is infinity uniformly that is a continuous function mind you, so this is the point that you have to carefully notice this is what we

introduce and in particular what happens is that this contains the set m of D the set of Meromorphic functions on D.

You take any Meromorphic functions on D, what is a Meromorphic function? A Meromorphic function as a holomorphic function on D minus an isolated set of points for which the given function have poles at which the given function has poles. So you take a Meromorphic function on D okay it is a (())(12:41) holomorphic function outside you know isolated set of points but at each of those isolated set of points in D the function has a pole but I can define the value of the function to be infinity at a pole because that is the limit I get as I approach a pole okay that is the definition of pole okay and therefore this a Meromorphic function becomes a continuous function in to C union infinity that is the point that is another critical point you have to understand okay by including the value infinity you are making the Meromorphic function continuous on the whole domain you are making it continuous even at the poles okay and do not confuse that continuity with the usual continuity because the usual continuity is with respect to complex numbers.

The target is complex numbers and you do not allow the value infinity okay but this continuity is different this continuity is with respect to the extended complex plane okay this is something that you have to clearly distinguish okay because you know a Meromorphic function in the usual sense cannot be continuous at a pole because at the pole it becomes infinity okay but and that is because you do not allow it to take the value infinity, you do not think of infinity as a value if you are only thinking of complex values but now since I have added the extra point at infinity the Meromorphic function becomes also continuous at infinity that is the point you must notice and then of course the further subset of H of D this is the subset of all holomorphic functions okay or analytic functions and of course you know holomorphic or analytic functions are also by default they are also included as Meromorphic functions.

So Meromorphic function is a function which can either be analytic or if it has singularities, the singularities must only be isolated and they must be only poles that is the definition okay. So the definition of Meromorphic includes the definition of holomorphic okay, so let me write that so this is m of D is Meromorphic functions and H of D is holomorphic functions and so you know basically what we want to do is we want to do topology on this set, the set of Meromorphic functions alright on the domain D and I told you this topology has to be

done with respect to I mean we say that we do topology while trying to study convergence okay, so the convergence that we should think of is normal convergence.

So the topology corresponds to working with normal convergence okay and what we saw last time is that if you take the domain to be the exterior of the unit circle okay if you take the domain to be the exterior of the unit circle and you take the sequence Z n, Z power n okay namely Z square, Z cube then that is a sequence in H of D that is the sequence of holomorphic functions and it converges normally to what? It converges normally to the constant function infinity which is in this set about this script C D C union infinity it is here okay and so I want to say the following thing, you must have studied this in the 1st course in topology but the idea is essentially the same namely that whenever you have a uniform limit okay the limit function is also continuous okay.

So that is the 1st thing that I want to that is the 1st thing I want to prove or recall at least and mind you here we are not working with uniform convergence on the whole domain. Whatever convergence we are working with is only normal convergence but that is good enough because you know normal convergence will also give locally uniform convergence, so locally you can still do the same thing that you would do if you had uniform convergence everywhere okay and that is good enough or local properties like continuity, analyticity, et cetera that is good enough okay.

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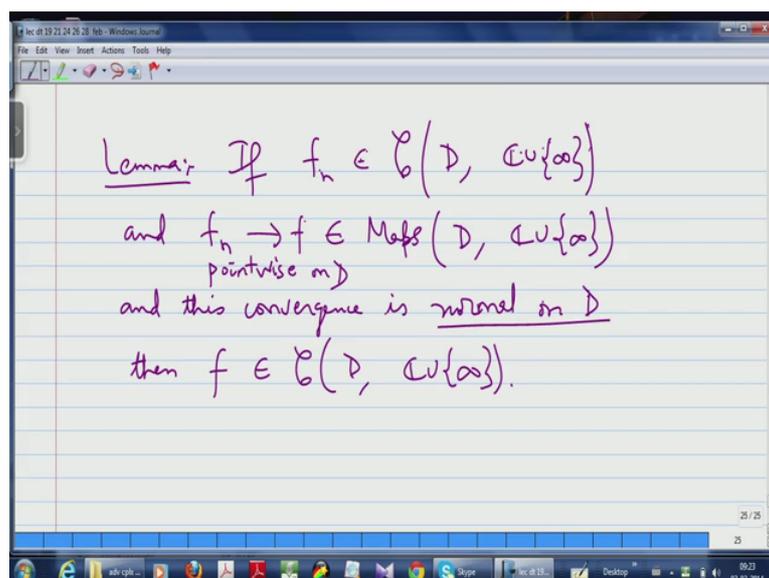
So here is a lemma; if f n is a sequence of continuous maps from the domain D into C union infinity and f n tends to f which is thought of as a map D to C union infinity so, so that means

