Real Analysis
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Lecture - 47
Uniform Convergence

We were to consider what is called Cauchy's criteria for uniform convergence of a sequence of functions.

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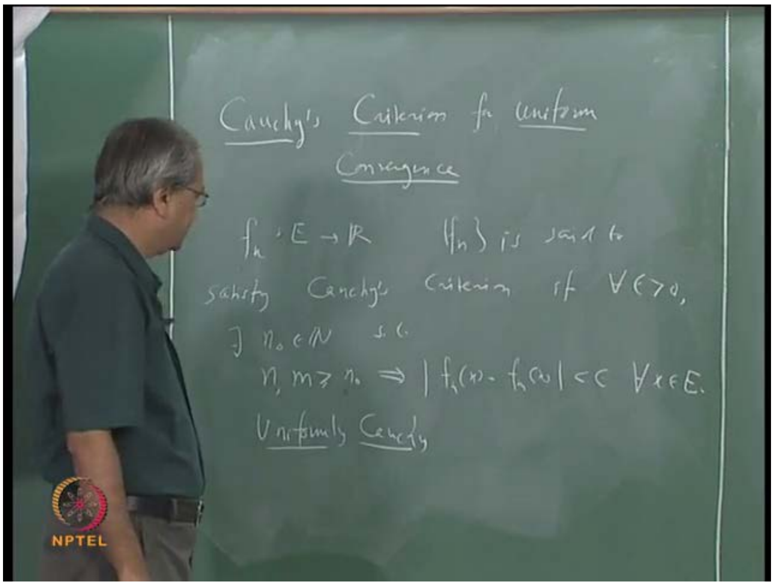

So, let us begin with that Cauchy's criteria for uniform convergence, so as usual we will take a sequence of functions f n from E to R and we shall say that this sequence of function satisfies this Cauchy's criteria. We say that this f n Cauchy's criteria if for every epsilon bigger than 0, there exists n 0 in N such that for all n and m bigger than or equal to n 0, this implies mod f n x minus f m x is less than epsilon. This should happen for every x in E suppose we had simply stopped here it would have simple mean meant that this sequence of real number f and x that is a Cauchy's sequence, but what we want is something more.

What should happen is that for every x in e this mod f n x minus f m x should be that means this n 0 should be again independent of x that that is n 0, what should work for every x such that mod f n x minus f m x is less than epsilon. Sometimes, this sequence is such a sequence is also called uniformly Cauchy, we also say that f n is a uniformly

Cauchy sequence. Now, what does what does the criteria say that if you if you are given a sequence of functions then that convergence uniformly if and only if it satisfies this Cauchy criteria.

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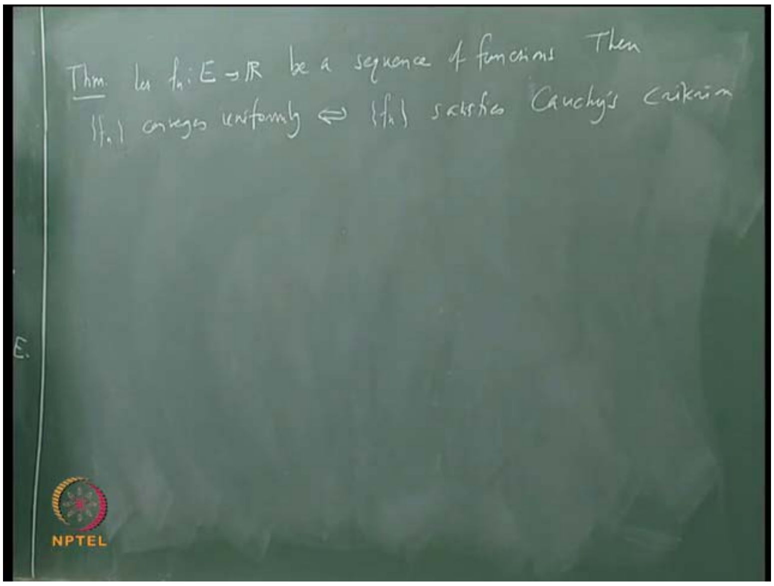

So, that is the theorem let f n from E to R be a sequence of functions, then f n converges uniformly if and only if f n satisfies Cauchy's criteria. In other words, if and only f n is uniformly Cauchy f n satisfies Cauchy's criteria, now before going to the proof of this, let me again say something which I said yesterday.

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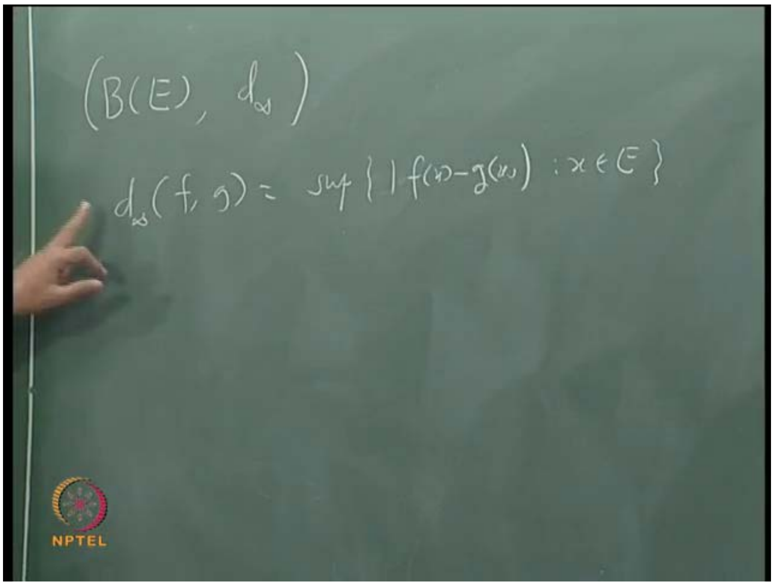

