Real Analysis
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Lecture - 50
Construction of Everywhere Continuous Nowhere Differentiable Function

So, as we started in the last class we shall see the details of the construction of a function, which is continuous everywhere in the real line, but not differentiable at any point and in order to prove certain things about that.

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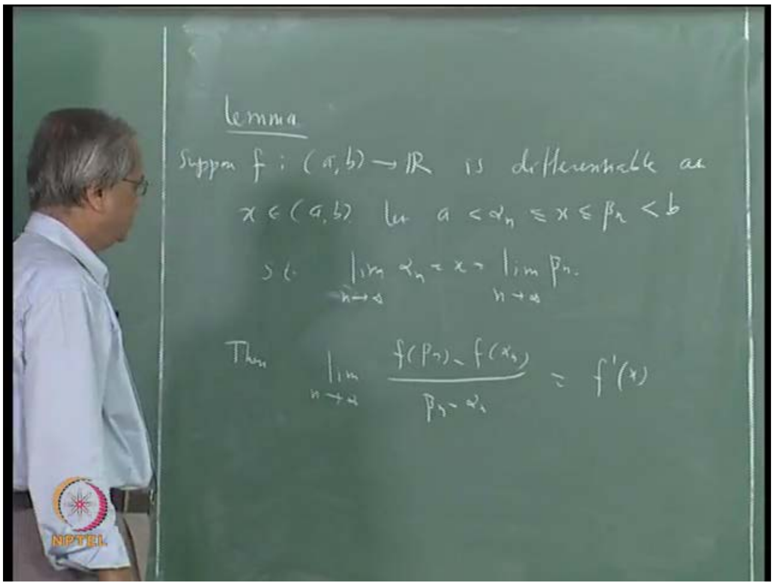

We had started with a preparatory lemma let us say, so it was let me, so suppose f is from a b to R, suppose f from a b to R is differentiable at some x in a b and let us consider two sequences. Let us say that a less than alpha n, alpha n less not equal to x, x less not equal to beta n and beta n less than b that is why we are considering two sequences alpha n and beta n such that limit both of them tend to 0. That is limit as n tends to infinity of alpha n is equal to x and similarly, limit of n tends to infinity of beta n is also x, then what we want to say is the following. Then limit as n tends to infinity of f at beta n minus f at alpha n divided by beta n minus alpha n this is equal to f prime x.

Now, what is the use of this lemma and why we are going through all this, the idea is very simple suppose we want to show that a function is not differentiable at some point then that can be done by constructing two such sequences alpha n and beta n. Showing that this limit does not exist, this limit does not exist that is the, that is the idea, so let us, now see how this is proved.

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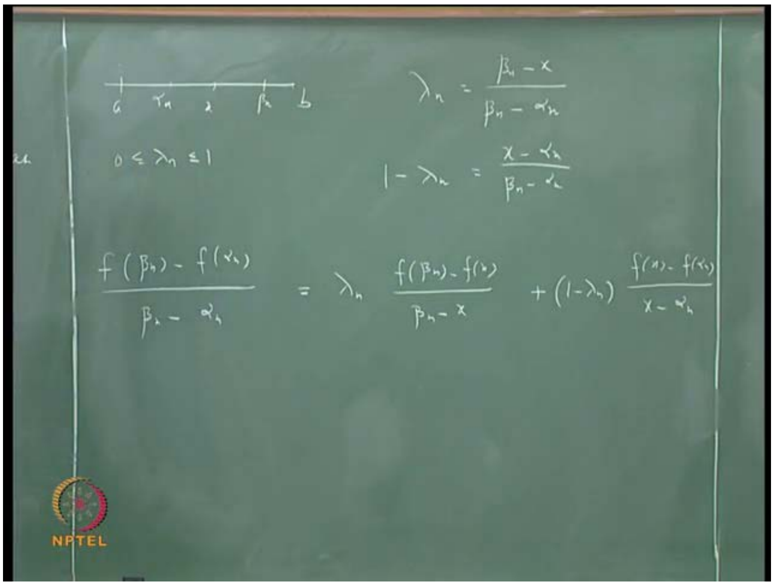

Let us it may help to draw some, so see here you have a let us say x and let us say b, so you have alpha n somewhere here and beta n somewhere, here let me call this lambda n as beta n minus x divide by beta n minus alpha n. Then you will see that if you look at 1 minus lambda n that will be see lambda n is this beta n minus f divided by this whole distance beta n minus alpha n. So, 1 minus lambda n will be x minus alpha n divided by beta n minus alpha n and for lambda n as well as 1 minus lambda n both lie between 0 and 1. So, 0 less than not equal to lambda n less not equal to 1 and similarly, for 1 minus lambda n, now what we observe is the following if you look at this ratio f beta n minus f alpha n divided by beta n minus alpha n.

We can write it as follows, we can write it as lambda n into f beta n minus f x divided by beta n minus x even this is the important step. Then plus 1 minus lambda n into f x minus f at alpha n divided by x minus alpha n I mean this is fairly elementary, all you need to do is put this value of lambda n. See what is lambda n, lambda n is beta n minus x divided by beta n minus x, so lambda n into this will be suppose you take lambda n and multiply to this what you will get is this.

