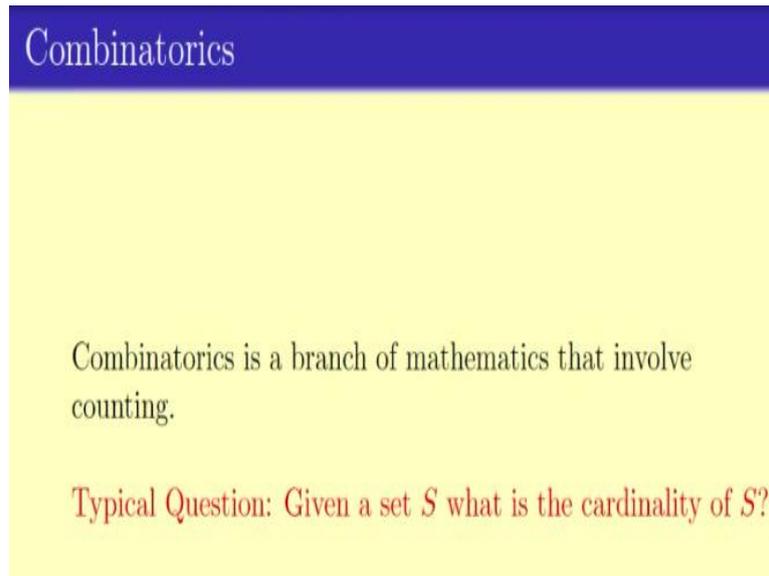


Discrete Mathematics
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Lecture - 35
Some Counting Problems

Welcome back. So we have been looking at counting.

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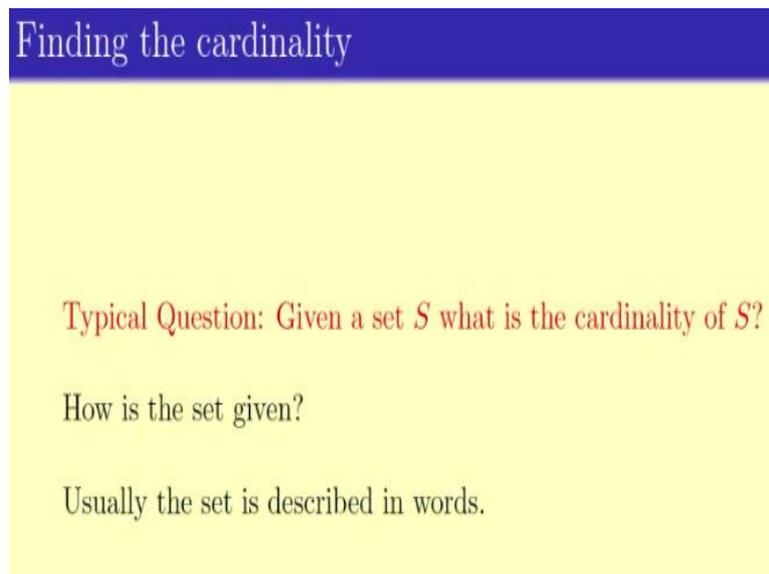
Combinatorics

Combinatorics is a branch of mathematics that involve counting.

Typical Question: Given a set S what is the cardinality of S ?

So this is, in the last few lectures, we have been looking at Combinatorics, which is a branch of mathematics that involves counting. A typical question that is asked is, given a set S , what is the cardinality of the set S .