We had defined yesterday this set B of E, this set of all bounded functions on E and on that we have defined the norm and metric induced by the by the norm and that metric. We have d infinity, so what is let us recall what is d suffix infinity that is d suffix infinity of f and g this is nothing but norm of f minus g that is, or which is same as supremum of mod f x minus g x for x in E and that is a that is a metric.

Is it saying that a sequence is uniformly Cauchy or saying that a sequence satisfies Cauchy's criteria is same as saying that this sequence is a Cauchy's sequence in this metric because what this means is suppose you take the supremum over x in E. Then, this says that the d between f n and f m is less than epsilon distance between f n and f m is less than epsilon in that metric. So, of course in any metric space if a sequence converges is a Cauchy's sequence, that is true in any metric space converge is true if it is a complete metric space.

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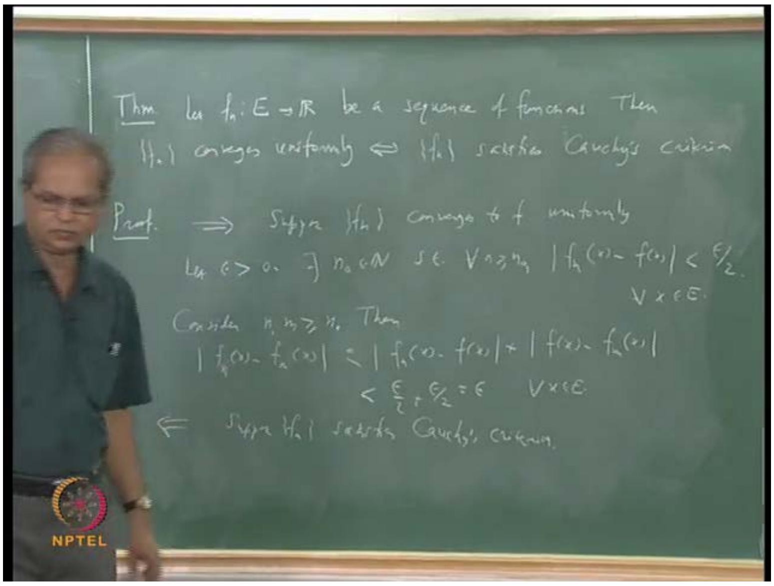

So, this theorem essentially says that this is a complete space, now let us go to the proof this part is relatively easy to prove that is if f n converges to f then it satisfies Cauchy criteria. Again, as I said to show that a sequence converges, it said is it is a Cauchy sequence that a straight forward thing, anyway let us let us go through it. Suppose, f n converges to some f uniformly, then we have to show that it is a Cauchy sequence, to show that it is a Cauchy sequence take some epsilon and show that such an n 0 exist. So, let epsilon be bigger than 0, then since we know than f n converges to f uniformly we can use that. There exist n 0 in n such that for all n bigger than or equal to n 0 we have mod of f n x minus f x less than I can say less than epsilon by 2 and this should happen for every x in E for every x in E.

This a standard way of doing and you take any n and m bigger than or equal to n 0 and look at f n x minus f m x, now consider mod of then mod of f n x minus f m x, how is to be done is clear add and subtract f. So, this is less not equal to mod of f n x minus f x plus f x minus f m x and each of this quantity is less than epsilon by 2, so this is less than epsilon by 2 plus epsilon by 2.

That is equal to epsilon and again since this n 0 was independent of x, this works for every x and E, so whatever we have done about that n 0 should work for every x n. So, this also is true for every x in E, so this shows that f n satisfies Cauchy's criteria that is what is meant by. We want to show is that for every epsilon there exist n 0 such that whenever n and m are bigger than 1 equal to n 0 mod f n x minus f m x is less than epsilon for every x that what we have shown. Now, next is the other way, now we assume that f n satisfies Cauchy's criteria and from that we need to show that it is a uniformly convergence sequence, so suppose f n satisfies Cauchy's criteria.

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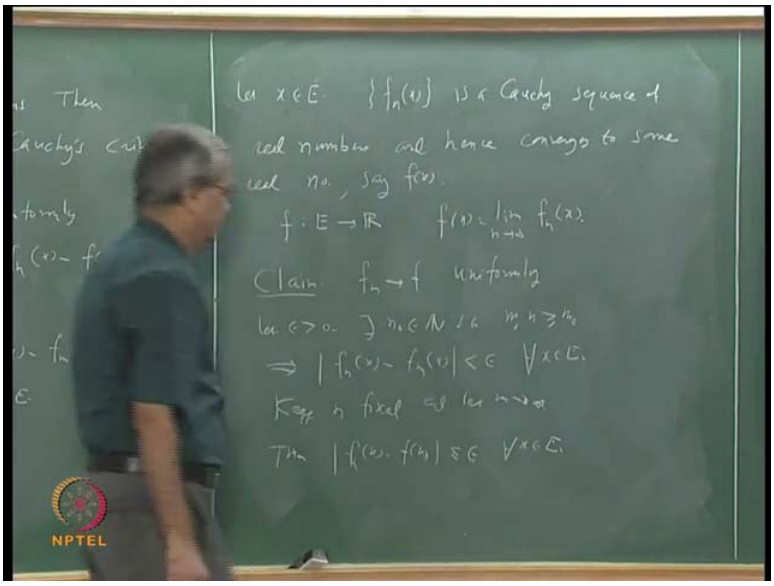