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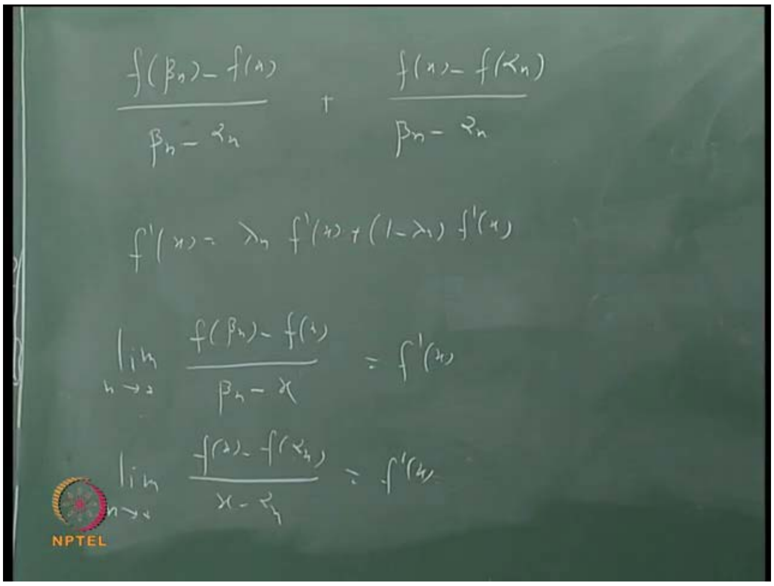

f beta n minus f x divide by beta n minus alpha n that will be the value of this lambda n into this and suppose you multiply 1 minus lambda n is this x minus alpha n divided by beta n minus alpha n. So, suppose you multiply 1 minus lambda n to this what you will get is plus f x minus f alpha n divided by beta n minus alpha n and if you take this sum of this that will same as this. So, it is elementary argument, what we can also say further is that is that I will subtract from both the sides f prime at x ,will subtract at both sides f prime at x. So, suppose I write this I will make say this minus f prime x is equal to I can say that lambda n into this minus f prime x plus 1 minus lambda n into this fraction minus f prime x.

Thus, all that I have done is this side I have written minus f prime x and that side I have written lambda n into minus lambda into f prime x minus 1 minus that is essentially. We are using this fact f prime x is same as lambda n, f prime x plus 1 minus lambda n f prime x I mean subtract this quantity from that side and the other quantity from the right hand side. Now, look the absolute value, so suppose I take the absolute value, here suppose I take the absolute value, here you can say that that is less not equal to absolute this is 0 less not equal to lambda. So, this is this is same as lambda n multiplied by absolute value of this plus 1 minus lambda n is also a positive quantity, 1 minus lambda n is also a positive quantity.

So, that multiplied by or at least not multiplied by this alright now if you look at the quantity is now that are occurring in the bracket this is f beta n minus f x divided by beta n minus x we know that as n tends to infinity this goes to 0. So, not this goes to this goes to f prime x remember we know this that is limit as n tends to infinity of f beta n minus f x divided by beta n minus x because beta n goes to x as infinity beta n goes to x limit of this is f prime x and this is something that we know.

Similarly, limit as n tends to infinity of f x minus f alpha n divided by x minus alpha n limit of this is also f prime x which is same as saying that limit of whatever you have. Here, in the bracket is 0 as n goes to infinity, but what about this lambda n and 1 minus lambda n you can say that they are bounded sequences, they are bounded sequences. So, you can say that lambda n is less not equal to n and 1 minus lambda n is also less not equal to n.

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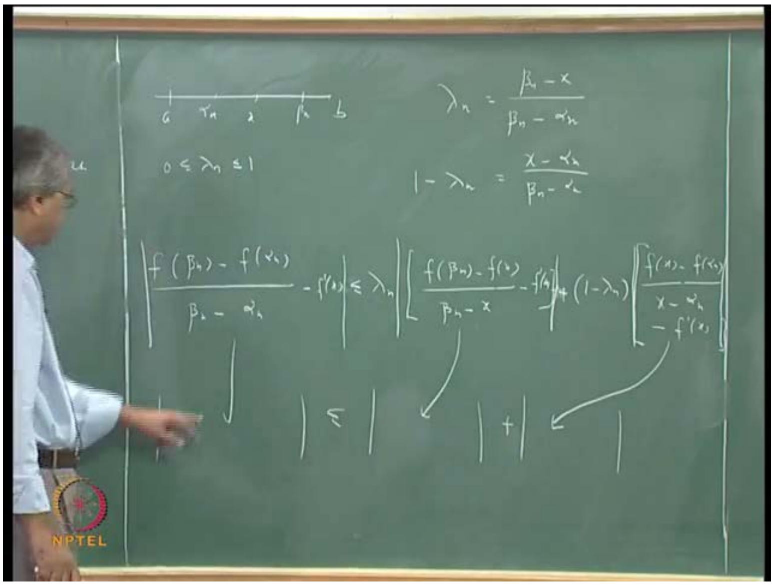

So, what I can say is that whatever, I will just write, here whatever you have, here in this bracket that is less not equal to lambda n is less not equal to 1 lambda n is less not equal to 1 that is less not equal to whatever you have. Here, in this bracket 1 minus lambda n is also less not equal to 1, 1 minus lambda n is also less, so plus the quantity in this bracket and each of this is going to 0 as n goes to infinity.

This is also goes to 0 and this also goes to 0 as n goes to infinity and hence we must have that whatever this quantity also goes to 0 as n goes to infinity and that is what we wanted to prove. That limit as n tends to infinity of this is equal to f prime x, now let us come to the main theorem that we or main theorem or what are the virtues of main examples.

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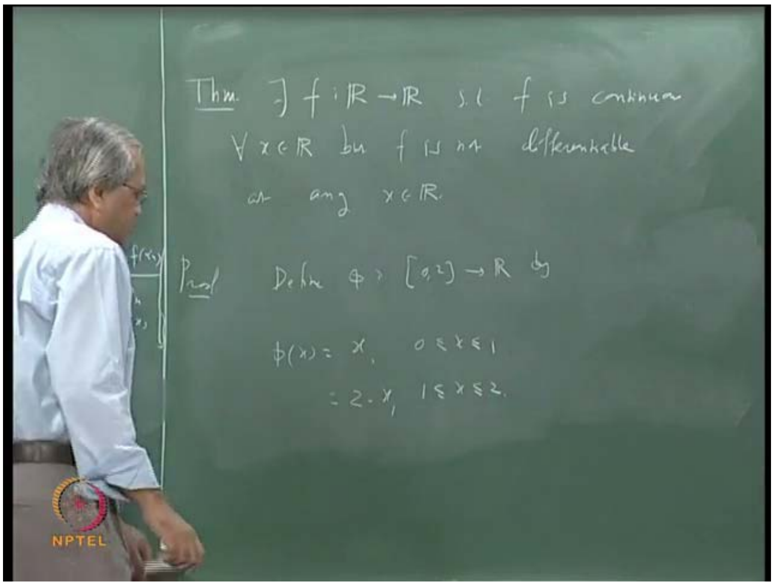

Anyways, let us state in the form of a theorem, what the theorem there exist a function f from R to R such that f is continuous for every x in R and, but f is not differentiable f is not differentiable at any x in R. Now, constructing of this function again as I said yesterday this is something is that is contrarily to what we into to do thing that is we cannot imagine such as function easily. It is not possible to draw a graph of this function let us say how once starts the construction, we first construct some auxiliary function, so let us call that, let me call at function phi.