when I say f n tends to f point wise, so this is point wise on D okay that means f n of Z tends to the value f of Z as n tends to infinity for each Z in D okay and this convergence is normal on D then f is also a continuous map D to C union infinity, so I am just saying that are normal limit of continuous map is continuous and why should this be true? Because you know uniform limit of continuous map is continuous and if you have a normal limit is actually a uniform limit locally and therefore the limit function is continuous locally but continuity is a local property therefore if something is continuous locally then it is continues.

If you have a global map which is locally continuous, continuous on every on sufficiently small open sets okay which cover the domain then it is continuous okay so this is a very the proof of this lemma is just trivial I mean if you assume the fact that the uniform limit of continuous functions is continuous okay, so this is something that you can easily deduce, so let me pinpoint the important fact here if you take a point in D okay then you can choose a sufficiently small disk surrounding that point to lie in D that is because D is a domain mind you D is an open connected set, so D is an open set so if you give me a point of D there is a whole disk surrounding that point which is lying inside D and if I make the radius of disk small enough I can ensure that even the boundary of that disk lies inside D okay.

Now if I include the boundary to the disk then it becomes compact set because it is closed and bounded set and on a compact set I know that this convergence is actually uniform because I have been given normal convergence. Normal convergence means at whenever you look at the convergence on compact subset it is uniform okay therefore if you give me any point I can find a sufficiently small disk on whose closure the convergence is uniform, so in particular it is also uniform on sufficiently small disk okay that means that the limit function on sufficiently small disk is continuous but then I can cover the whole space by such small disk, so the limit function becomes continues everywhere okay that is the proof alright. So I want to write down the proof I have told you in words, now comes the very important theorems, so here is the theorem, the theorem is that so let me tell you in this lemma we have been only worrying about continuous maps alright, now you can ask this question because you know in the previous slide.

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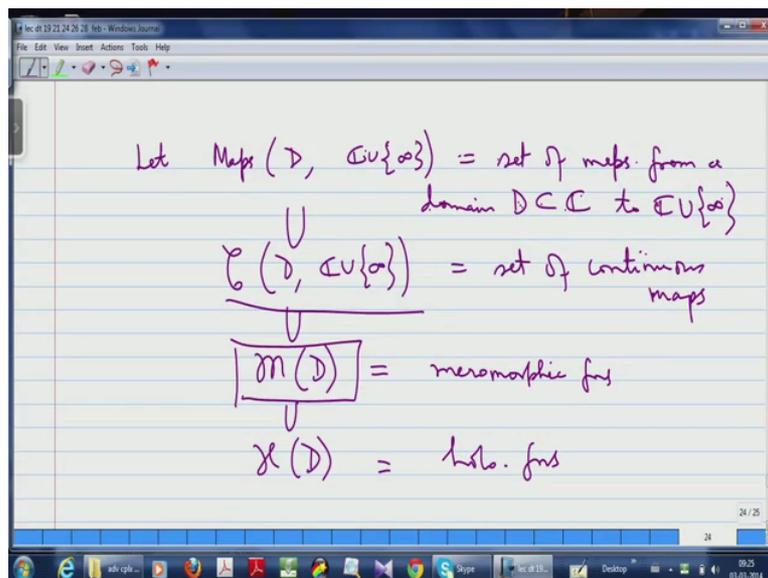
If you look at the previous slide I have giving you these 2 subsets is one subset which is m of D which is the Meromorphic functions and then there is this other subset which is H of D which is a holomorphic function and then you can ask what will happen to the limit function if the given sequence of functions is in m of D or H of D, so you can ask so what we have just now proved in this lemma is that a normal limit of continuous function is continuous alright that is what the lemma says.