Now, first of all what we can do is that take any x in E and consider this sequence f n x, consider this sequence f n x than that is a Cauchy sequence because for each in E this is satisfied. So, it is same as seen this is a Cauchy sequence of real numbers, so then f n x is a Cauchy sequence of real numbers of real numbers and real line is complete. So, once a sequence is Cauchy sequence of real number it should converse to some real number that real number we shall call f x. Hence, converges to some real number and we call that real number f x, this is this can be done for each x in this can be done for each x in e.

So, that means this defines a function from E to R, so that defines a function f from e to r and what is the how the function is related to the sequence f n each f x is the limit of this f n x each f x is a. So, it is given by this f of f of x is equal to limit and that is to infinity f n x, but at this point we have only said that it is point wise converges because how did we construct it? We started with some x and said that the corresponding sequence is Cauchy sequence of real numbers and hence that has a limit that limit we have taken and that limit we have called f x.

So, with this construction we have only got a function f which is which is a point wise limit of this sequence f n, but that is not what we want what we want is a limit should be uniform limit. So, obviously if at all uniform limit exist that has to coincide with point wise limit it cannot be some different function, so we should try to should this is a uniform limit. So, that means we get one this is we can we can write this as a, f n converges to f uniformly f m converges to f uniformly. Now, let us take again some epsilon bigger than 0 than what we can say is that since we already know that f n is uniformly a Cauchy sequence that is f n satisfies Cauchy criteria.

Using that there exist some n 0, so there exist there exist some n 0 n such that m and n bigger than or equal to n 0 one this implies mod of f n x minus f m x less than epsilon and again this should happen for every x in E. Now, the argument which is given after this is a fairly standard one what we say after this is true for every m and n bigger than or equal to n 0. So, what we do after this is keep n fixed or one of them either m let us say suppose keep n fix and let m vary let m vary means consider all possible values of m bigger than or equal to n 0.

That means let m go to infinity if m goes to infinity what should happen this should go to f x this should go to f x, but this inequalities true for every m that is mod f n x minus f m x is less than epsilon is true for every m. So, in the limit at the most what could happen is that this may become less than or equal to ok. So, this is what, so we can say after this keep and fixed keep and fixed and let m tends to infinity then since it is because of this we know that if m tends to infinity f m, f m x goes to f x, f m x goes to f x.

So, this will go to f x and hence this will become n epsilon of course, this that mod f n x minus f x less than or not equal to epsilon, but you can say this less not equal to does not matter. Here itself we are going to take some number smaller than epsilon and then we could have got less than epsilon so that a minor thing so this means that f n converges to f uniformly.

That is given epsilon we have found some f 0 such that whenever that n is bigger than that n 0 mod f n x minus f x is less than epsilon of course. This again happens for every x in e is that clear, so again let me again comment at that what this theorem actually means is that space b E which we have a taken is a complete metric space.

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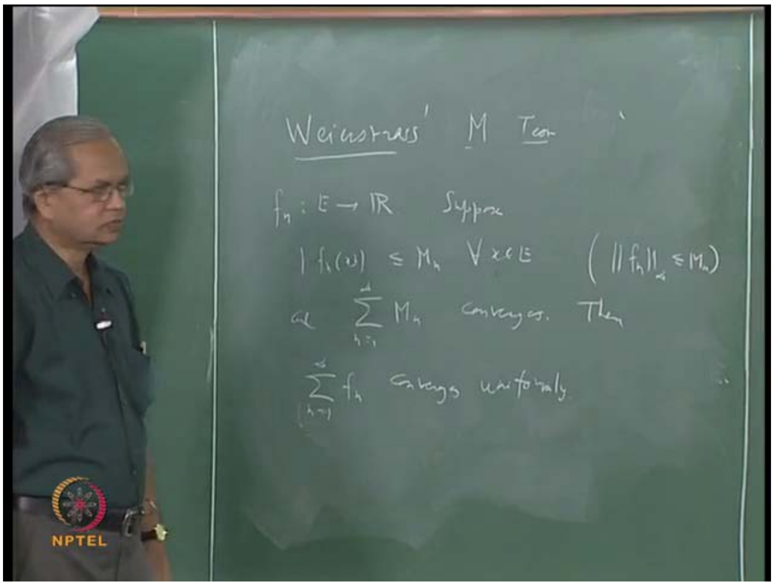

Now, translating this to the convergence of a series we get one useful criterion for convergence of a series it is called Weierstrass m test, Weierstrass is a name. So, Weierstrass m test this is a test which helps in deciding easily about certain kinds of series which are uniformly convergent or not. So, again what we do is let us let us first see what the test says the test is as follows again we take a sequence f n f n from E to R and suppose we have some numbers m n.