Let us say define phi from first I will define in the interval 0 to 2, 0 to 2 to R by phi of x equal to x for 0 less not equal to x, less not equal to 1 and equal to 2 minus x, for 1 less not equal to x, less not equal to 2 I think it will help in drawing the graph of this function.

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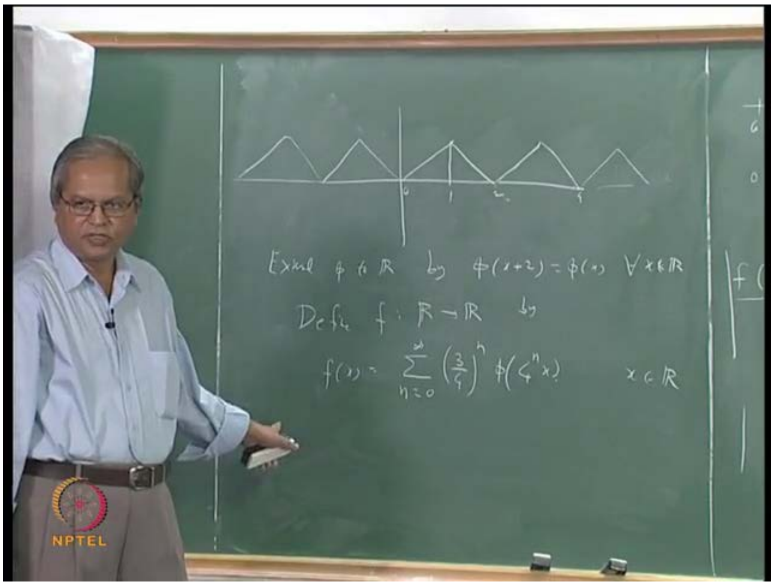

Let us, so suppose this is, this is 0 and that is 2 this is 1, and then between 0 to 1 is nothing but x, is nothing but x and between 1 to 2 it is 2 minus, that means it takes the value 1 at 1 and 0 at 2 again needs a straight line. So, it is this fairly straight for function it is continuous everywhere not differentiable, here not differentiable at x equal to 1 then what we do is that we extend this function to the whole of R by making it periodic. So, will simply say extend phi extend phi to R by assuming this by phi of x plus 2 is equal to phi x for every x in R this is such a thing is called a periodic extension.

If a function is defined in some interval, you just repeat the values of the function at all other intervals of the same length. For example just use this formula phi of x plus 2 is equal to phi x that will say what is the function in the interval 2 to 4, so this extends the function. You are given the function from 0 to 2 this will first extend it from 2 to 4, use the same thing then you will get function in 4 to 6, 6 to 8 etcetera. Similarly, this side this will minus 2 to 0 then minus 4 to minus 2 etcetera that is what is called periodic extension of the function.

But, of course this still is not a required function because you know that if this function is continuous everywhere it is not differentiable at integer points, it is not differentiable at integer points, but what we want is that it should not be differential at any x. Now, we construct that function and define f from R to R, define f from R to R as follows f of x is equal to sigma n going from 0 to infinity 3 by 4 to the power n and multiplied by phi of 4 to the power n x, now this expression has to be exact.

Here, we cannot afford to make any mistakes otherwise sub sequence calculations will all go wrong, this is for each x in R and, now what we want to claim is that this f is the required function this f is the required function that this is continuous everywhere. But, differentiable at no point in R, let us first dispose of the question of continuity, here you have the function phi of 4 to the power n x suppose you call this functions as f n.

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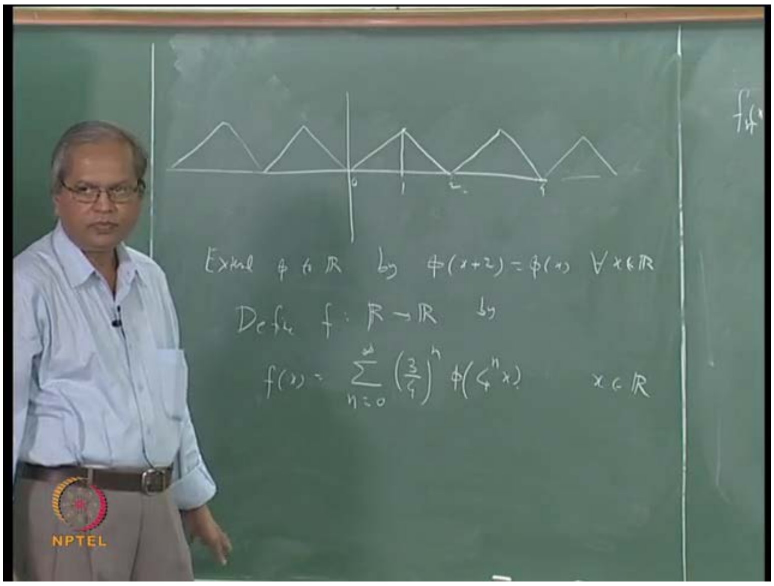

Suppose you call this function as f n, f n is phi of 4 to the power n x f n of x is equal to that each of this f n is continuous, each of this f n is continuous is this is a uniformly convergent series why is that uniformly convergent, look at mod f n x is less not equal to 3 by 4 to the power n.