Now we can ask is a normal limit of Meromorphic functions Meromorphic that is one question can ask then the other question you can ask is, is the normal limit of holomorphic that is analytic functions analytic you can ask that but you know you already have seen an exception, we have seen an example of a normal limit of holomorphic analytic functions which is going to the constant function infinity okay. Now this is the only exception that is the beautiful thing, so in the case of an analytic function and that is a theorem, so if you are looking at sequence of holomorphic or analytic functions suppose it converges normally to a limit function then the limit function is either completely holomorphic that is completely analytic or it is completely infinity it is a constant function infinity and there is nothing in between okay.
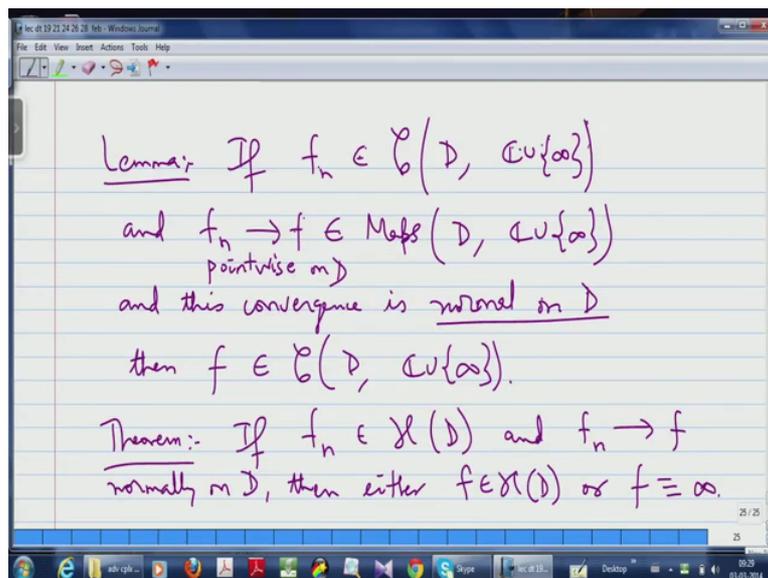
So this is very good behaviour, so what it means is that you know if you are the exception whenever you are working with respect to the spherical metric, that is you are working with respect to extended complex plane namely you are allowing the value infinity okay then you must allow the exceptional case that is sequence of holomorphic function tends to the constant function infinity uniformly on compact sets okay that is the only exception that is what the theorem says, so for example you do not have this further pathologies like you know you have a sequence of analytic functions and in the limit you get a Meromorphic function. Suddenly a pole pops up in the limit you know such kind of horrible behaviour does not happen okay.

Now that is very important that tells you that you know the behaviour is very good right so otherwise you would have been worried if you can find a sequence of analytic functions okay which is converging normally to a limit function and the limit function suddenly starts having poles you know of course intuitively you do not expect that to happen but how do you prove that such a thing does not happen and that is what the theorem says okay. So let me write that down theorem: if f n is a sequence of holomorphic functions on H of D and f n converges to f normally on D then either f is also holomorphic or f is identically infinity that is all, you do not get the intermediate is of f b in Meromorphic okay.

So this is the theorem and this is intuitively this looks fine but the big deal with serious mathematics is to (())(24:20) some statements which are intuitive and then the bigger deal is to really prove them you giving prove is very important part of mathematics. So let us go to this so the proof requires a couple of things 1st of all the proof requires the so-called

invariance of the spherical metric with respect to (())(24:45) , so let me explain that so that is one ingredient of the proof, the other ingredient of the proof is so-called Hurwitz's theorem which probably you have seen it in the 1st course in complex analysis but I do not expect many people to have seen it in the 1st course in complex analysis, so I will tell you what the theorem is, it is a pretty simple theorem it has got to do with uniform convergence and it has got to do with the argument principal which you might have seen okay, so let me start with this start with the invariance of the spherical metric with respect to inversion.