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Finding the cardinality

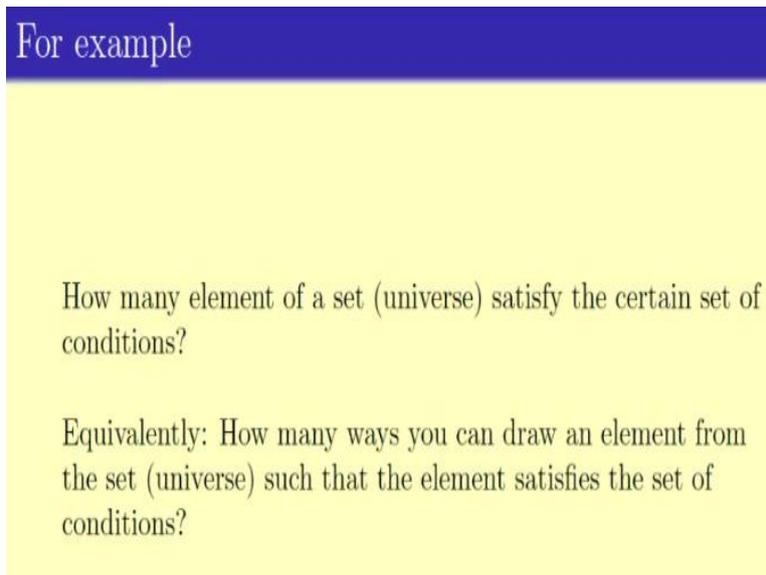
Typical Question: Given a set S what is the cardinality of S ?

How is the set given?

Usually the set is described in words.

Now, the main question is that, how is the set given. In most of the time, the set is not given explicitly, meaning elements of the set are not given explicitly. Instead it is described in words and hence the set S is kind of understood, but the elements of the set S is hard to understand or to innumerate. And, so the question of counting the number of elements in the set is a valid important and sometimes challenging problem.

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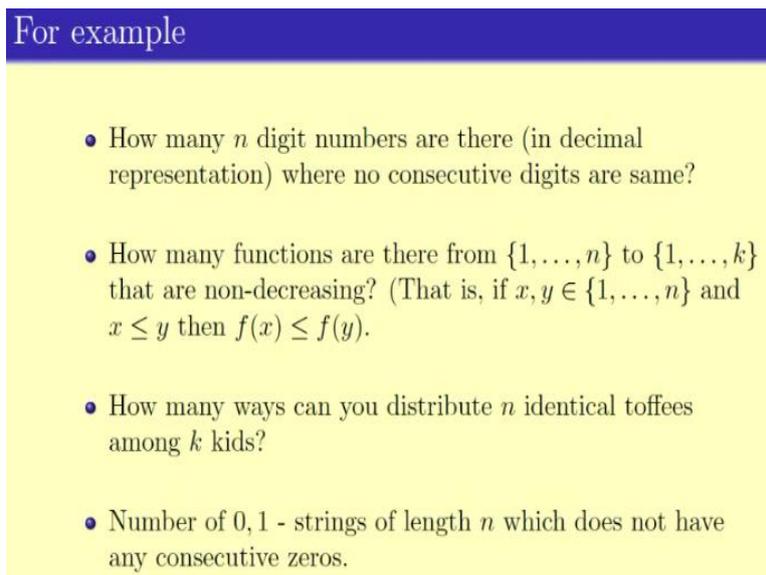
For example

How many element of a set (universe) satisfy the certain set of conditions?

Equivalently: How many ways you can draw an element from the set (universe) such that the element satisfies the set of conditions?

For example, one can ask things like, how many elements in the universe satisfy a certain set of conditions, or equivalently, how many ways can you draw an element from the universe, such that, the element satisfies the set of conditions.

