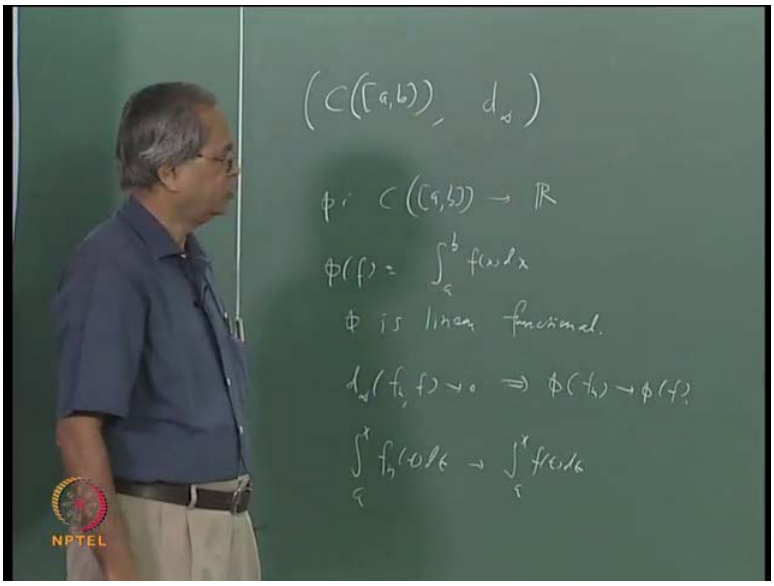
Real Analysis
Prof. S. H. Kulkarni
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 49
Uniform Convergence and Differentiation

So, in the yesterday's class we proved that if you consider a sequence of functions and if each function f n is integrable and if f n converges to f uniformly then f is also integrable. The sequence of numbers integral f n that will converge to integral of f, let us also see the interpretation of this results in language of this metric spaces.

(Refer Slide Time: 00:37)



We have already seen that, for example if you take this space C a b that is space of all continuous function with this matrix d suffix infinity. We have seen that our earlier result about the uniform convergence and continuity that implies that this is a complete metric space and we have also observed quite early that convergence, uniform convergence is basically convergence this metric.

Now, let us see how our yesterday's result of integration can be interpreted in this, now we also know that this C a b is a vectors space. So, one can define a map phi from this C a b to R, let us talk of a Riemann integrals only for the time being as follows at define phi of f as integral a to b f x d x. Then you already know that this is a linear functional, in fact have approved that if following the various theorems of the integration that we have proved phi of f 1 plus f 2 is phi f 1 plus phi f 2 phi of alpha times f will be alpha times.

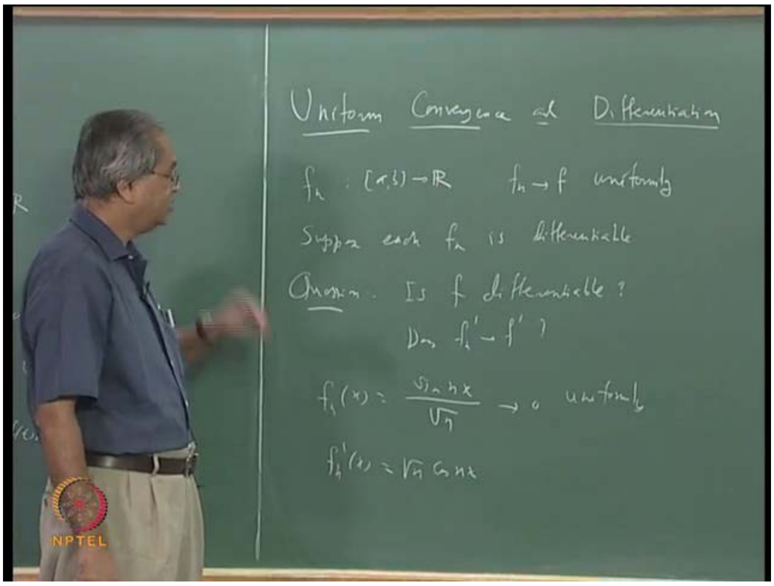
So, phi is a linear function phi that is of course that is something that is already know phi is a linear functional and what did we proved yesterday that if f n convergence to f n uniformly, f n convergence to f n uniformly then integral of f n converges to integral of f. In other words, if f n converges to f in this d infinity that is, that is d infinity f n f tends to 0 this implies phi of f n tends to phi of f, remember phi f and phi f n are numbers phi f and phi f n. So, that means if f n converges to f in this metric then phi f n converges to phi f, now what is the meaning of this it means that phi from this C a b to R is a continuous function.

That is one of the ways of describing continuities is that whenever x n converges to x f of x, n converges to f of x that is equivalent of saying that f is continuous. So, that is what we have said, here whenever f n converges to f in this metric phi f n converges to phi f that is same as saying that phi is a continuous function. Since it is already a linear functional it means that the phi is a continuous linear functional, then there is one more thing. Here, while this while describing the results about the integrals we have take b this upper limit as b, but that is not very much essential one can let this upper limit vary also.

For example, I can consider integral a to x f n t d t, suppose I can say that integral a to x f n t d t and then this will converge to integral a to x f t d t, if f n converges to f uniformly. Whatever you take x in a b then integral a to x f n t d t will converge to integral a to x f t d t, so it means that this upper limit need not big it can be any number. Now, whatever you say about the upper limit, one can say the same thing about the lower limit also instead of a, you can take some other number.

So, consider integral from y to x f n t d t where y and x are any numbers lying in a b that will converge to integral y to x f t d t, so there is nothing very particular about this numbers a and b. In other words, you should take any sub interval of the a b that integral of f n over that sub interval will converge to integral of f over that same sub interval that is about the integration.

(Refer Slide Time: 04:48)



Now, let us go to the discussion of uniform converges and differentiation and differentiation one can ask an analogous question. Here, suppose we have a sequence f n, f n from a b to R and suppose f n converges to f uniformly, f n converges to f uniformly then suppose each f n is differentiable. Suppose each f n is differentiable then what we might except that if each f n is differentiable then we can expect, we might expect that f also is differentiable and not only that we should also expect that f n prime converges to f prime. We have already seen that this is false as far as point wise converges is concern, but which is also true is that this is the question is suppose each f n is differentiable.