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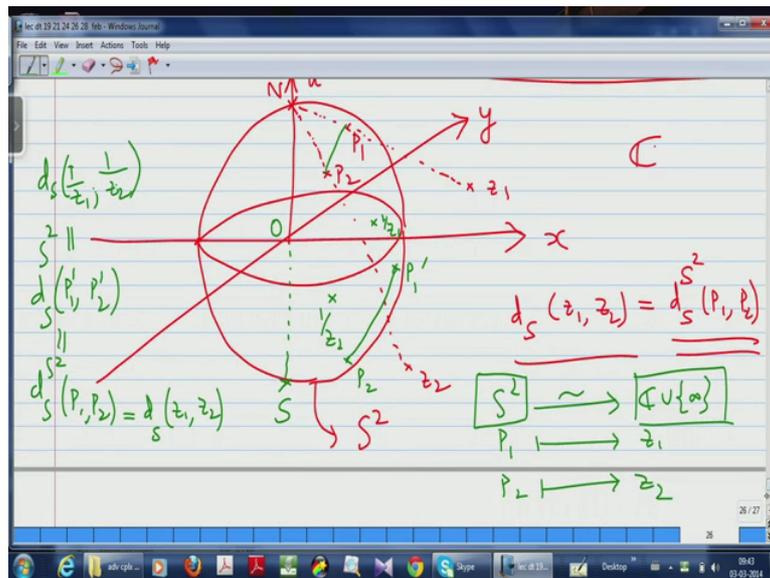
So let us again recall, so let me put this here may be I will use a different color, so invariance of the spherical metric with respect to inversion, this is a very important fact, so here is my so here is what I am going to say so you see you have this so let me again draw this stereographic projection, so here is the this is x y plane this is the complex plane which is identified with the x y plane and then you have this well you have the Riemann sphere again okay and of course there is this 3rd axis which…so here is the 3rd one this is u because of course you know I am reserving I reserving Z to be x plus i y, so I called in the 3rd axis as u and now you see you take 2 points Z 1 and say Z 2 okay on the complex plane and in fact you know then you get the stereographic projection on the Riemann sphere.

So you get these points so you get this point P 1 and you get this point P 2 okay and of course you know you get P 1 by so this is the North pole and you join North pole to P 1 then it should go and hit Z 1 you join North pole to P 2 and it has to go and hit Z 2 okay, so this is the definition of stereographic projection and what is the spherical distance between Z 1 and Z 2 this is actually the spherical distance well let me call this as a S 2 this is the real 2 sphere

this is a standard topological notation, so let me put this put S 2 here between P 1 and P 2 and what is a thing I have written this on the right.

This DS superscript S 2 is actually the geodesic distance, it is the distance along that great circle passing through P 1 and P 2 you know if you have a sphere and if you take 2 points of the sphere if there are 2 distinct points as only one big circle, circle of largest radius that passes through those 2 points and lies on the sphere that is called the great circle passing through those 2 points okay and then what you do is that you have these 2 points on a circle and they divide the circle into a smaller portion, a larger portion in general and you take the length of the smaller arc okay, so that is what this distance here is and that is defined to be equal to the distance of the spherical distance between the point Z 1 and Z 2 okay and well this is spherical distance.

Now the point is that you know I need also to write something here, so let me do the following thing let me take another color, so you have also so if I write it out in a so let me write it somewhere here, so you have S 2 and you have this stereographic projection with C union infinity, so C union infinity is extended complex plain there is a point at infinity added to that set of complex numbers and this infinity goes to the North pole mind you under the stereographic injection, so what I have written here is the stereographic projection and so here P 1 goes to Z 1 and P 2 goes to Z 2 okay and this is the situation that we have but on C union infinity there is automorphism okay there is a self-isomorphism, there is an isomorphism at least as a set and what is that isomorphism? That is inverse okay so on C union infinity you have the map Z going to 1 by Z.

Z going to 1 by Z makes sense on C union infinity where you declare 0 to go to infinity and you declare infinity to go to 0, so it interchanges 0 and infinity but for points which are different from 0 and infinity you know for a complex number which is different from 0 and infinity of course when we say complex number infinity is not allowed, so if it is nonzero complex number than it is reciprocal is also a nonzero complex number, so the point is that on this right side here which is the C union infinity there is this map Z going to 1 by Z and this map Z going to 1 by Z is in fact a homeomorphism because you know it is continuous and it remains continuous even if you give C union infinity of course you know Z going to 1 by Z is continuous in the punctured complex plane that is if you take the complex plane and remove the origin Z going to 1 by Z is actually you know holomorphic isomorphism because

it is an injectable holomorphic map and you do not have to go too deep because it is inverses itself for Z going to 1 by Z the inverse map is itself okay.

We apply the map twice you get identity okay, so it is a holomorphic isomorphism it is an analytic isomorphism C minus 0 to C minus 0 but the point is if you include 0 then you have to also include infinity and you have to send 0 to infinity and you have to send infinity to 0 and therefore you get Bijective map of the extended complex plane and that map is actually a homeomorphism okay with respect to the one-point compactification topology on C union infinity we can check that, so but you see this Z going to 1 by Z which is inversion is a homeomorphism on the extended plane.