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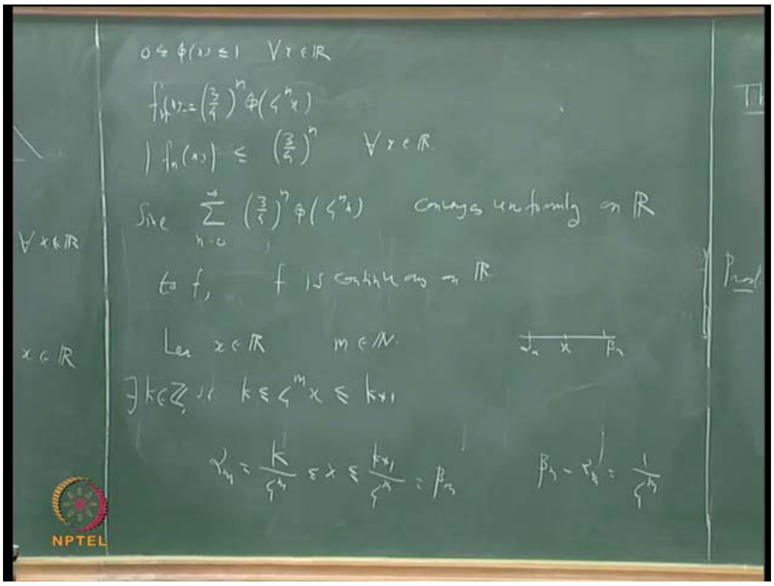

That is mod f n x is, let us see remember again we are what we are using the function phi takes the values between 0 and 1, so each f n x this is true 0 less not equal to f n x less not equal to one for all x in R. Hence, of course f n x is not the let us, let us this is 3 by 4 to the power n and into phi 4 to the power n x. Hence, let us just use this say this, here 0 less not equal to phi x less not equal to 1 for every x in R that is clear from, here that is clear from, here and hence mod f n x is less not equal to 3 by 4 to the power n for every x in R and sigma 4 to the power n is convergent. So, by Weierstrass m test the series converges uniformly, Weierstrass m test the series converges uniformly and we have said that once the series converges uniformly.

If each function occurring is continuous this sum is also a continuous function so we can then since sigma n going from 0 to infinity this 3 by 4 to the power n phi of 4 to the power n x converges uniformly. Remember converges uniformly on R, converges uniformly on R to f since this happens f is continuous on R, so remember in proving all the properties of f we have to obviously use the properties of phi because that is how we have started f is constructed using phi. Other functions which every all functions which depend on the starting function phi, so we have proved that f is continuous everywhere, we need to show that f is not differentiable at any x f is not differentiable at any x.

So, let us take some x in R and if we want to prove that it is not differentiable at R, we are going to use the tactic given by the previous lemma will construct two sequences.

Suppose I call them as alpha m and beta m because alpha should be less not equal to x, beta m should be bigger not equal to x both should converge to 0. But, f beta m minus f alpha m divided by beta m minus alpha m that is not a convergent sequence, suppose we show that, suppose we show that then this will be proved. To do that, let us take some m at some natural number and look at 4 to the power m x 4 to the power m x, now this is some real number, this is some real number.

So, you know that given any real number you can always find integer K such that that real number lies between K and K plus 1, that something you can always do. So, you can say that there exist an integer K that is there exist K in Z such that K is less not equal to this, less not equal to K plus 1. Now, what I will do is that I will take this K by 4 to the power m as alpha m and K plus 1 by 4 to the power m as beta that is or which is same as saying that K by 4 to the power m. This is less not equal to x and this is less not equal to K plus 1 by 4 to the power m and I will take this as alpha m, so alpha m is K, K by 4 to the power m and beta m is K plus 1 by 4 to the power m.

Now, is it clear that alpha m is less not equal to x, that is obvious x is less not equal to beta m, that is also obvious what is that other thing that we need to say that alpha m converges to x and beta m also converges to x is that also clear. In fact if you look at beta m minus alpha m see if you look at beta m minus alpha m beta m minus alpha m that is nothing but 1 by 4 to the power m 1 by 4 and that goes to 0.

That means, what is happening, what is happening x is in the, let us say x is in this interval alpha m to beta m and this interval is shrinking and going to 0 the length of this interval is shrinking and going to 0, so alpha m also goes to x and beta m also goes to x. So, if I need to show that f is not differentiable at x what we need to show is that f beta m minus f alpha m divided by beta m minus alpha m that is not a convergence sequence or it does not converge to anything that will mean that f is not differentiable.

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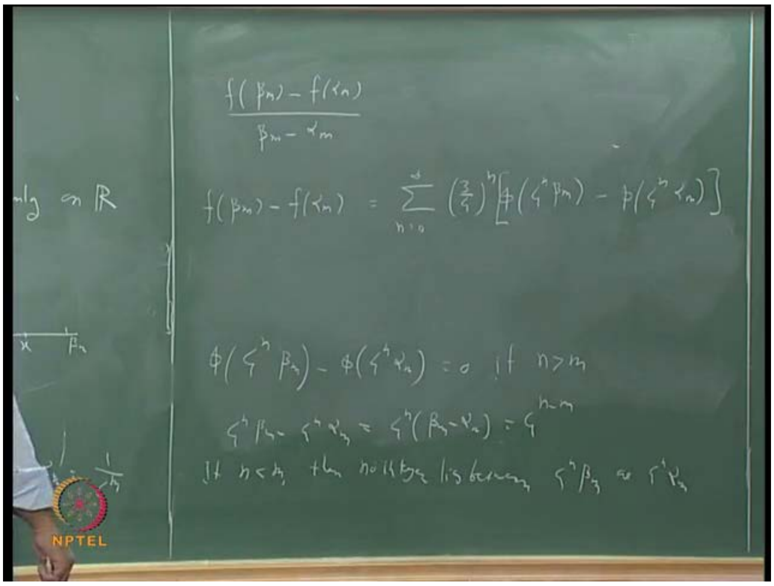

That means, we have to look at this sequence f beta m minus f alpha m divided by beta m minus alpha m and we have to show that this sequence is not convergent this does not converge to anything and if that is shown it will mean that f is not differentiable at x that is the idea. Now, to calculate that let us first look at the numerator, let us just see what happens to this f beta m minus f alpha m because as far as beta m minus alpha m is constant we already know that is 1 by 4 to the power m, so about that there is nothing much to be calculated.