Let us say such that mod of f n x is less not equal to m n for all x in e, so suppose in our language this is same as norm of f n is less not equal to m n because norm of f n is less not equal to m n for x in E. This is essentially same as saying that if you look at this norm of f x suffix infinity this is less not equal to m n. So, in in particular you can take this m n as this number itself you can take this m n as this number itself and this is something that you can do whatever this is the sequence given once f n is bounded. We can always talk of this we can always take this m n, but what we want is something more.

Now, m n is sequences of real numbers m n is a sequence of real numbers, so suppose this happens and if you look at this series sigma m n, n going from 1 to infinity converges. Then, the functional series sigma f n that is series of functions then sigma f n n going from 1 to infinity converges uniformly that is what again recall what is involved. There should exist some number m n in fact this will be a non negative real number such that every x in e mod f n x less not equal to m n and the series sigma n m should be convergent that does the series of positive numbers. Then, that is very easy to check in the convergence of the series of positive numbers, so that can be checked very easily and then if that happens we can conclude from this Weierstrass m test that sigma f n converges uniformly.

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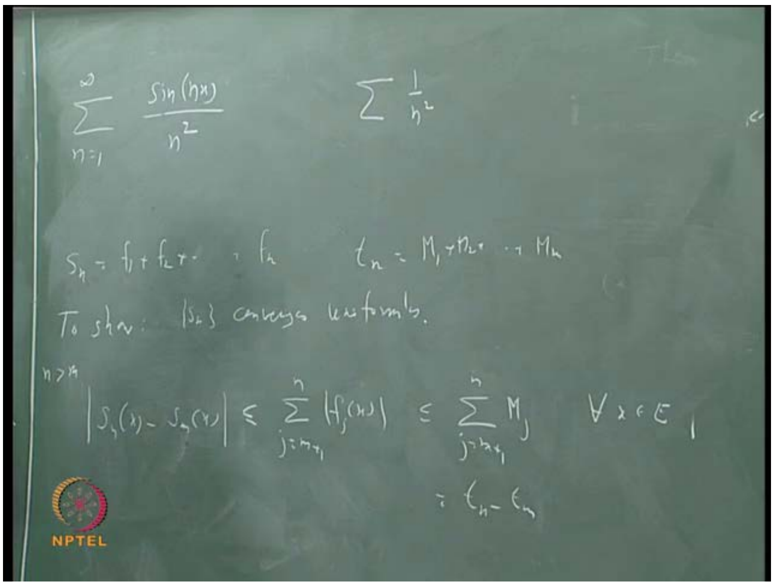

For example, suppose something like this you take the series sigma n going from 1 to infinity. Let us say sin n x by n square that is a series of functions and what is the obvious choice of m n here 1 by n square because each of this mod say that x is less not equal to 1. So, each of this is less not equal to 1, 1 by n square and sigma 1 by n square convergence that is a well known thing sigma.

So, this series of functions converges uniformly and this way you can check the convergence, uniform convergence of a series in a fairly easy manner. So, this is the idea of Weierstrass test this numbers m n you can in most of the cases again, remember we do not have to have m n to exactly equal to this. It can be any number bigger than or equal to this, all that we need subsequently is that sigma m n should be convergent.

So, let us look at the proof of this test it depends basically on what we have proved just now that a sequence convergence if an only which satisfies the Cauchy's criteria. So, there are as you know anything that you want to talk about the series has to be stalked in terms of its partial sums. So, let us say let us take S n is equal to f 1 plus f 2 plus n and similarly let us look at the partial sum of this let us look at the partial sum of this. Suppose, I call that as t n, these are the number m 1 plus m 2 etcetera plus m n, now to show that this series converges uniformly it is same as saying that this sequence S n converges uniformly right that is what i want to show to show s n converges uniformly.

Now, by using the last theorem, we have seen that to show that sequence convergence uniformly it is sufficient to show that it satisfies the Cauchy's criteria. So, that is what we shall show that it satisfies Cauchy's criteria, now to look at the Cauchy's criteria we will have to look at S n minus S m. We have to show that for every epsilon there exist n 0 such that n and m are bigger than or equal to m 0 mod on S n x minus S m x is less than epsilon. So, first of all let us calculate what this S n x and S m x are S n x minus S m x this is same as S n x is because S n x is same as f 1 plus f 2 plus f n and S m x is same as f 1 plus f 2 plus f n.

So, let us say n bigger than m, so suppose we take n bigger than m, then this will be same as f m plus 1 plus f m plus 2 etcetera up to f m plus n. We can say that this is same as sigma let us say j going m plus 1 to n f j x, let us take the absolute value, then this will be less not equal to mod f j x what do you know about this mod f j x that mod f j x. This is not equal to m j for every x in E, so I can say that this is less not equal to sigma, sigma j going from m plus 1 to n m j for every x in it.

Now, what can you say about this number sigma j going from m plus 1 to m n, we know that sigma m n is convergence sigma m n is convergence that is same as saying that t n is in fact this is nothing but this number is nothing but t n minus t m. Since, sigma m n is convergent is same as saying that t n is convergent, but if t n is convergent it is Cauchy it is a sequence of real numbers. Now, that argument after this is standard what you can say is that given any epsilon you can always find n 0, since that whenever n is bigger than or equal to n and m are bigger than or equal to n 0 t n minus t m is less than epsilon.