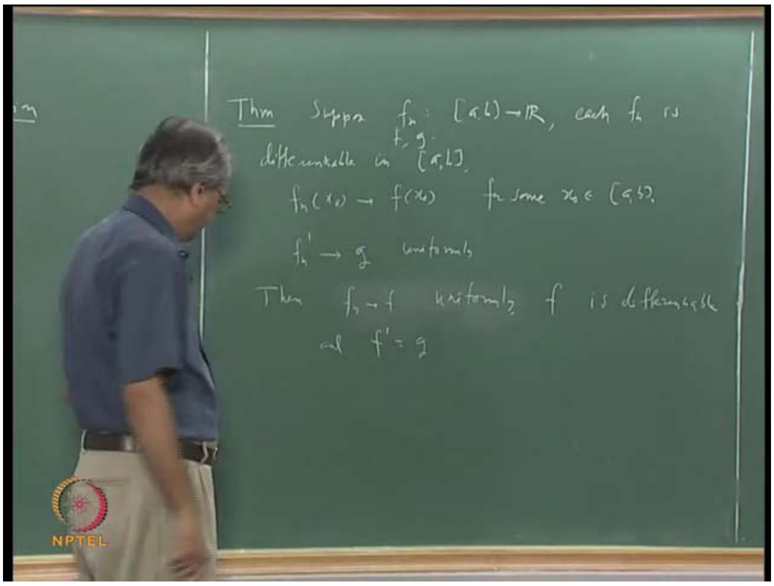
The question is this is f differentiable is, f differentiable and thus f n prime converges to f prime in whatever sense point wise uniformly whatever and answer to both the questions are no. That even if f n converges to f uniformly it does it is not necessary that f must differentiable and even when f is differentiable this may or may not happen. It is fairly easy to construct examples in fact subsequently we will prove a very powerful theorem, perhaps you may have already heard of that theorem. It is called Weierstrass theorem, Weierstrass theorem says that given any continuous function you can find a sequence of polynomials which converges to that continuous function uniformly.

So, you can obviously find the function which is continuous, but not differentiable and such a function also you can a sequence of polynomials. Now, every polynomials is differentiable, but the limit will not be differentiable also it can happen that f n converges uniformly. But, f n prime need not be convergent all f n prime need not be convergent all that is that is fairly easy to show suppose we take this example f n, f n at x is let us say sine n x by root n, sine n x by root n. Then is it, is it obvious that this tends to 0 uniformly whatever interval you take, this tends to 0, uniformly you can take m suffix n will be less not equal to 1 by root n.

In fact m suffix n will be and at m n tends to 0, so that is easy to show, but what about f n prime x. That will be n, n cos n x divide by root n, so that will be root n cos n x root n cos n x and you can say that this becomes unbounded even if you take x is equal to 0,, this will this sequence is simply root n. So, this is not a converges sequence at all and, so as far the derivatives are concern then things are somewhat bad that is even with uniform convergence you cannot assure various things. So, for the derivatives what we have to do is that we have assumed something more namely that not only that f n comma just to f uniformly we have also assumed that f n prime converges uniformly.

Once you assume that f n prime converges uniformly, that is fairly strong assumption then you do not need, even this we can in fact simply assume that f n converges to f point wise or even at one particular point also that will be sufficient. So, let us just see what is the what does the theorem say.

(Refer Slide Time: 09:42)



The theorem is the following, suppose f n are functions from a b to R each f n is differentiable f n is differentiable in a b, as far as the convergence is concerned we will say that two things for some x naught f n x naught converges to f x naught. Let us say that is f n x naught converges to f x naught, for some x naught in a b and f n prime converges to g uniformly f n from we have to take this 3 function. Also f n, f and g all these are functions from a b to R f and d, so what are the assumptions each f n is differentiable if at some point f n at x naught comma just to x naught then f n prime converges to g uniformly.

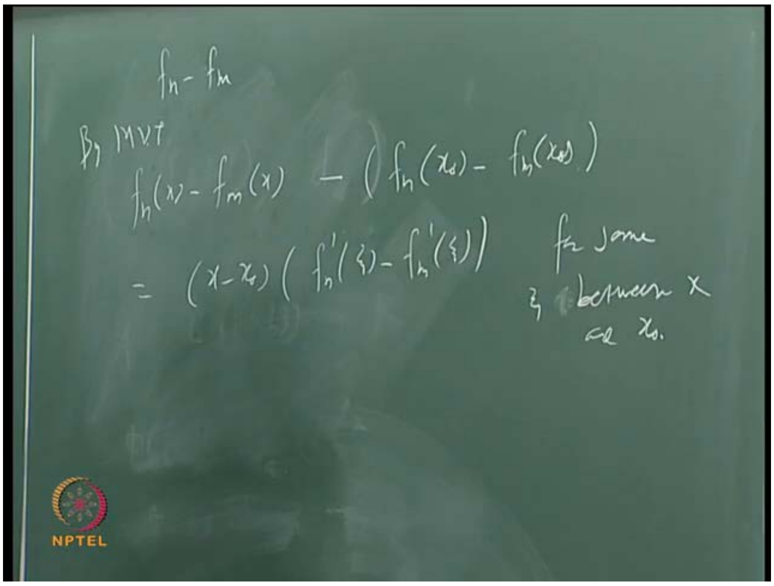
Where f n prime converges to g uniformly then what is the conclusion, conclusion is that if that is the case f is f is differentiable and its derivative is same as g. Then these are all hypothesis then f is differentiable, f is differentiable. In fact even before that I would say that, since that is not we have assume f n then f n converges to f uniformly, f n converges to f uniformly f is differentiable and f prime is equal to g.

Actually if we make one more stronger assumption, suppose we also assume that f n prime is also continuous for each n then the proof becomes very easy. Then all we can do is that we can use this previous theorem we can, since f n x naught is not converges to f x naught for any x you integrate from x naught to x. So, consider x integral of x naught to x f n prime t g d t that goes to g t d t and then show that because of the earlier theorem integral of x naught because we have seen that upper limit and the lower limit can be anything.

So, I can take integral of x naught to x, instead of f n I will take f n prime by this theorem it will mean that integral of f n prime t d t will converges to integral g t d t. Then it is easy to show that that is same as an integral of f n prime between x naught to x is nothing but f n of x minus f n of x naught and we have already known that f n of x naught comma is just to f n of x naught. So, using that you can clearly easy to show that f is differentiable and its derivative is same as g, but we if do not assume that f n is f prime is continuous then the proof is slightly involved.