Now, this is by definition, now here there is some complications will start this is sigma n going from 0 to infinity this 3 by 4 to the power n and phi of 4 to the power n beta m, phi of 4 to the power n beta m phi of 4 remember phi of 4 to the power n beta m. Similarly, minus f at alpha m will be similarly minus this into phi of 4 to the power n alpha m I will just write this thing together minus phi of 4 to the power n alpha m.

Now, what we have to do after this is that we have to calculate the values of these numbers phi of 4 to the power n beta m minus of phi 4 to the power n alpha m. Let us just look those things first and this will depend on what is n and how m and n are related to each other, what happens if n is less than m ,what happens if n is bigger than m what happens if n is equal to m, etcetera it depends on that.

Let us just look at those quantities first phi of 4 to the power n beta m minus 4 to the power n alpha m, now even before going to that let us just look at this number what 4 to the power n, not this it is not it is minus phi of 4 to the power n alpha m. That is what we require right suppose I look at the number 4 to the power n beta m minus 4 to the power n alpha m that is same as 4 to the power n into beta m minus alpha m 4 to the power n and that is, so that is nothing. But, 4 to the power n into beta m minus alpha m and beta m minus alpha m is nothing but 1 by 4 to the power m beta m minus alpha, m remains 1 by 4 to the power m.

So, I can say that this is same as 4 to the power n minus m, this we can say always whatever be n and m we can always say that 4 to the power n beta m minus 4 to the power alpha m that will be always equal to 4 to the power n minus m. Now, suppose n is bigger than m suppose n is bigger than m then this difference is some power of 4 some power of 4, but phi is a periodic function with period 2. So, that means phi of 4 to the power beta m is minus, this will be 0 if n is bigger than m remember what is the property of phi.

Phi of x plus 2 is equal to phi x, phi of x plus 2 is phi x for every x, similarly phi of x plus 4 is phi x, phi of f plus 6 is, so if the difference between the two arguments of phi is a multiple of two difference between phi values must be 0. Now, consider the case when n is bigger than m, n is bigger than m the difference between this number and this number we just coming. Here, that is a positive power of 4, positive power of 4 and hence the difference between this must be 0 and hence the difference between this must be 0.

So, we can say that this is 0 if n is strictly bigger than m, if n is strictly bigger than m, if n is equal to m what happens n is equal to m then we can say that let us consider this n equal to m case. Let us first consider n less than m if it is n less than m then it means it is a negative power of, negative power of 4, if it is a negative power of 4. We can say that between these two numbers that is again what are the numbers beta m will be not only just negative powers power of 4 it is it is like this beta m is 4 to the power 4 to the power n beta m is nothing but 4 minus 4 to the power n minus m.

So, if the difference is the negative power of 4 we can say moreover we also know that those are not the integers, see the point is this. If you take any we are looking at the properties of phi, we are looking at the properties of phi what I want to say is that if n is strictly less than m, if n is strictly less than m then no integer lies between these two numbers 4 to the power n beta m and 4 to the power n alpha m. Let us see how this follows, what is what will be the 4 to the power n alpha 4 to the power n alpha m it will be remember, let, I think it will be clear if I we actually calculate that.

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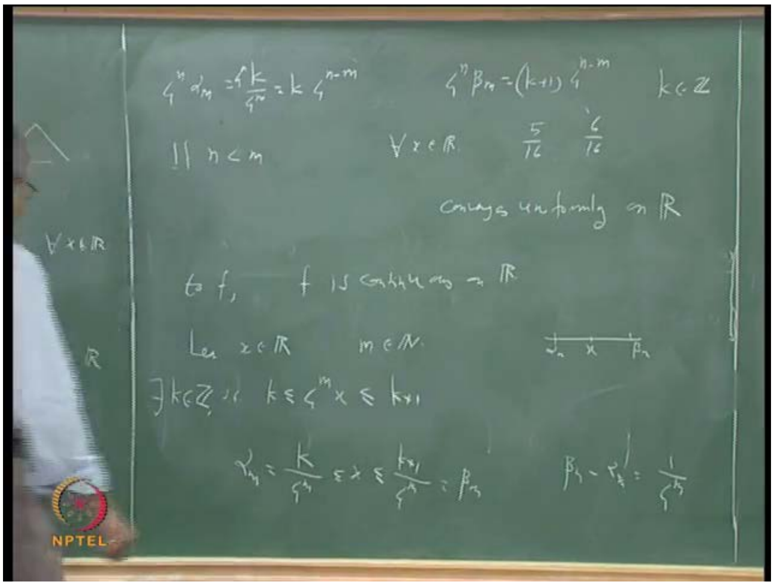

What will be the 4 to the power n alpha m alpha m is K to the power m, so it is K by 4 to the power m into this 4 to the power n, so it is nothing but K into K into 4 to the power n minus m, K into 4 to the power n minus m. Similarly, 4 to the power n beta m that will be k plus 1 into 4 to the power n minus m where k is some integer remembers K is an integer. So, if n is, now let us come up look at this statement if n is less than m then between these two numbers no integer lies between these two numbers, no integer lies how does, how does that follow. For example, you just take the case let us say, let us say let us say n, n minus m is minus 2, let us say n minus m is minus 2 and let us say K is, K is let us say 5.