If both t n minus t m is less than epsilon is shows that mod s n x minus s m x is less than epsilon for every x is e which is same as saying that this s n satisfies Cauchy's criteria. Hence, convergence uniformed that is fine, so far we have just discussed what is meant by uniform convergence of sequence or series and how to check in particular Cauchy's criteria. This Weierstrass m test is a very practical and useful test and using this you can check the uniform convergence of the series of functions in several cases.

Now, let us go back to our earlier problem what we wanted to know was that if each f n has certain properties than does it follow that the limit f also has the same properties at least under the condition of uniform convergence. We have seen that under the point wise convergence that is not true, so let us now check about uniform convergence. Now, if you look at the properties like continuity, then see till now we have taken this set E to be any arbitrary set, we need not bother about what that set E was, but to consider the properties like this. If we want to talk about the limit or continuity etcetera than the whole thing has to happen inside some metric space right now we will take set e to be a sub set of a metric space.

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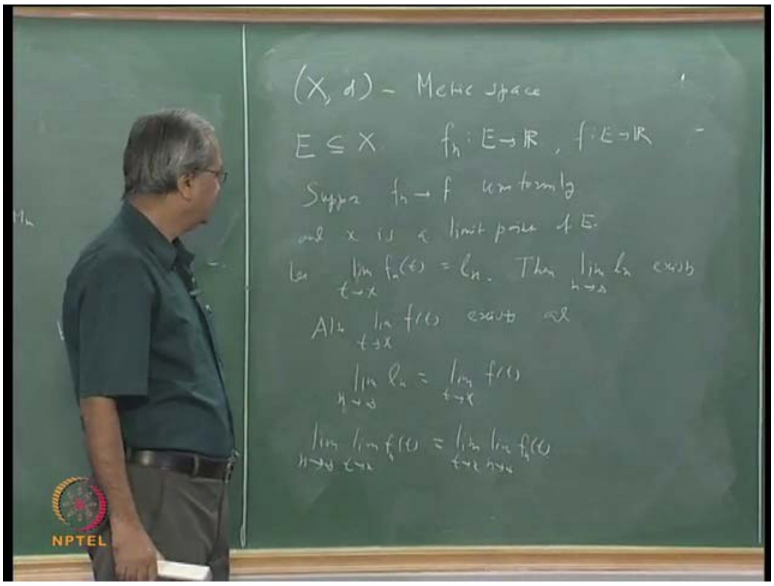

So, let us say that let us say that x d is some metric space and of course, subsequently when we want to discuss the differentiability integrity etcetera. Then, we will have to take it as subset of a real line in fact an interval, but for the continuity still we can take any arbitrary set in a metric space. So, suppose E is a subset of x and f n is E to r f n and f is also from E to R and suppose f n converges to f uniformly. Then, what we want to say is that if each f n is continuous then f is also continuous, but instead of proving that directly since the definition of continuity involves the limits we shall prove something about the limits.

Then, whatever we want to say about the continuity will follow as a corollary, so let us say that suppose this f n has a limit as some point. Now, to talk about the limit suppose I take this point x and i want to turn the limit as f n tends to let us say as t tends to x that this x does not have to belong to E. You know that for talking about a limit the point at which you are talking about the limit does not have to be in the domain, but what it must be at least a limit point because in every neighbourhood the function should be defined.

It need not be defined at the point x etcetera, so let us say that suppose let suppose f n tends to f uniformly and x is a limit point of E and x is a limit point of E and let limit suppose f for each f n limit of f n t as t tends to x exist. Suppose, f n limit of f n t as t tends to x exist and suppose we call that limit l n l suffix n than what we want to say is the following that this sequence l n convergence this sequence l n convergence. Suppose, l is that limit, then limit of f t s t tends to x also exist and that limit is same as this limit of this l n, so then the limit l n as n tends to infinity exist that this is a convergence sequence.

Also, limit of f t as t tends to x exist and both the limits are the same and limit of l n as n tends to infinity is same as limit of f t as t tends to x that is limit of l n as n tends to infinity. This is n as n tends to infinity is same as limit of f t as t tends to x, suppose we write this again l n and f t in there full form because l n is nothing but limit as t tends to x of f n t. So, suppose I write that here that is limit left hand side is same as this limit as n tends to infinity of l n is limit as t tends to x of f f n t.

This is equal to limit as t tends to x and f t is nothing but, limit of f n t as n tends to infinity because that is what we assumed that f n convergence to f. So, f t is nothing but limit as n tends to infinity of f n t, so if you look at the last equation you will understand what I said write at the beginning by discussing about uniform convergence what we are saying is that if f n convergence to f. Uniformly, this limits can be interchanged here, you are taking limit as t goes to x first and then limit as n tends to infinity, whereas here it is other way here you are taking limit as n tends t infinity first and then taking the limit as t goes to x of the same quantity f n t.