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For example

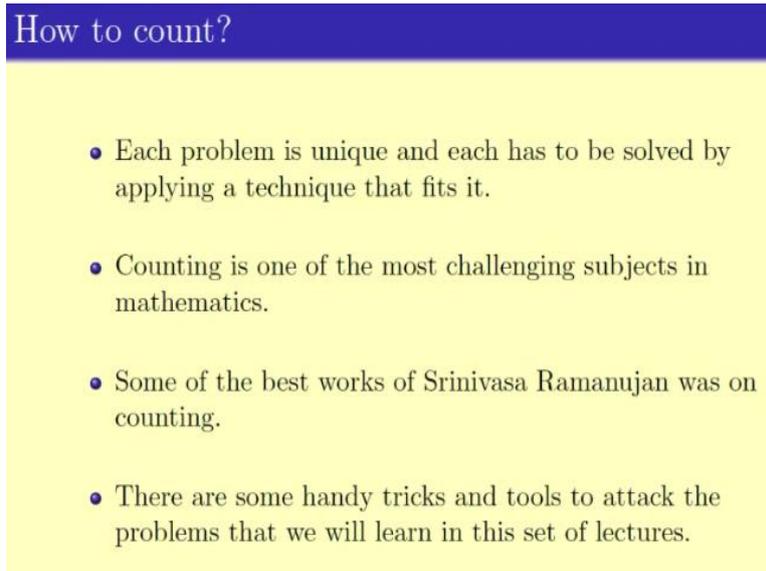
- How many n digit numbers are there (in decimal representation) where no consecutive digits are same?
- How many functions are there from $\{1, \dots, n\}$ to $\{1, \dots, k\}$ that are non-decreasing? (That is, if $x, y \in \{1, \dots, n\}$ and $x \leq y$ then $f(x) \leq f(y)$).
- How many ways can you distribute n identical toffees among k kids?
- Number of 0, 1 - strings of length n which does not have any consecutive zeros.

Some example that we have been thinking about is, the number one, how many n digit numbers are there, in the decimal representation, where no consecutive digits are same. We

will be seeing the proof of, or the answers to all of this question in this video. Second one is, how many functions are there from one to n to one to k , that are non-decreasing. That is if x and y are two numbers and if x is less than y , then f of x is less than or equal to y .

The third is, how many ways can you distribute n identical toffees among k kids. And the last one is the number of 0.1 strings of length n , which does not have any consecutive zeros.

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How to count?

- Each problem is unique and each has to be solved by applying a technique that fits it.
- Counting is one of the most challenging subjects in mathematics.
- Some of the best works of Srinivasa Ramanujan was on counting.
- There are some handy tricks and tools to attack the problems that we will learn in this set of lectures.

Now, the problem of counting as you have possibly seen by now, understood by now, that every problem is unique and there one can apply different techniques to solving a problem, and you have to apply the technique that fits the problem. In fact, counting is one of the most challenging subjects in maths. And, there are some of the best works of the great Srinivasa Ramanujan's works on counting. And there are some handy tricks and tools that we teach in set of lectures, that one can use to solve sets.

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Counting for selection

Selecting k objects from n objects

	Order Important	Order NOT important
Without Repetition	$\frac{n!}{(n-k)!}$	$\frac{n!}{k!(n-k)!}$
With Repetition	n^k	$\frac{(n+k-1)!}{(n-1)!k!}$

So here is the thing that we found out for the counting for selection, namely if you have to select k objects from n objects, then depending on certain things, namely depending on whether repetitions are allowed, meaning whether the same object can be chosen more than once. And whether the order in which we pick the k objects matters, we have works the various answers. Couple of videos earlier, we did prove all these numbers.

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Problem

How many ways to distribute n balls in k bins?

- Are the bins distinguishable or are they indistinguishable
- Are the balls distinguishable or are they indistinguishable
- If the balls are distinguishable then does the ordering of balls in the bins matter?
- Can some of the bins be empty?
- Are there any other restrictions.

Another important question here is how to distribute n balls into k bins. And there are a few problems that one to ask, namely, are the bins distinguishable or are the bins labelled, or are they not. Are the balls distinguishable or are the balls labelled. If the balls are distinguishable, then does the ordering of the balls in the bins matter. Can bins be empty, and are there any other restrictions.

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Counting for Distributing

Distributing n items among k bins.