Then this will be 5 by 4 and that will be 6 by 4, that will be 6 by 4 not 4 6 by 4 square, that is 5 by it will something 5 by 16 and 6 by 16. If n minus m is minus 2, we are considering the case when n is less than m we are considering the case when and that similarly for any other integer it will be, it will be 5. Suppose K is 5 it will be 5 by some power of 16 and this will be 6 by some not some power, 5 by some power of 4 and 6 by some power of 4 between these two numbers there is no integer between these two numbers, there is no integer, now what is the significance of this.

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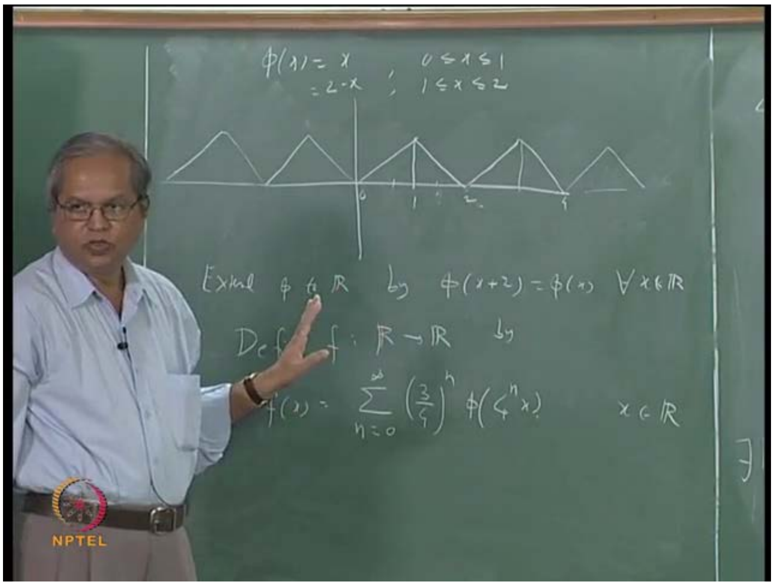

If between these two numbers there is no integer that means, for example something like this is not possible this is alpha m and that is beta m this is not possible. Similarly, this is alpha m that is beta m that is also not possible, what is possible that alpha m and beta m both lie either for example on this part or on this part. Now, which means that alpha m and beta m both are given by I remove that definition of phi x, let us again recall that what phi x, phi x was x 4 and 2 minus x for 1 less not equal to x, less not equal 2.

Remember what is the property of phi 1, property is that it takes the values with 0, 0 and 1 it takes the value 1 at all the odd integers it takes the value 0 at all the even integers and between these integers it takes the value either x or 2 minus x. So, suppose both are either in the interval 0 to 1 or 2 to 3 or 4 to 5 etcetera then both will be x or if both are in the interval 1 to 2 or 3 to 4 or 5 to 6 etcetera. In that case, both will be 2 minus x, in that case both will be 2 minus x.

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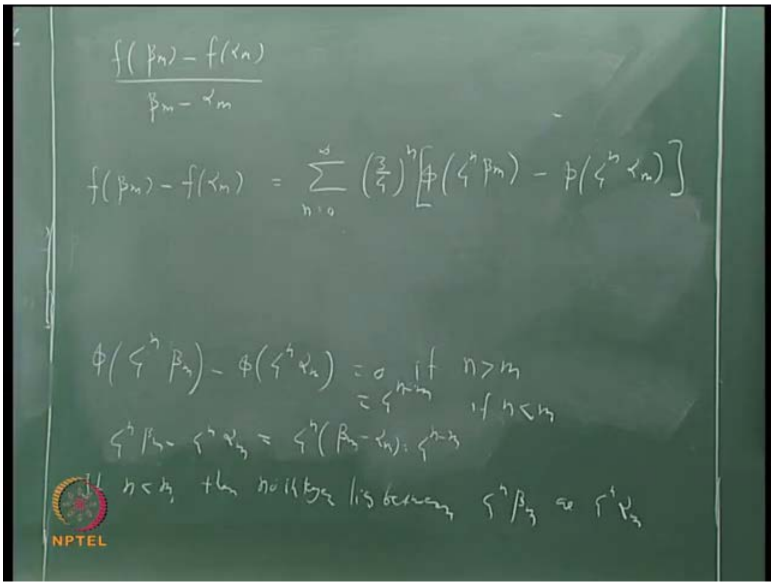

So, that means the difference between their phi values will be same as the difference between the two that is phi of this 4 to the power n beta m minus phi of 4 to the power n alpha m. That will be same as the difference between the two if n is less than m that is what we want to say if, so is equal to I will say that this is equal to 4 to the power n minus m I will remove this if n is less than m what happens. If n is equal to m, if n is if n is equal to m then again if you if you look at this was let me it is 4 to the power n minus m that is 4 to the power 0, 4 to the power 0.

So, in that case see remember it will mean that alpha m is K and beta m is K plus 1, alpha m is K and beta m is k plus 1. In that case, it will mean that the difference between the two values is just 1, either is 1 or 2 or 2 or 3 because it is remember phi is 0 at all odd integers and it is 1 at all even integers. So, between K and K plus 1 one of them will be even and one of them will be, one of them will be, so in that case the difference will be 1, so which is covered by this same formula if you take n is equal to m this will be 1.

So, I will say this is if n less not equal to m, now we have all the things that are required for what we want to do, so first of all since is this clear whatever we have done so far of course it is fairly elementary. Only the thing is you have to little careful about all these values and every place we used the properties of this properties of this phi. So, first thing is that since it is 0 between for n bigger than m it means that this series will go up to 0 to m, this series will go up to only 0 to m.