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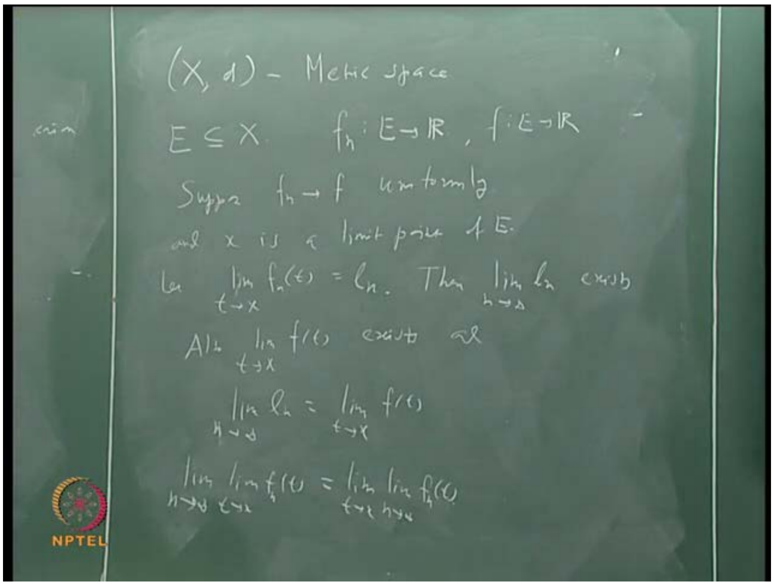

Now, let us prove these things one by one first thing to prove is that l n convergence that is limit as n goes to infinity, but l n is a sequence of real numbers, l n is a l n is a sequence of real numbers. So, to show that it convergence it is sufficient to show that it is Cauchy alright and that we can show by using that since f n convergence to f uniformly it is also uniformly a Cauchy sequence.

So, let us let us start with that so let us say that let epsilon be bigger than 0 than there exist n 0 in n such that n and m bigger than or not equal to n 0 this implies mod f n t minus f m t this is less than epsilon for every t in E right for every t. Now, since this is true for every t in e we can let t goes to x this inequality satisfied for every t, so it will be we know that limit of this as t goes to x exist that is l n limit of this as t goes to x exist and that is l m. So, l n minus l m may not be strictly less not equal to epsilon in the limit in the limit at the most the inequality need not remain strict, but it will remain less not equal to.

So, we can say that letting t tends to x letting t tends to x we have mod l n minus l m is less not equal to epsilon. Now, that shows that this is a Cauchy sequence that shows this is a Cauchy sequence, so this implies that l n is a Cauchy sequence l n is a Cauchy sequence. Since it is a Cauchy sequence of real numbers it is a Cauchy sequence of real numbers and hence it should converge to some real number and hence converges and hence converges to some subset real number l belonging R.

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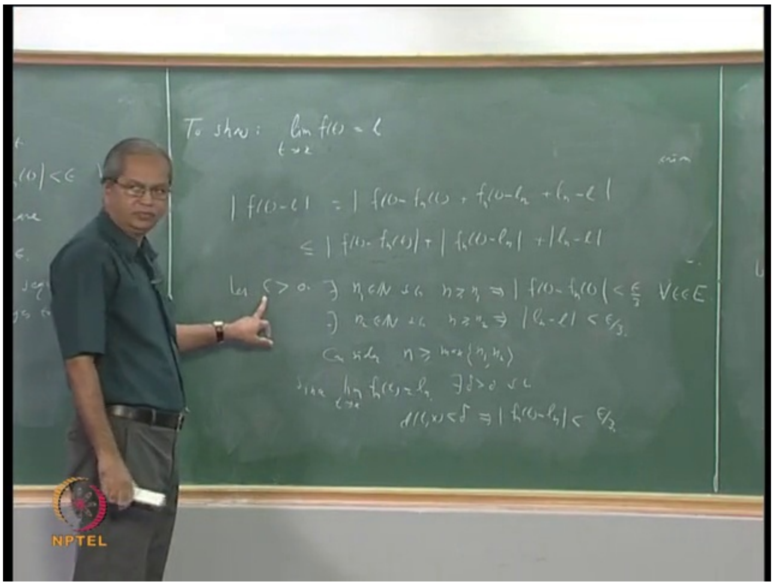

So, we have shown this first requirement that limit of l n exist and suppose now and we have called that limit as l, now what remains to show that that number l is same as this limit of f t as t tends to x let us go to that. So, to show l is equal to or let me write in this way limit as limit of f t as t tends to x is equal to l. By the way, it is a standard convention in analysis, you would have come across that is this equation means that this limit exist and is equal to l it means both the thing that limit exist. The limit is equal to l that is the meaning of this symbol, so once we show this it will is it clear that once we show this than everything is proved that once. We show this this equation is proved this is nothing but say re written, we already written the thing in a different manner.

So, this is the thing, now the proof of this is again follows a very standard technique used quite frequently analysis sometimes known as 3 epsilon proofs or epsilon by 3 proof etcetera. Basic idea is you just add and subtract some terms see what is the idea we want to show that the limit of l limit of f t as t tends to x is equal to l. Now, by definition what does it mean that given epsilon you should be able to find some delta says that whenever distance between t and x is less than delta distance between f t and l is less than epsilon. Now, that means that means what does it mean practically we want to show that this difference f t minus l is small this difference can be made arbitrarily small that is what we want to show by taking t as close to x as you want.