In the sense we need some more work to do and that is what we shall, we shall do again the idea is that let us, let us take the things one by one, first thing is we want f n converges to f uniformly, f n converges to f uniformly. Then we have seen that one of the ways to show that sequence of function converges uniformly is that you show that it satisfies Cauchy criteria you, sorry it satisfies Cauchy criteria then what do we know about f n. We only know that f n, at x naught if converges at other points we do not know to show that it satisfies Cauchy criteria what will I do to show is that.

(Refer Slide Time: 14:31)



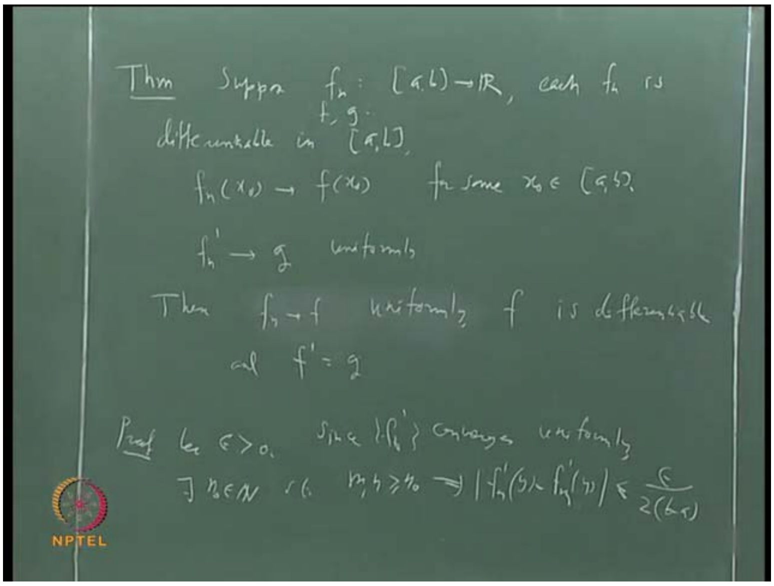
That is if you look at f n x minus f m x we have to show that this quantity can be made arbitrary small for by choosing n and m sufficiently large. That small number should be independent of x, in other words what I should show is that let us say that mod f n x converges f n x is less not equal to some alpha m for every x in a b. Then that alpha m goes to 0, but what we know is that f n x naught converges to f x naught, so if I take f n x naught minus f m x naught that is going to be small. So, our idea is basically we add and subtract and, similarly what we know is that f n prime converges to g uniformly, so this f n prime satisfies the Cauchy criteria.

So, instead of f n f n x minus f m x, if I have taken f n prime x minus f m prime x that that is easy to make that particular f n is arbitrary small, but basically they use all these facts. So, what is the idea will take this minus this f n at x naught minus f m at x naught suppose we are able to show that this can be made arbitrary small an independent of x. Then is it clear that then it will be in this is also small because I am only adding the small quantity, here I am adding a small, but what can we say about this.

These is nothing but if you look at the function suppose if we look at the function f n minus f m if you look at the function f n minus f m. It is a difference between the values of this function at x and x naught it is the difference between and, moreover this is a differentiable function. So, one can use mean value theorem, one can use mean value theorem and say that this will be nothing but x minus x naught into the derivative of this at some point lying between x and x naught. So, we can say that it is same as by mean value theorem we can say that by Lagrange's mean value theorem this is same as x minus x naught into f n minus f m.

Derivative of this at some point zi let s u say which is same as saying that f n prime zi minus f m prime zi for some zi in zi between x and x naught let me see because I do not know which is bigger and which is smaller between x and x naught. Now, the idea is this we know that this quantity f n prime zi minus f m prime zi that can be made arbitrary small. That will be arbitrary small independent of this zi whatever it is zi because we know that f n prime converges uniformly, f n prime converges uniformly. So, this satisfies Cauchy criteria so let us start from this that is he starting point of the proof.

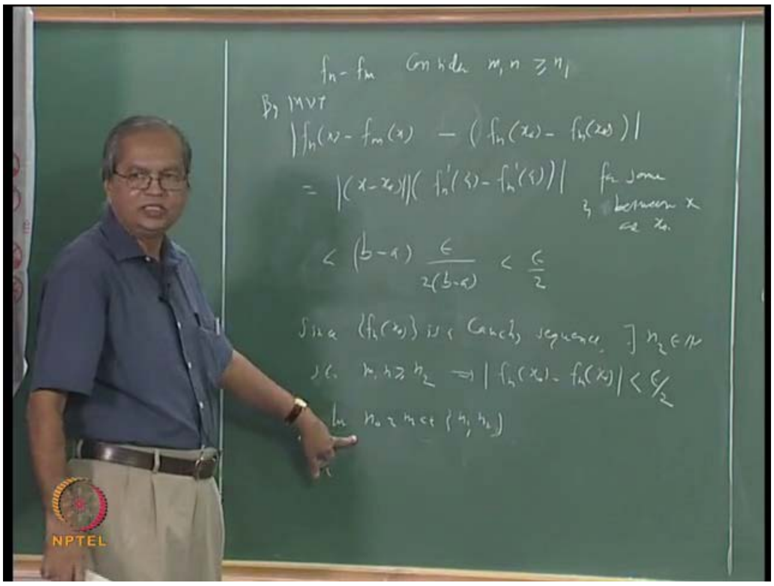
(Refer Slide Time: 18:00)



So, we can say that, let me, let me write it here, so let us say that let epsilon be bigger than 0 since f n prime converges uniformly, since f n prime converges uniformly. You can say there exist n 0 in n such that m and n bigger not equal to n 0, m and n bigger not equal to n 0 this implies mod f n prime. Let me call it y f n prime y minus f m prime y this can be made less not equal to any arbitrary number for every y in m I will take that number as because here we had to add and subtract this quantity.

So, it will make that less than epsilon by 2, but here the multiplication is there also by x minus x naught, so this will remain bounded by b minus a. So, we will say that this is, we shall we shall choose n 0 such that whenever m and n are bigger not equal to n 0, this is less than epsilon by 2 times b minus a. Now, consider some m and n which are bigger than not equal to this n 0, consider m and n bigger not equal to this n 0.