Now if you transport that is via the stereographic projection you will get you will get homeomorphism of the real sphere S 2 which is a Riemann sphere okay because you know if 2 spaces are isomorphic then if 2 topological space are isomorphic that is holomorphic if one space you have a automorphism a homeomorphism, self-homeomorphism then this isomorphism will transport it and give rise to a self-homeomorphism to the other space okay, so this inversion which is a homeomorphism of the self-homeomorphism of the extended plane will give you a self-homeomorphism Riemann sphere okay and guess what it is? You know what address it is, you can check it, it is actually nothing but rotation of the Riemann sphere okay it is the rotation of the Riemann sphere about the x axis okay and you can see that that is because you see you take any point on the x-axis okay you take any point on the x-axis say for example it the point 1 on the x-axis that corresponds the complex number 1 okay.

Where does it go (())(33:43) it goes back to 1 alright and the point minus 1 goes back to minus 1 and you know the stereographic projection is such that for every point on the unit circle, the point on the Riemann sphere is the same as the point on the unit circle for the stereographic projection, this geographic projection induces a bisection on the unit circle because the unit circle lies also on the complex plane, it lies also on the Riemann sphere on S 2 and this stereographic projection fixes point wise it fixes the unit circle, so this inversion is going to induce some homeomorphism of the Riemann sphere that is going to fix plus 1 and minus 1 okay and look at what happens to a point at infinity and (())(34:37) infinity goes to 0 and 0 goes to infinity okay.

Now what does this translate to the Riemann sphere you see on the Riemann sphere infinity corresponds to the North pole, 0 corresponds to the South pole okay the point 0 corresponds to the South pole because you see the 0 is here and you know what is the stereographic

projection of 0 it is you have to take the lines joining the North pole to 0 and then look at the unique point on the Riemann sphere where this line hits and that will be the South pole okay, so 0 on the complex plane corresponds to South pole on the Riemann sphere. So you know this inversion which sends 0 to infinity and infinity to 0 on the extended plane when you translate into the Riemann sphere it will send North pole to the South pole and South pole to the North pole.

Now you can imagine this map what it is doing? it is switching the North and South poles and it is fixing plus or minus 1, so it is a rotation about the x-axis that is what is happening okay. You can use your analytic geometry and you can actually write out equations and check that this is actually a rotation by 180 degrees of this sphere with respect to the x-axis okay and now what does this tell you, this tells you that if I take the point Z 1 and Z 2 okay and I take the spherical distance between them and if I take the inverse points 1 by Z 1 and 1 by Z 2 and take the spherical distance between the inverse points that will be the same because this spherical distances are being measured by looking at the corresponding points on the sphere and the inversion corresponds to rotation of the sphere but if you take any 2 points on the sphere, the spherical distance between those 2 points that is not going to change if I rotate the sphere that is invariant under rotation of the sphere.

So the moral of the story is that this spherical distance between Z 1 and Z 2 is the same as spherical distance between 1 by Z 1 and Z 2. In other words the spherical metric is invariant for the inversion that is a very important fact which you are going to use the proof okay. So let me write this down so let me say let me write somewhere here, so diagrammatically what you are going to have is that so this is Z 1 and let us say that this is Z 2 sorry this is so if this is Z 1 then this is…so 1 by Z 1 is going to lie somewhere here and if this is Z 2 then 1 by Z 2 is going to lie somewhere here and this 1 by Z 1 and 1 by Z 2 are going to correspond to points P 1 prime and P 2 prime on the Riemann sphere and the fact is that the spherical distance between…the spherical distance on S 2 between P 1 prime and P 2 prime is the same as spherical distance on S 2 between P 1 and P 2 because P 1 prime and P 2 prime are just gotten from P 1 and P 2 by rotation by 180 degrees and so in other words I am saying that if I draw this arc here and if I draw this arc here they are of the same length.

Mind you this is a prospective drawing, so they do not really look to be of the same length when I draw it on this but they are of the same length essentially and well what you see this guy here on top of it is the spherical distance between 1 by Z 1 and 1 by Z 2 and this guy here

the bottom is the spherical distance between Z 1 and Z 2 is okay and this spherical distance is invariant under inversion and the way I have drawn it I have taken Z 1 and Z 2 in the complex plane but you can make one of them or even both of them infinity, it still works okay. So the moral of the story is therefore so let me also mention this P 1 goes to Z 1, P 2 goes to Z 2 and of course, so let me write it here P 1 prime goes to Z 1 prime, P 2 prime corresponds sorry so Z 1 prime is actually 1 by Z 1 and P 2 prime goes to Z 2 prime which is 1 by Z 2 okay.