	Items Indistinguishable	Items Distinguishable	
		Ordering inside bin matters	Ordering inside bin don't matter
Bins Labeled (can be empty)	$\binom{n+k-1}{k-1}$	$\frac{(n+k-1)!}{(k-1)!}$	k^n
Bins Labeled (can't be empty)	$\binom{n-1}{k-1}$	$\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} \frac{(n+k-i-1)!}{(k-i-1)!}$	$\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} (k-i)^n$
Bins Unlabeled	?	$\frac{\left(\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} \frac{(n+k-i-1)!}{(k-i-1)!}\right)}{k!}$	$\frac{\left(\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} (k-i)^n\right)}{k!}$

Now, at least of the first four such restrictions we looked at all the possible cases. And we have got this particular matrix, namely whether the bins are labelled or not, whether the items are distinguishable or indistinguishable. And if the items are distinguishable, then does the ordering matters or does not matter. Now, we have one of them that was left unsolved, namely this one, right. Namely whether bins are labelled, when the bins are unlabelled and items are indistinguishable. How many ways can you solve this.

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Partition of numbers

How many ways can you write the integer n into sum of positive integers?

We denote the number by $P(n)$

A similar question is: How many ways can you write the integer n into sum of k positive integers?

We denote it by $P(n, k)$.

$$P(n) = \sum_{k=1}^n P(n, k)$$

Now this one is related to how many ways can you write the integer in the sum of positive numbers. Now why is it so, let us see, so I have, say n balls. And I want to break it up into bins. But I do not care, the labelling of the bins. In other words, I want to say that, okay, I want to put two of them in one bin, one of them into other bin, maybe three of them in the third bin, maybe two of them in this another bin, and so on.

So what matters is that, I have one bin here so neglect it, complete this one. So this is a breakup of this set of thirteen balls into five teams. I do not care about labelling. So in other words, it basically looks like I have one bin containing one, two bins containing two, one bin containing three and one bin containing four. I do not care about the ordering, so in other words, what I get. I get one, two, two, three, four.

And I know that the sum of them must be equal to, so this is twelve balls actually, yeah, twelve. So, this is one way of splitting twelve balls into four bins. Now I could have done a separate one, for a, maybe I could have, what I could have done is I could have split this one, and I could also have maybe caught hold of this also. So, in that case I could have got three bins of two each, two plus two plus two, and two bins of three each, right.

This is three, and this is three, and one, two, three, here, and this is the bin. So, the number of ways I have writing this twelve as sum of integers is specifically the number of ways, I can split this n indistinguishable balls into exactly k bins, well none of the bins are empty, right. So we denote the number by P_n , and the question is that, how many balls, how many ways can you write the integer n as sum of k positive numbers.

So this is where we do not want the bins to be empty, and we denote it by P_{nk} . And as you can realise that P_n equals to sum over P_{nk} , where k equals to one to n . So, when we ask, how many ways can we split n indistinguishable balls into distinguishable bins, what we mean here is, so k indistinguishable bins, so this number is actually P_{nk} . Now, this P_{nk} is of course a very, very well spreaded problem. And in fact Srinivasa Ramanujam worked on this problem quite a lot.

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How many ways can you write the integer n as sum of positive integers?

Srinivasa Ramanujan had given an expression that is asymptotically similar to $P(n)$. Thus $P(n) \sim$

$$\frac{1}{\pi\sqrt{2}} \sum_{i=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn} \left(\exp \left[\frac{\pi}{k} \sqrt{\frac{2}{3} \left(n - \frac{1}{24} \right)} \right] \right)$$

where, $A_k(n) = \sum_{0 \leq m < k; (m,k)=1} e^{\pi i [s(m,k) - \frac{1}{k} 2nm]}$

and where, $s(m, k) = \frac{1}{4k} \sum_{j=1}^{k-1} \cot(\pi j/k) \cot(\pi jm/k)$

And, why we do not have a easy way of calculating $P(n)$, Srinivasa Ramanujan came up with the formula, that is very close to $P(n)$. We call it asymptotically similar. And this formula is extremely complicated. I want to show you the formula, and this will help you to appreciate how hard the problem is, of calculating the number of ways one can distribute n indistinguishable balls into k indistinguishable bins.

So here, Srinivasa Ramanujan gave an expression, which was basically this term, which not only has some weird objects like exponential of time. So this term typically new, this is the number e power $\frac{\pi}{k} \sqrt{\frac{2}{3} \left(n - \frac{1}{24} \right)}$, okay. And not only have this thing, it also has things like a derivative and from A_k , what is A_k ?