(Refer Slide Time: 19:51)

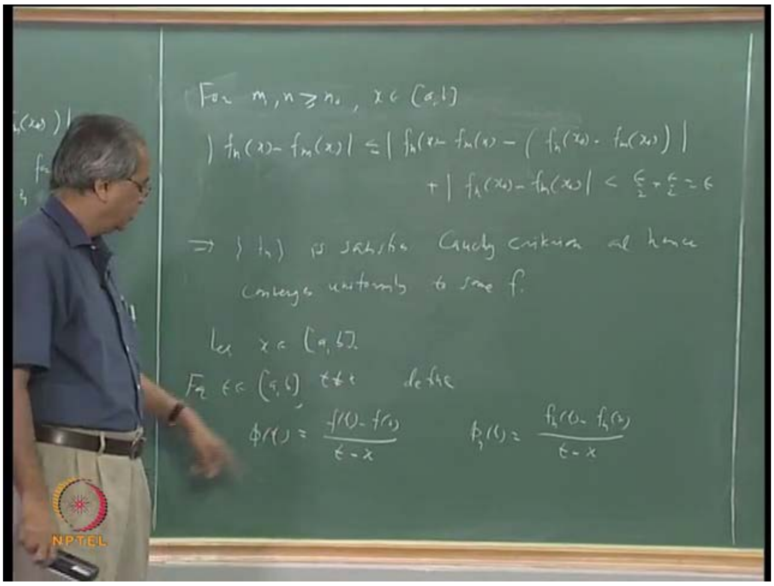


Let me take it, here consider m and n bigger not equal to n 0 then for those m and n we have done all these calculations of course this is true for any m and n. But, now will be apply for those m and n bigger not equal to n 0, i will take f 2 value everywhere. So, this mod x minus x naught x and x naught both are points at the interval a b, so mod x minus x naught is less not equal to b minus a mod x minus x naught is less not equal to. So, this less not equal to b minus a and this is less than epsilon by 2 into b minus a, so which is the whole things is less than in fact this is strictly less.

So, I can take it strictly less than epsilon by 2 alright what next we can say that since f n converges to f x naught I can again do the same I can find, I find some. Let us say n 0 prime such that, here itself let me call it b n 0 prime, so let us this is say n 0 double prime such that this I think this primes are unnecessary complicating. Let us call it n 1, let us call it n 1, so consider this m and n bigger not equal to n 1 what I would say is that there will exist n 2 such that whenever m and n are bigger than n 2, f n x naught minus f n x naught is less than epsilon by 2.

So, what is the argument since f n x naught since the argument is f n x naught is a convergence sequence hence it is a Cauchy sequence. Since f n x naught is a Cauchy sequence, Cauchy sequence there exist into n such that m and n bigger not equal to n 2 or this implies mod of f n x naught minus f m x naught is less than epsilon by 2. What is to be done after, this is clear you take n 0 as maximum of n 1 and n 2, let n 0 as maximum of n 1 and n 2 then for m and n which is bigger not equal to this n 0 both of this inequalities then satisfies.

(Refer Slide Time: 22:52)



Let us continue here, so what we say is for m and n bigger not equal to n 0 consider a term and x is a b consider mod f n x minus f n x and all that we do is add and subtract f n x naught minus f n x naught add and subtract. So, this is less not equal to mod f n x minus f m x minus f n at x naught minus f m at x naught and plus mod f n at x naught minus f m at x naught. Each of this is less than epsilon by 2, this is less than epsilon by 2 because of these considerations, here remember the first part is less than epsilon by 2 because of this second assumption f n prime converges to g uniformly.

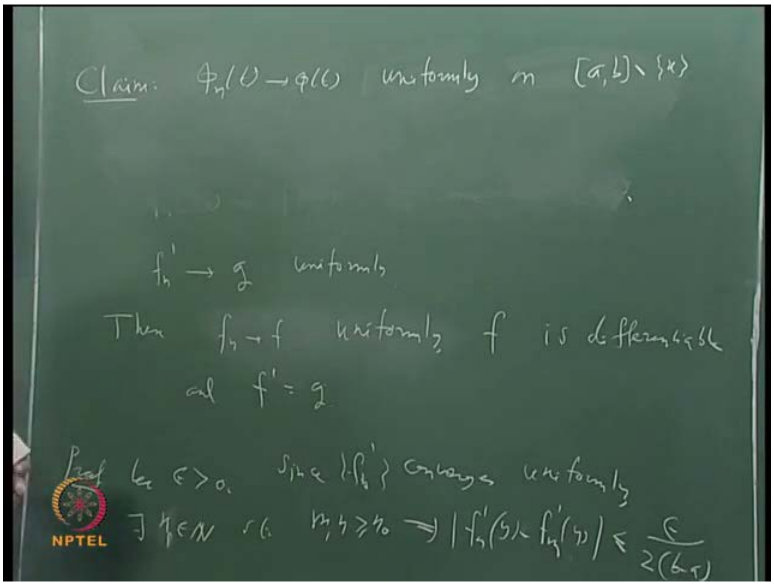
Second is less than epsilon by 2 because of the first assumption that f n x naught is converges to f of x naught, so this is less than epsilon by 2 plus epsilon by 2 which is epsilon. So, what we will show that given any epsilon there exist n 0 such that wherever m and n is bigger not equal to n 0 mod f n x minus f m x is less than epsilon for every x in n which is same as saying that f n is uniformly Cauchy. So, this implies that f n is uniformly Cauchy or f n satisfies Cauchy criteria, Cauchy criteria and hence converges uniformly, and hence converges uniformly to let us say some f.

So, converges uniformly to sub that is whatever it function it converges that function we shall call that function we shall call f. Now, what remains is to show that this f is differentiable and its derivative is same as that function g, what is the function g. It is the function to which this f n prime converges uniformly that is what is remaining to be shown. Now, to show that let us proceed as follows, so let us, let us take some x in a b, let us take some x in a b and we shall procedure a function for let us for t in a b, for t in a b and t naught equal to x for t in a b and t naught equal to x define several function.

First function I will define is phi of t, phi of t that we will define as follows, if t minus f x divided by t minus f t minus, f x divided by t minus x and I will define phi n of t minus f n x divided by t minus x. You can understand why these functions are defined because we are interested in differentiability, so to show that we want to show that f is differentiable we want to show that f is differentiable. That is same as showing that limit of phi t as t tends to x exist limit of phi t as t tends to x exist what about phi n t, in case of phi n t we know that limit of limit of phi n t as t tends to x exist.