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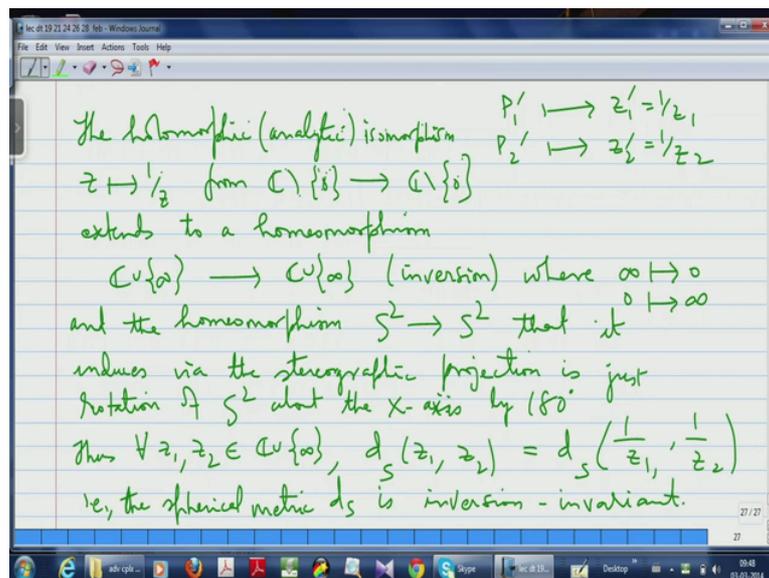
So let me write the statement here so the holomorphic that is analytic isomorphism Z going to 1 over Z from C minus 0 to C minus extends to homeomorphism C union infinity to C union infinity and this is inversion where you send infinity goes to 0 and 0 goes to infinity and the homeomorphism that it induces via the stereographic projection on S 2. S 2 to S 2 that it induces via the stereographic projection is just rotation of S 2 about the x axis by 180 degrees okay, so this is the fact that you need to check. You can write it out it is a very simple exercise in analytic geometry if you want but the point is that distances on the sphere will not change if you rotate the sphere, so the upshot is the thus or all points Z 1, Z 2 in C union infinity the spherical distance between Z 1 and Z 2 is the same as the spherical distance between 1 by Z 1 and 1 by Z 2.

So this is the fact that we want, so it uses the fact that if you…distances on the sphere will not change if you rotate the sphere and the rotation of the sphere by 180 degree is about the x-axis corresponds to actually it translates to inversion for the extended plane okay, so the spherical metric is inversion variant, so let me write that in words i.e. the spherical metric d S

is inversion in variant okay, so this is one fact that we need to use okay I need to use this fact this fact will be used in trying to prove our theorem that you know if you have a sequence of analytic functions on a domain. Either it converges to an analytic function or it converges to the constant function infinity provided you assume the convergence is normal many form on compact subset okay. So this is one fact, the other fact is the very important Hurwitz's theorem okay which I will try to explain next.

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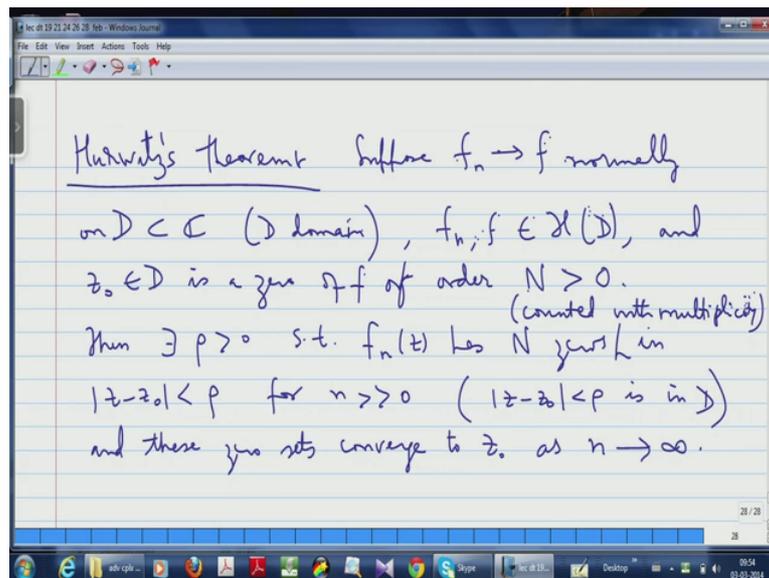
So the next thing that I needed is Hurwitz's theorem that is another thing that I need Hurwitz's theorem okay, so what is this Hurwitz's theorem? So basically the theorem is very simple, what the theorem says is that it tells you something that is very believable. What it says is that you know if you take a normal limit of analytic functions, holomorphic functions then of course you know the limit function suppose a limit function is complex value not it doesn't take the value infinity, so we are in the setup of an undergraduate course of complex analysis where all complex function takes only values in C okay and you are not allowing the value infinity okay.