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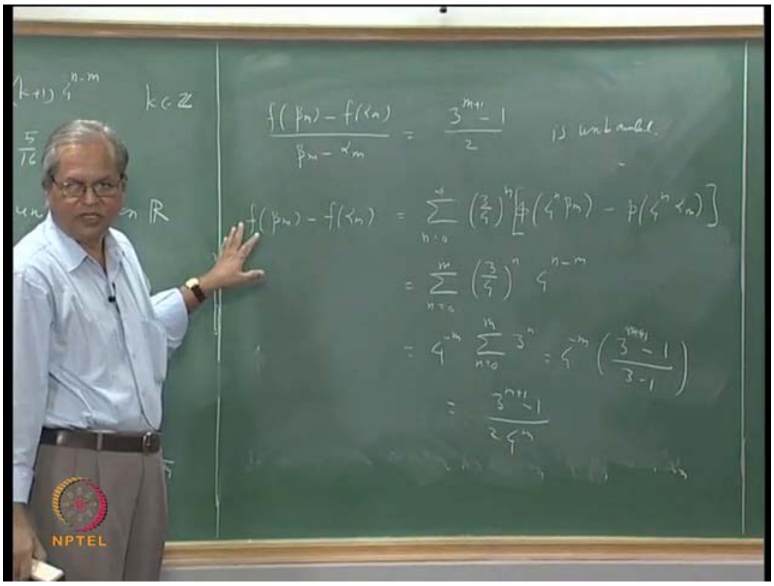

So, this is same as sigma n going from 0 to m because after for n bigger than m the value is 0 that is what we have seen already n bigger than m value is 0 and for n less than m value is given by 4 to the power n minus m. So, this is nothing but 3 by 4 to the power n and multiplied by 4 to the power n minus m I think, now we have everything that we require, once we come to this step everything after this is fairly easy. Now, this is just a geometric progression, geometry progression also finite number of them, so we can easily sum that before that we make one simplification.

Since, here we have 3 by 4 to the power n and, here you have 4 to the power n we can, we can, in fact this we can cancel that and we can say that this is nothing but and this 4 to the power minus m. We can take outside the summation say because sum is with respect to n, so we can say that this is same as 4 to the power minus m into sigma n going from 0 to m 3 to the power n right that is correct. So, this is nothing but geometric series with the common ratio 3, first term is 1 last term is same, so we can easily sum that, so this is nothing but 4 to the power minus m multiplied by.

It will be 3 to the power n plus, sorry not n plus 1 m plus 1 minus 1 divided by 3 minus 1 and that is same as see this is 2. So, this is same as 3 to the power m plus 1 minus 1, this divided by 4 to the power m and 2 into 4 to the power m that 2 is there, that is f beta m minus f alpha m, that is f beta m. Now, what we want is this divided by beta m minus alpha m, but what about beta m minus alpha m that is nothing but 1 by 4 to the power m, so this is nothing but and we want to divide by that. So, once we divide this 4 to the power m will cancel this 4 to the power m will cancel, so will get f beta m minus f alpha m that is same as 3 to the power m plus 1 minus 1 divided by 2.

All that we need to show is that this sequence is not convergent, this does not converge to anything as m goes to infinity is in fact it is not a bounded sequence at every convergence sequence should be bounded. But, it is easy to see because 3 to the power m plus 1 will explode as for a large value of m, so this is not is unbounded, this is unbounded. So, this does not converge to anything and, so what does it mean that how did we come to all this that that if f where differentiable at x this sequence should have been convergent and it will converge to the value f prime x.

But, does not happen that means f is not differentiable at x, f is not differentiable at x and we have taken any arbitrary x in R. So, that means f is not differentiable at any real number, so this function f which we have constructed by this fashion whatever we have written, here that is continuous for all values of x, but not differentiable at any x. So, this was just a construction of a function all these discussions started with various applications of this uniform convergence etcetera and I mentioned that we shall also discuss one more theorem.

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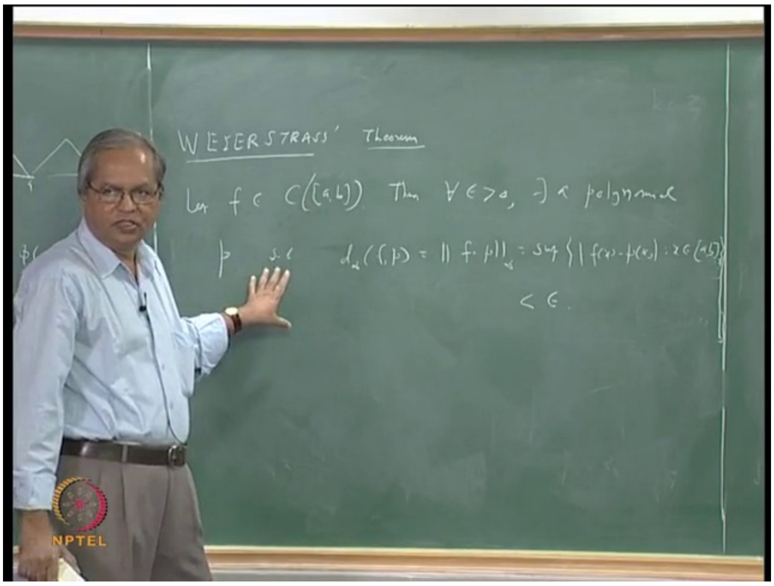

Namely the very famous theorem of Weierstrass which says that if you take a continuous function on a closed and bounded interval then you can approximated by polynomials. In other words, you can find the sequence of polynomials which converges uniformly to that polynomial to that given function.

So, that is what is called Weierstrass theorem, Weierstrass theorem what the theorem says is that suppose you take a continuous function. So, let us say that let f on some interval let f belong to C a b then you can always find the sequence of polynomials on a b such that that sequence p n converges to f uniformly, that sequence p n converges to f uniformly. We can write the same thing in other words then we can say that then for epsilon bigger than 0 there exist a polynomial p such that distance between f and p. That is, this d infinity f p or which we same written as norm of f minus suffix infinity or let us make it even precise that is same as supremum of mod f x minus p x for x in a b.

This is less than epsilon in our, so if you take this epsilon as 1 by n you can find a corresponding polynomial p, p n such that distance between f and p n is less than 1 by n and that sequence, p n will converge uniformly to the given function f that is the idea. It also there are various implications of this theorem, one implication is this suppose you look at the set of all polynomials it follows from this theorem that set is dense in C a b. The set of all polynomials is dense in C a b, second thing is have you, have you heard of a separable metric space what is that, there should exist a countable dense subset if a, if a metric space has a countable dense subset it is call separable.