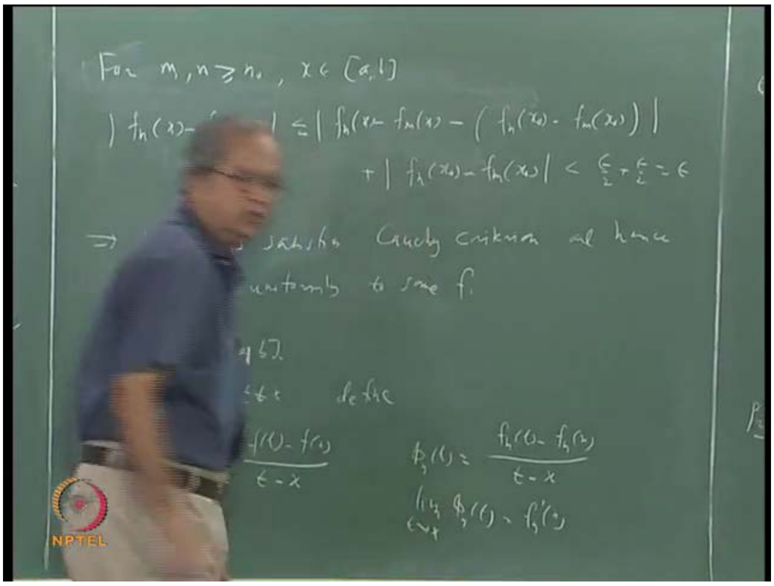
That is same as saying that f n is differentiable that is something we have assume, so this limit is exist and that is something that we know already. So, let me just say that know the idea is as follows suppose we show that phi n converges to phi uniformly in this interval a b except at t naught equal to x. Then of course that then this x is a limit point of this a b minus x, remember our theory about the limit let me, let me first say what we trying to do.

(Refer Slide Time: 28:18)



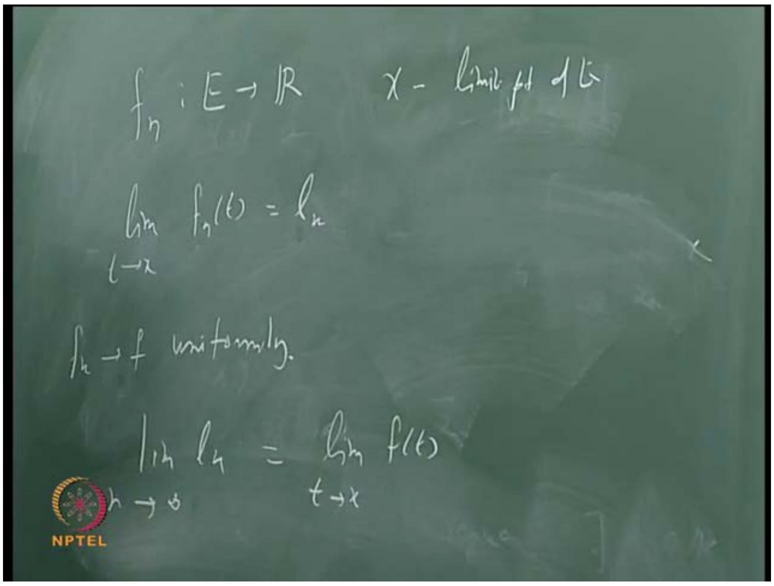
We will claim this you can, I will say that phi n t converges to phi t uniformly not on the whole interval a b, but in the interval a b minus x uniformly on the interval a b minus this x singleton x. Let me remind that out theorem about the limit we have said that if f n converges to f uniformly on e and x is a limit point then limit of f n as t tends to x goes to limit of f x as t tends to x. So, suppose this is proved, suppose this claim is true then one can say that limit of phi t as t tends to x and see limit of limit of phi n t as t tends to x is nothing but f n prime at x is that clear that is limit of phi n t as t tends to x is nothing but, f n prime x.

(Refer Slide Time: 29:42)



Let me remind you that theorem once again, in case you have forgotten.

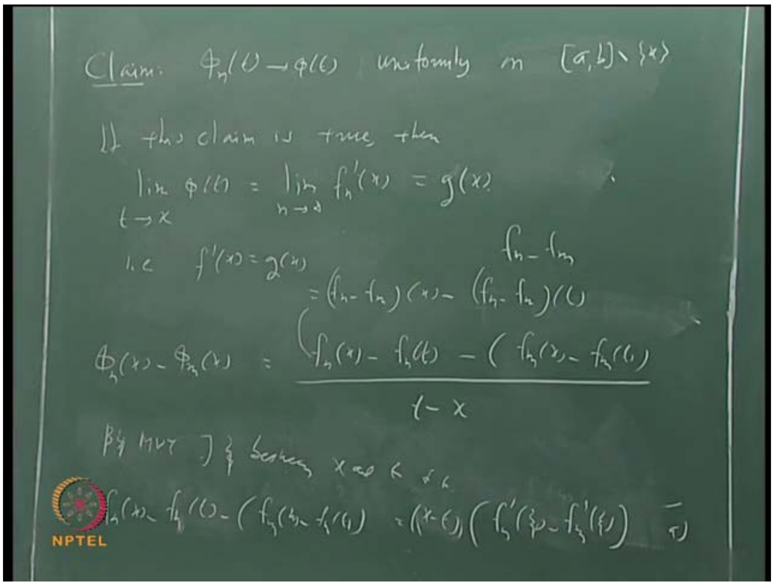
(Refer Slide Time: 29:49)



The theorem was the following f n even before proving the theorem we have taken the functions f n from e to e to r and t x was a limit point of e and we have said that limit of f n t as t tends to x. We have taken these numbers at some l n, we have taken these numbers at some l n and what was the assumption that f n converges to f uniformly. If f n converges to f uniformly then we have say was that limit of this l n as n tends to infinity exist also that is that limit is same limit of this l n as n tends to infinity exist.

Limit of this f t as t tends to x exist and that limit is same as this limit this was our, in fact our first theorem of the uniform convergence. This is same as limit of f t as t tends to x we are going to use that theorem for the functions phi n and phi.