Now, we know nothing about the difference mod f t minus l, but we know the difference about several other things by whatever we have assumed so basically we add and subtract those term. For example, we know that mod f t minus f n t can be made small for example, I can write this we know that mod f t minus f n t this can be made small because f n f n convergence f uniformly. Then, we also know that f n t minus l n that can be made small because limit of f n t as t goes x is silence so if you go t to x this can be made small. Then, finally if you look at l n minus l this also can be made small if if l is is close to if n is sufficiently large the difference between l n and l is small, so what we can say.

Now, is this for any t for ant this much is true that mod f t minus l is less not equal to mod f t minus f n t plus mod f n t minus l n plus mod l n minus l sp. We want to choose delta in such a way that this becomes less than epsilon, so we shall make the arguments in such a way that each of this term becomes less than epsilon by 3. If you make each of the whole thing will become less than 3 epsilon, that also does not matter so that is why such a thing is called a 3 epsilon proof or epsilon by 3 proof. Whatever basic idea here is that you add and subtract some terms, now let us look at one by one when will this be small mod f t minus f n t this will be.

So, we will begin with this let epsilon bigger than 0 be given we want to make this whole sum less than epsilon, so we will try to make this whole thing this less than epsilon by 3 this less than epsilon by 3 and this less than epsilon 3. Now, out of this this and this will become less than epsilon by 3, if n is sufficiently big, so that choice will make first that choice will make first.

Now, we can say that there exist let me call n such that n bigger than or equal to n 1 implies mod f t minus f n t is less than epsilon by 3 is less than epsilon by 3, but one more thing is important this happens because f n tends to f uniformly. This happens for every t in E this happens for every t in E, then since l n convergence to l we also say that there exist m 2 in n such that n bigger than or equal to n 2 implies mod l n minus l is less than epsilon by 3. Now, you should take some n which is bigger than or equal to both of this maximum of n 1 and n 2, then for that n both this inequalities will be satisfied, so consider some such n.

So, let so consider some n bigger than or equal to maximum of n 1 and n 2, then for that n this is less than epsilon by 3 for every t in E and this is also less than epsilon by 3, what remains is this what is to be done about this. Now, we will now look at only for that particular n this n which we have chosen this n which we have chosen for that n look at f n t minus l n now f n t minus l n will become small. If t is sufficiently close to x if because that is what we have assumed that limit of f n t as t goes to x, now what you do is choose delta in such a way that whenever distance between t and x is less than delta.

This difference is less than epsilon by 3, so we can say now that for this n since limit of f n t as t goes to x is equal to l n, there exist delta bigger than 0 such that wherever the distance between t and x is less than delta distance between t and x is less than delta. This implies mod f n t minus l n is less than epsilon by 3, now the proof is more or less over what. So, given epsilon we have found a delta such that whenever distance between t and x is less than delta mod f n f n t minus l n is less than epsilon by 3 mod f n t minus l n is what is this n.

This n or what is this f n this some number which is bigger than this n one and n two this n one and n two we already chosen. Now, once this is the case since for this n is bigger than 1 equal to n 1 and n 2 for that particular f n mod f t minus f n t is less than epsilon by 3 this is less than epsilon by 3 and this is also less than epsilon by 3.

So, this whole thing is less than epsilon so that means we have given epsilon we have found a delta such that whenever distance t and x is less than delta mod f t minus l is less than epsilon. That is same as saying that that is same as showing this ok, so that proves this theorem completely again what is the theorem if the each of f n has a limit as t goes to x then f also has a limit as t goes to x and that limit is nothing but limit of this l n.

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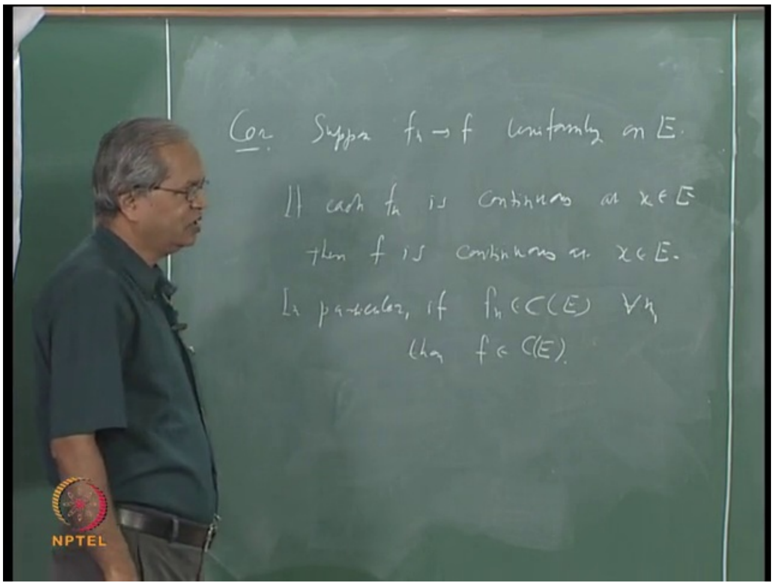

Now, as a simple corollary we can say that if each f n is continuous then f is also continuous, so we can say that as a corollary of this under a same hypothesis that is what is hypothesis that suppose f n tends to f uniformly. Suppose, f n tends to f uniformly then in fact this can be even at a point, but let us about a point if f is continuous if f if each f n is continuous f n is continuous. Either we can say at some point x in e or at all point x in e it makes no difference in the proof, let us say if each f n is continuous to begin.