But A_k is this object, which is $e^{\pi i s(m,k) - \frac{1}{k} 2nm}$, where i is the complex number minus square root of minus one $s(m,k)$, now what is $s(m,k)$? $s(m,k)$ is this expression, now this shows how complicated it is. If this number is not exactly correct, it is very close to being correct, expression for $P(n)$. And a few years later, Ramanujan, Hardy and Rademacher, got a formula, which was the exact formula for $P(n)$.

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How many ways can you write the integer n as sum of positive integers?

The following was the expression for $P(n)$ given by Hardy-Ramanujan-Rademacher

$$\frac{1}{\pi\sqrt{2}} \sum_{i=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn} \left(\frac{1}{\sqrt{n - \frac{1}{24}}} \sinh \left[\frac{\pi}{k} \sqrt{\frac{2}{3}} \left(n - \frac{1}{24} \right) \right] \right)$$

where, $A_k(n) = \sum_{0 \leq m < k; (m,k)=1} e^{\pi i [s(m,k) - \frac{1}{k} 2nm]}$

and where, $s(m, k) = \frac{1}{4k} \sum_{j=1}^{k-1} \cot(\pi j/k) \cot(\pi j m/k)$

And it was instead of exponential they replaced it with, so they just made some changes to this part, and keeping everything else same. So inside this is the expression for P_n , how did they come about with this particular formula, that is clearly a hard problem, okay, it is clearly very hard way of getting it, but couple of things to note here. This shows, how hard this counting problem can be.

And also shows that it is of such importance that, the great men like Srinivasa Ramanujam and all have worked on this problems. In fact, this is one of the greatest work of Srinivasa Ramanujam, of proving that P_n is actually this number. So, this brings us to the kind of the end of this whole thing.

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Counting for Distributing

Distributing n items among k bins.

	Items Indistinguishable	Items Distinguishable	
		Ordering inside bin matters	Ordering inside bin don't matter
Bins Labeled (can be empty)	$\binom{n+k-1}{k-1}$	$\frac{(n+k-1)!}{(k-1)!}$	k^n
Bins Labeled (can't be empty)	$\binom{n-1}{k-1}$	$\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} \frac{(n+k-i-1)!}{(k-i-1)!}$	$\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} (k-i)^n$
Bins Unlabeled	$P(n, k)$	$\frac{(\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} \frac{(n+k-i-1)!}{(k-i-1)!})}{k!}$	$\frac{(\sum_{i=0}^k (-1)^{i+1} \binom{k}{i} (k-i)^n)}{k!}$

Where we can now know how many ways can we distribute n items into k bins. Now with this done, let us try to solve the problems, that we had in our minds.

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Problems

How many n digit numbers are there (in decimal representation) where no consecutive digits are same?

$$\begin{array}{l} \underline{a_0} \ a_1 \ a_2 \ \dots \ a_{n-1} \\ 9 \times 9 \times 9 \ \dots \times 9 \\ = 9^n \end{array} \quad \begin{array}{l} a_i \in \{0, 1, \dots, 9\} \\ a_0 \neq 0 \end{array}$$

So, to start with, let us see how many ways, how many n digit numbers are there in which there are no consecutive digits are same. Now, how does the n digit number works, n digit number is something like, I have some number a_0, a_1 , till n minus one, right, where a_i are either zero, one till nine and a_0 , okay. Let me imagine that a_0 is not equal to zero, because if a_0 is zero then I do not have a n digit number, right.

So a_0 can have one to nine, any one of those possibilities. Now, what can a_1 have, a_1 can have any of the same possibilities, but it cannot have the number that a_0 have. So if a_0 is one, then a_1 can have anything like zero, two, three, four, five, six, seven, eight, nine. If a_0 is five, then a_1 can be zero, one, two, three, four, six, seven, eight, nine. So for every possibility of a_0 , a_1 can have nine possibilities.