(Refer Slide Time: 31:07)



Suppose we show that phi n t converges to phi t uniformly and limit of phi n t as t tends to x axis that is same as f n prime x, so if this claim is prove if this. Let us if this claim is proved or if this claim is true if this claim is true then what must happen is that then limit of phi t as t tends to x this limit should exist. This limit should exist and that should be same as limit of this f n prime x as n tends to infinity limit of f n prime x as n tends to infinity remember this f n prime x is something like this l n f n prime x is takes the place of this l n.

That is limit of f and t as t tends to limit of phi n t as t tends to x is f n prime x, but, we have already assumed that f n prime converges to g. So, this right hand side is nothing but g prime x, right hand side is nothing but g prime x, sorry g of x I am sorry you are right it is nothing but g of x. So, what it says it that this limit exists and that limit is same as g of x which is same as saying that f is differentiable. Saying that this limit exist is same as saying that f is differentiable and that limit is same as g of x is saying that f prime x is equal to g g x that is that is f prime x is equal to g of x.

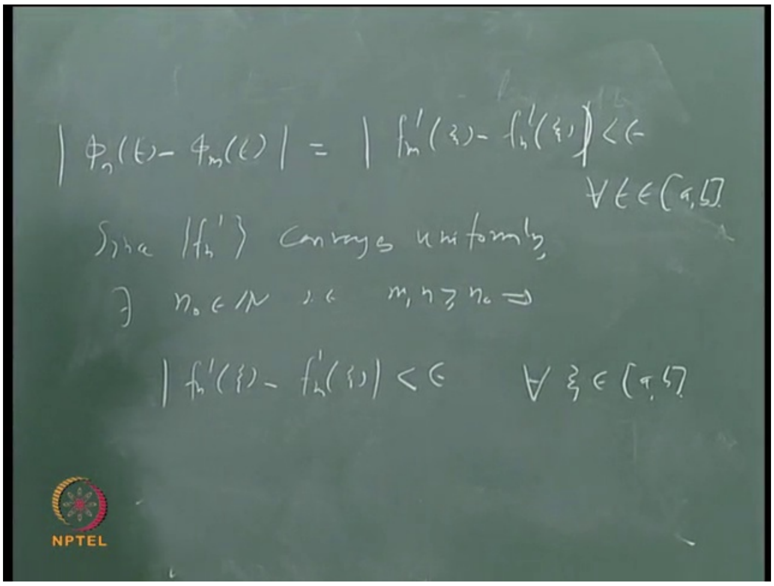
So, what remains, now what remains is to prove this claim we assume this claim is proved then whatever we wanted would be proved first this claim is true, once the claim is true the proof or theorem is over. Now, what does what is involve, here we want to show that phi n converges, see remember phi n t converges to phi t point wise there is no problem about that, about that. There is no problem because we have already shown that f n comma just to f uniformly, so f n t will converge to f t f n x converges to f x.

So, only thing reality to show is that phi n uniform convergence and we have the same thing we have done in the earlier that is we shall show that phi n is a satisfies Cauchy criteria phi n, satisfies Cauchy criteria. So, if we consider let us say phi n x minus phi m x phi n x minus phi m x then that is same as f n x minus f n t then minus f m x minus f m t this divided by t minus x. We shall use the same thing or same technique which we have used earlier, namely mean value theorem because if look at this f n x minus f n t and f m x minus f m t these are nothing.

But, the values of the function f n minus f m at x and t if you look at again the function f n minus f m you have f n x minus f m x and then minus f n t minus f m t. So, what we can say is that by the same by mean value theorem we can say that, by mean value theorem there exist a zi between x n t such that such that f n x minus f m t. Then minus f m x, sorry f n x minus f n t minus f n x minus f n t that is nothing but t minus x into f n prime at zi minus f m prime at zi, we are applying mean value theorem to this function f n minus f m.

Remember, here we have f n minus this whole thing can be written as this numerator is nothing but f n minus f m at x minus f n minus f m at t what we are saying this will be x minus t fine. This will be x minus t fine, it will not matter because we are going to take absolute values, so what, so finally what we will get is this t minus x will cancel with that x minus t with a negative sign. So, since we are going to take absolute value what we will get is this.

(Refer Slide Time: 37:09)

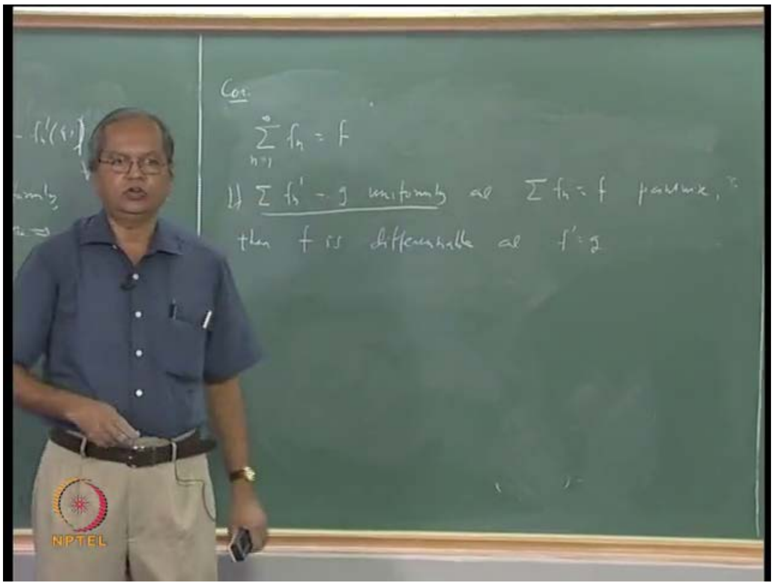


Mod phi n x minus phi n, sorry mod phi n t minus phi n t this will be equal to mod f m prime zi minus f n prime zi for some zi lying between t and x. Do you agree after applying mean value theorem and after cancelling this t minus x and x minus t etcetera. Now, what is to be done after this is clear we already know that f n prime converges to g uniformly, we already that f n prime converges to g uniformly. So, we can always find given epsilon, we can always find some let me just write since f n prime converges uniformly there exist n 0 in n such that m and n bigger not equal to n 0.

This implies mod f n prime zi minus f m prime zi is less than epsilon for zi in a b that is important for all zi in a b. So, going back here since this is less than epsilon it will mean that this is also less than epsilon for since this happens for every zi this is less than epsilon for every t in a b, for every t in a b. It is basically an application of mean value theorem once again and this shows that this sequence f n is uniformly Cauchy, the sequence f n is uniformly Cauchy.