So, using Weierstrass theorem one can show that C a b is separable, how does one show that we have just, now we have seen that the set of all polynomials is dense in C a b, of course set of all polynomials is not countable. If you take that because there are real coefficients and the real number themselves are uncountable, so set of all polynomials is not a countable set. But, you can take polynomials with rational coefficients polynomials, with rational coefficients that will be a countable set and that will be dense in the set of all polynomials which is dense in C a b.

So, that is that is how one shows that C a b is separable, now to look at the proof of this first important thing to we discussed, here is that we can change this interval a b to any other interval it does not, it does not depend on what is the interval. Hence, we can as well take the interval a b as the interval 0 to 1 that is the, that is the first observation

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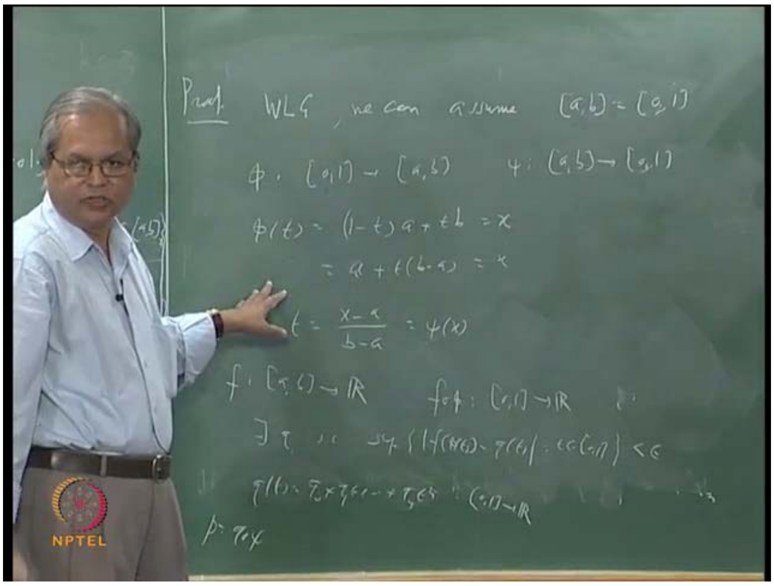

So, as we say, as we use this expression earlier without loss of generality, without loss of generality we can assume that this a is equal to 0 and b is equal to 1 or what is same as saying a b is equal to 0 to 1 of course let us in order to avoid the triviality. Let us take a less than b if a is equal to b then there is just one point in that interval and it that there is only one, then the C a b will be same as R, C a b will be same as R and set of all polynomials also will be same as R. So, that is a trivial case, now why we can assume of course one thing we can say that this two intervals are homeomorphic.

But, of course that does not say why polynomials will be preserved, but in this particular case we can give actual homeomorphism. What is that we can say that either let us say define phi form 0 1 to a b by phi of t just take what is called the convex combination of this phi of t as 1 minus t times a plus t times b call this as x phi of t is equal to x. Then it is obvious that phi is a continuous function and phi is also a bijection, phi is also a bijection we can also prove that phi is a homeomorphism we cannot in fact in this case.

But, in even otherwise you should know that if you are given two compact metric spaces any continuous bijection is a homeomorphism, but anyway in this case we can also see if since this phi t is equal to x. We can also write t terms of x and what will be that this is, this is same as suppose I want to write t in terms of x it will be this will be a minus a plus t into b minus a that is equal to x. That is same as saying that t is equal to x minus a divided by b minus a t is equal to x minus a divided by b minus a and suppose I call this let us say psi x. So, phi is from 0 1 to a b and psi is from a b to 0 1 phi and psi both are homeomorphism and they are inverses of each other phi composed with psi is identity.

Similarly, psi composed with phi is also identity, now what is the argument why we can take the interval to be 0 1 we are given a function f on a b, we are given a function f on a b. If I look, if I look at f compose see f is function from a b to R f is function from a b to R and phi is function from 0 1 to a b, so if I look at f compose with phi that will be function from 0 1 to R. That is f compose with phi, f compose with phi that will be a function from 0 1 to R, that will be a continuous function f compose with phi will be a continuous function.

So, suppose we have proved the theorem, for the interval 0 to 1 it will mean that for this given epsilon there will exist some polynomial q let us say q which is define on 0 to 1 such that f, f compose with phi minus q. The distance between that is less than epsilon that is it will, what it will mean is that there exist there exist q such that supremum of mod f of phi t minus q of t supremum taken over t in 0 to 1. That is less than epsilon where the polynomial q is defined on 0 to 1, where the polynomial q is defined on 0 to 1. So, q is a polynomial let us say, let us say q t is some polynomial q t is something like say q naught plus q 1 t etcetera plus q n t to the power n and this is define from 0 1 to R.

But, what we can, but what let us see, now actually want a polynomial not form 0 1 to R, but from a b to R then what we can, so is that you can, now consider composition of q with psi. You can, now consider a composition of q with psi q composed with psi will be from a b to R, only question is whether q also will be suppose I call that p as q compose with psi. So, what will be that, so p of x will be q naught plus q n t etcetera for the, for the t you substitute this for each t you write x minus a divided by b minus a for t square it will be x minus a square divided by b minus a square.

So, suppose you do all those things it will be a polynomial in x, it will be a polynomial in x and suppose you go f of phi t is nothing but f of x phi t is x. For q of t q of t is nothing but q, q of t is nothing but p of psi x, q of t is nothing but p of phi x and, so that also will be less than epsilon. So, in short you can work out in details yourself what is to be understood as given any continuous function on the interval a to b you can construct the continuous function on the interval 0 to 1 and vice versa.

If the function is a polynomial then the transform function is also a polynomial, transform function is also a polynomial and that is why we can we can straight away start assuming that the given interval is the interval 0 to 1. So, it is enough to prove the theorem for that case, so what is the proof in that case that we shall see in the next class.