And what about the next ones a_2 , same, because we are not allowed to have two consecutive digits. So whatever that a_2 is, a_1 is a_2 can have anything other than a_1 . So again there are nine possibilities. And by this way, all this n terms can have nine possibilities. So, the answer to this question is product of all of them, which is nine to the power n . So, the number of eight digit numbers in decimal representation, where no consecutive digits are same is nine to the power n , okay.

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Problems

How many functions are there from $\{1, \dots, n\}$ to $\{1, \dots, k\}$ that are non-decreasing? (That is, if $x, y \in \{1, \dots, n\}$ and $x \leq y$ then $f(x) \leq f(y)$).

$$\frac{(n+k-1)!}{(k-1)! n!}$$

And moving on to the next problem, this is an interesting problem, how many functions are there from one to n to one to k , that are non-decreasing. So, let us forget the non-decreasing part, let us first ask how many functions are there from one to n to one to k . Now, how do I calculate that, any of this function can send any this, function can send first item to any of this k points, right. It can also send second item to any of the k points.

It can send the third item to any of the k points. So, the total number of ways you can get this function is, k times k times k like this n times, which is k power n . So, this is the number of functions that are there from one to n to one to k , good. Now if I add the question, that number of increasing function. What does increasing mean? Increasing means that if x is less than y , then f of x is strictly less than f of y .

Now how many such functions are there from one to n to one to k . Now I will say here, that of course, if k is less than n , it is less than n , then I cannot have an increasing function, why? Because how can I have a increasing function, because of one goes to one, two goes to two, three goes to three, I have to keep on increasing, then n has to go to something like n , but n is greater than k . I do not even has enough terms there, then the number is zero.

And if here if the opposite is true, if k is bigger than or equal to n , then what happens? If k is bigger than or equal to n , it means that I can pick n elements from this 1 to k and once I have picked some n elements out of this 1 to k , then this unique way of sending this 1 to n to the set of n elements. Why? Because this 1 has to go to the smallest element in n , two has to go to the second smallest element, set A .

So I say I have set A , so it is A is a set subset of 1 to k of size n , so size of A is n , then there is only unique way of sending this set 1 to n to the set A . Because I have to send the first element, the element 1 to the smallest element in A , second one will be second smallest element in A and so on. So the number of ways, number of increasing functions is the same as the number of ways I can select n objects from 1 to k .

So the number is equal to k choose n . Now let us name this problem a bit more complicated, by asking the main question, which is this question, how many non-decreasing functions. Now, again here, this is of course what is written. This can be less than, x is less than or equal to y , then f of x is less than or equal to y . Now again, the idea is simple. Once I think a subset is A , okay and this time with replacement.

The size of A equals to n , but A can have multiple copies, right multiple copies of the same element. So once I pick the subset A with replacement from 1 to k , then I know there is an unique way of sending 1 to x to A once again. So the number here is basically number of ways of getting n objects from k objects with replacement and how many ways can I get it. So this is equal to choosing n objects from the set 1 to k with replacement.

And how many ways can we get it? Now I quickly take you back to this old slide and this number is basically what we are asking for the way in which selecting k objects, actually I am asking how to select n objects on k objects, but this is just the opposite of that with repetitions and I do not care of the ordering, right, so it is this number. So in our case, it will be this number with k and n shift.

So I go back where I started, this n plus k minus 1 factorial by k minus 1 factorial times n , okay. Now let us move on to the third problem.

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Problems

How many ways can you distribute n identical toffees among k kids?

$$\frac{(n+k-1)!}{(k-1)! n!}$$

The third problem is how many ways can you distribute n identical toffees among k kids and this is again, I have done in many times, is n plus k minus 1 factorial by k minus 1 factorial times n , okay. Now let us move on to the last problem.

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Problems

Number of 0,1 - strings of length n which does not have any consecutive zeros.

The last problem states the number of 0,1 string of length, which does not have any consecutive zeros. This problem is quite a challenging problem. I let you guys to think about this problem. We will come back to the next class with this problem and this we will help us to start a completely new way of calculating. Thank you.