Hence, it follows again that phi n converges to phi uniformly we already know that phi n converges to phi point wise, now we got phi n converges uniformly. So, the limit function must be same as phi and as we have already observed once this claim is prove the proof of the theorem is this over. So, this says about the relationship between uniform convergence and the differentiation and as in the earlier case we can also ask similar questions about the series that is till, now we have taken f n converges to f.

(Refer Slide Time: 39:53)

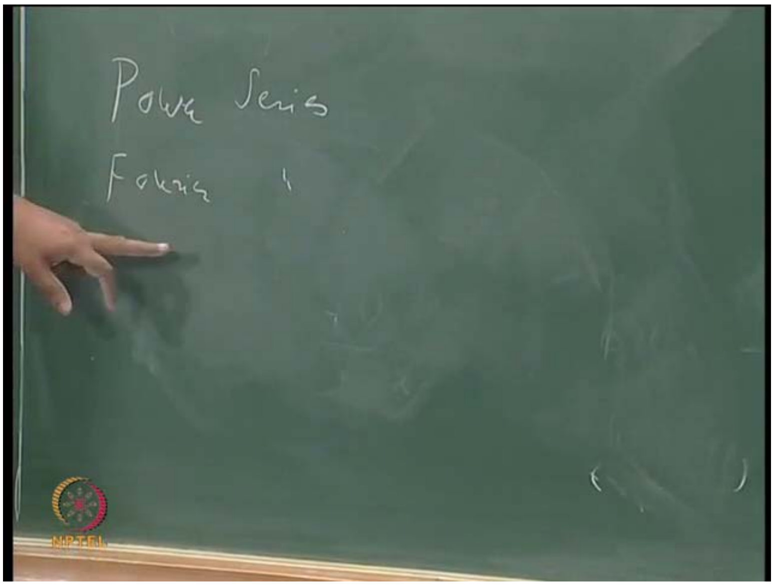


Suppose we look at the series like this sigma f n converges to f sigma n going from about to infinity suppose, here the convergence is uniform. But, then also you cannot say immediately that you can differentiate the series term by term it does not follows at sigma f n convergence to sigma f n prime converges to f n. We need additional condition, namely that sigma f n prime should also converge uniformly and this original series should converge at least at one point then we can differentiate the series term by term. So, let us take record those observation, so corollary is this if sigma f n prime converges to f uniformly let us say not f.

Now, let us call this g uniformly and sigma f n is equal to f point wise, let us say actually it is enough to say at least at one point by using that theorem then f is differentiable, then f is differentiable. Its derivative is nothing but g, its derivative is nothing but g and f prime is equal to g in other words sigma f n prime is equal to f if sigma f n is equal to, f sigma f n prime is equal to f prime. Which basically means that you can interchange the operations, summation and differentiation that you can do provided this happens sigma f n prime equals to g?

That is the series obtain by differentiating each term must be uniformly convergent, then only you can interchange the summation and differentiation. Now, let us see how all these theorems which we have considered about the uniform convergence and various uniform convergence and continuity. Uniform convergence and integration and differentiation can be used, there are several uses of this there are two major things that are coming into picture those are certain kind of series and in proving that those series are uniformly convergent take new continuous functions, new differential functions etcetera.

(Refer Slide Time: 42:25)



One set is what is called power series something that you have heard of already power series and other type of series are what are called Fourier series. Now, these are also meting like power series, but here the function involved are polynomials and, here the functions involved are what are called trigonometric polynomials.

So, these are the positive results that will follow from the discussion of uniform convergence there is also, some one very interesting result which in some sense is negative. But, it is interesting in the sense that it says something different from our usual intuitions says and that is a following, that is if you look at by if you say that something is a continuous function you are expected to think.

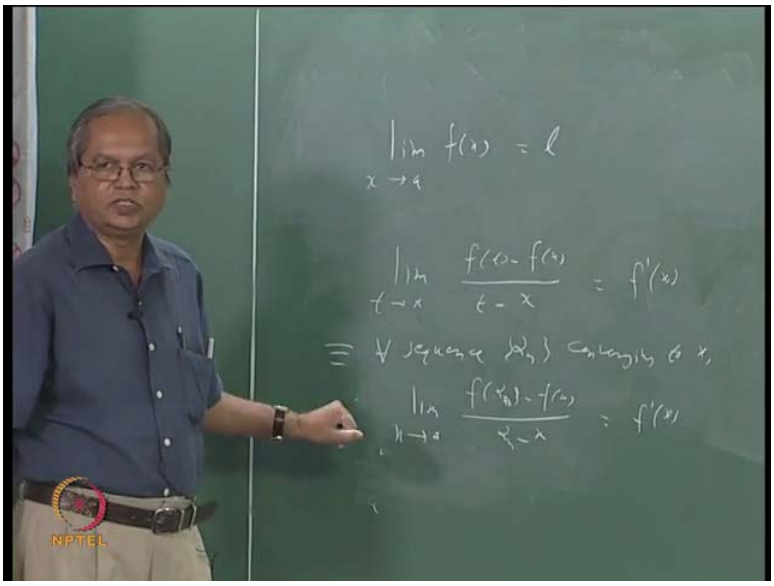
(Refer Slide Time: 43:25)

That it is graph will be something like a curve, so if you have that kind of idea then it is then it will appear that whenever a function is continuous it has to be differentiable also, it has to be differentiable also. But, you already know that that is not true the graph can be something like this, so its function can be continuous, but not differentiable. This is a well know example of modulus, suppose you want a function which is continuous, but not differentiable at two points when one can have a function like this.

Suppose more than two points it can be a function something like this that is, so it is fairly easy to construct a function which is continuous everywhere let us say. But, not differentiable at two points, three points etcetera, but suppose one asks a question does there exist a function which is continuous everywhere but, not differentiable. Anywhere that does, there exist a is that, is that question clear does there exist a function which is continuous for all x in R, but not differentiable for any x in R.

Then we will say that our usual intuitions does not help us in answering that question and one feels that such a function may not exist at all. But, the answer is yes there is a function does there exist a function which is continuous everywhere, but not differentiable at any point and the construction of such function requires everything. We have done, so far about make some preparation by the way it is not very difficult constructing that function is not very difficult what is involve is little bit of calculations. Some use of all these theorems which we have met. Now, to begin with let me first make a few observations one observation which we have already made, let me repeat it once again.