So suppose the sequence of analytic functions converges normally to a function which takes values only in C then you know already that this limit function is already analytic there is something that you know I have already explain to you that a normal limit of analytic functions is analytic provided you make sure that the limit function takes values only in C okay this is again something that you have seen in the 1st course in complex analysis basically using Morera's theorem and Clausius theorem okay.

So the limit function is analytic, now what Hurwitz's theorem says is that if you take a 0 of the limit function okay and mind you the limit function is analytic and you know an analytic function has isolated zeros unless it is identically 0 and a non-constant analytic function if you take the zeros they are isolated there is a very important fact and for example it is as powerful as identity theorem which says that if 2 analytic functions are having the same values on a convergence set of points even at the limit including the limit which is also a point of analyticity then the functions have to be identically the same that is identity theorem and that is equivalent to this theorem that the zeros of an analytic functions or isolated, a non-constant analytic function zeros are isolated okay.

So you take normal sequence of analytic functions, you take the limit function it is analytic and you take a 0 of the limit function then what Hurwitz's theorem says is that this 0 of the limit function is from zeros of the functions of the original sequence. In fact if you take a 0 of the limit function it has to have certain order you know the zeros always have a certain order this way you will have a 0 of order 1 or order 2 and so on, so if the order of 0 is capital n Hurwitz's theorem says that because this limit function has come as a normal limit of a sequence, what it will say is that beyond a certain stage each of the functions in your sequence will also have n zeros.

The same number of zeros with multiplicity as the 0 of the limit function that you are looking at and these zeros and this will happen in a neighbourhood of that 0 of the limit function and what will happen is slowly as n tense to infinity these zeros will cluster and cluster and cluster and come closer and closer and in the limit they will all (())(47:29) the 0 of the limit function. So basically what Hurwitz's theorem says is that the 0 of the limit function is actually coming as limit of zeros of the original functions beyond a certain stage and the way this is done is very precisely the sense that even multiplicities are taken care of that is essentially what Hurwitz's theorem is okay. This is how you say it in words but of course when you technically write it down it looks pretty more difficult okay.

So now let me write this statement of the theorem one suppose f n converges to f uniformly normally on D, D domain and f n and f are all holomorphic functions on D and Z naught belonging to D is zero of f of order n greater than 0 then there exist a row such that f n of Z has n 0 in mod Z minus Z naught less than go for n sufficiently large, so when I say n zeros counted with multiplicity. In this disk for n sufficiently large and of course mod Z minus Z naught less than row is in D, so you have to choose row small enough that is such a row and

further and these zeros sets converge to Z naught as an tends to infinity, so this is Hurwitz's theorem in notations okay.

So what it says is that 0 Z naught of the limit function, the limit analytic function f is the limit point of zeros of the functions in the sequence and the fact is that even the multiplicities are taken care of. Mind you multiplicity is a very important because you know multiplicity is you have to count zeros with multiplicity, you cannot just count zeros just as points it may be a 0. A function may have a 0 at a point but it may have a certain multiplicity that is the order of 0 you have to count the order also okay. So that is why the multiplicities are important these n zeros they need not be n distinct points there could be lesser than n points with some points having zeros of higher order okay, so multiplicity is very important, so this is Hurwitz's theorem okay and the prove of this theorem essentially uses the argument principle okay. It uses argument principle and it uses uniform convergence okay. So I will try to give a proof of that in the…I will give a sketch of that proof in the next talk.