(Refer Slide Time: 45:30)

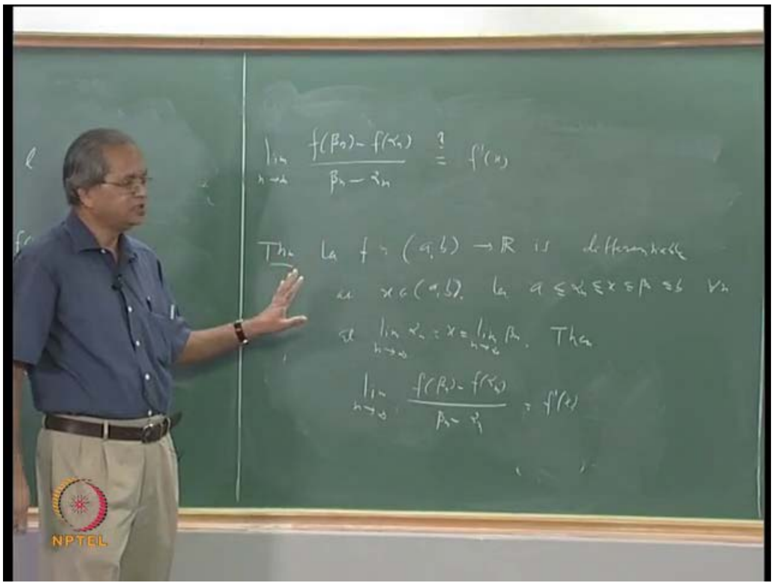


You know that saying that some limit of f x as x tends to a suppose this is equal to n we have already seen that this is equivalent of saying that if you take any sequence x n converging to a then f of x n must converges to l. If you take any sequence x n converging to a f of x n must converge to l suppose I have applied this concept to derivatives, suppose I apply this concept to derivatives suppose f is differentiable at x.

What is the meaning of that, it means that limit of let us say again we will say let the limit of t minus f x at t minus x limit as t tends to x exist suppose f is differentiable then this is same as this limit is same as f prime x, this limit is same as f prime x. How do I convert this using this earlier idea, this will mean that if I take any sequence suppose I call sequence alpha n, suppose I take any sequence alpha n converging to x then f of alpha n minus f x divided by alpha n minus x limit of that as n tends to infinity.

That should be same as f prime x that means let us, let us suppose if that means this is, so this is equivalent to say that for every sequence alpha n alpha n converging to x limit of f at alpha n minus f x divided by alpha n minus x limit of this as n tends to infinity. This limit should exist and that limit should be same as f prime x, now this very easy that follows immediately by applying this principle. But, what we want is, now this a slight modification of this what I want is that suppose instead of one sequence alpha n, suppose we take two sequences alpha n and beta n and suppose both of them are converging to x.

(Refer Slide Time: 47:58)

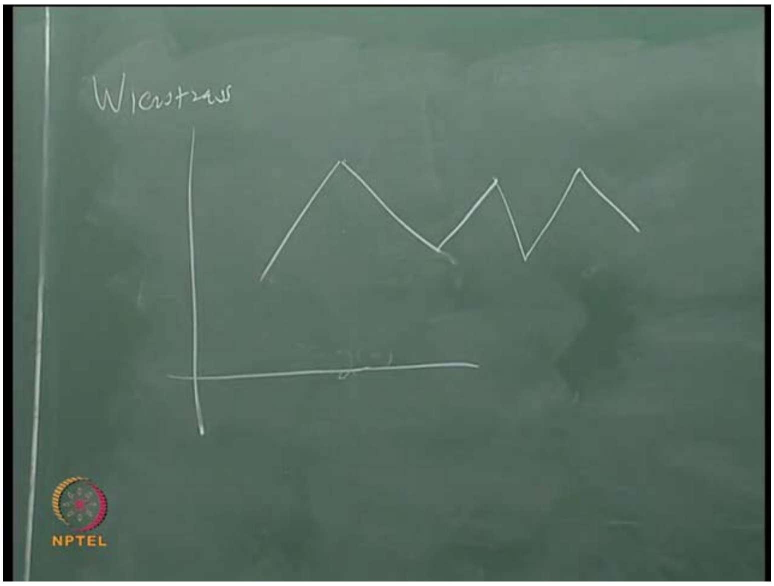


I look at this ratios f at beta n minus f at alpha n divided by beta n minus alpha n and take the limit of that as n tends to infinity. Then what are beta n and alpha n, beta n and alpha n are two sequences converging to x beta n and alpha n are two sequences converging to x. Now, the question is whether the limit of such thing is also same as f prime x that is the question and the answer is in general this is not true if you take alpha n and beta n as totally arbitrary sequences converging to x this may or may not happen. But, if you choose alpha n and beta n in a particular way this is true and what is that particular way let me just explain it.

Now, let us say this for theorem let us say that let f suppose f is defined on some interval a b open interval because we are talking of differentiability we will take open interval f from a b to R is differentiable at let us say some x in a b. We take two sequences alpha n and beta n in the following manner, let a less not equal to alpha n less not equal to x, less not equal to beta n and less not equal to b. That is all alpha n are less not equal to x and all beta n are bigger than or equal to x for all n and both of them converse to x and limit of alpha n as n tends to infinity.

This is x and limit of beta n as enters to infinity is also x then this is true then limit as n tends to infinity of beta n minus f at alpha n divided by beta n minus alpha n this is equal to f prime as and we shall first prove this. Then we shall use this to construct that function which is continuous everywhere, but differentiable nowhere, in fact that was a very famous function first constructed by a Weirstrass.

(Refer Slide Time: 50:54)



Who is famous for several things and this function construction of this function because this said something totally opposite to whatever is the usual intuition of everybody. That led to deeper thinking about what how the function as to be defined etcetera because till that time everybody use to think that function means something defined by a formula. Functions means 2 x cube plus 3 x square of f x is equal to sine x or functions means something defined by that kind of formula.

But, after this example came everybody thought that the concept of function has to be looked at more carefully. Then we have the modern definition of functions after several people contributed to that definition we shall not go to that kind of history. But, we shall, we shall look at construction of this function in tomorrow class because today the time is already over.