Let us talk of some point continuous at some x in e this time I am taking x in E, there I have taken just a limit point x in E. Then, f is continuous at x in E and in particular if this happens for all points that is that means that f n is continuous at all points in E f is also continuous at all points in E. So, in particular if we have used this notation c of E set of all continuous real valued functions on e that is C of E that is f from E to R f is continuous.

So, instead of saying inverse that each f n is continuous at all points x in E I will simply say if f n belongs to E, for all n f n belongs to E for all n then f belongs to E not f n E c of f n c of E for all n. Then, f belongs to c of E in fact this says something about this space c of e, but I will come to that little later first of all let us prove this corollary you can say that proving this corollary is nothing really different. Suppose, let us let me again look at this instead of saying, now here instead of saying that x is a limit point of e we are now taking x belongs to E, we are now taking x belongs to E.

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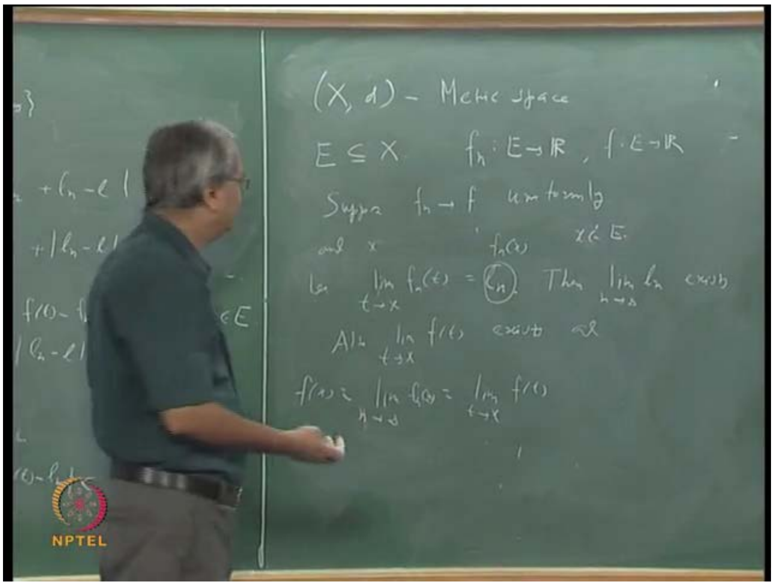

Now, what is the meaning of saying that f n is continuous at x it means that this limit f n t as t tends to x that is nothing but f n of x. This l n this will be changed to f n of x and what is the conclusion it means that the limit of f n x exist the limit of f t as t tends to x exist. That limit that limit is nothing but see remember this l n is nothing but l is nothing but f n of x limit n tends to infinity that is limit as t tends to x of f t that is same as limit n tends to infinity f n of x, but what is limit n tends to infinity f n of x.

That is same as f x because we have assumed f n converges to f, so this is nothing but f of x, so that means limit as t tends to x of f t is same as f of x which is same as saying that f is continuous at x. So, essentially all that we do is we apply this theorem at the point x, we apply this theorem at the point x and then get that if each f n is continuous f also must be continuous only. You have to use is this if f n is continuous at x is same as saying that l n is nothing but f n of x and just use that here you get the limit of f t as t tends to x is f x.

That shows that f is continuous at x, further if this happens for all points x in E, then f also must be continuous at every x in e that is nothing saying nothing extra that is once we show that f is continuous at f. Each f n is continuous at some x in E, then f is also continuous at x in E, so if the first statement is true for every x in e this also true for every x in e that is all ok now coming back to this notation.

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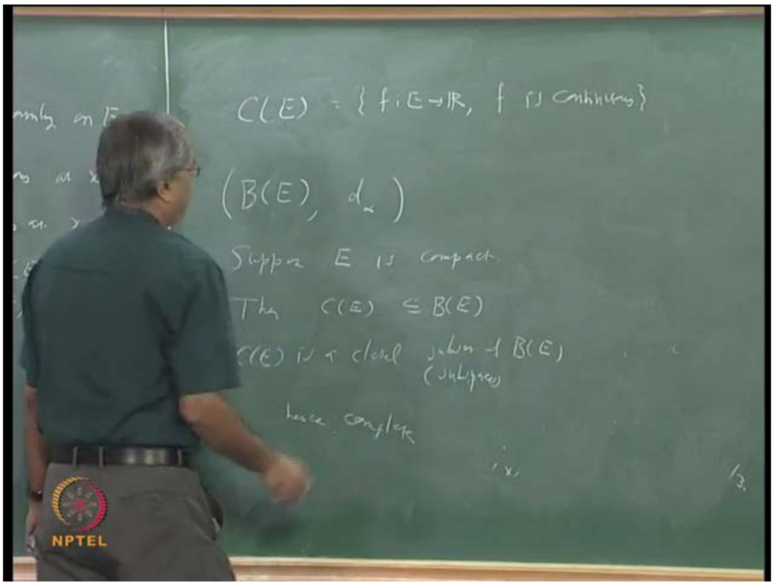

Let me again go back to this metric space set A of all functions which are bounded in E with respect to this metric d suffix infinity, we have shown that this is a complete metric space. That was the beginning, now in general if you take any arbitrary set E in a metric space and then every continuous function need not be bounded. If you put an extra condition that E is a compact set then you know that every continuous function is bounded.

Now, let us consider that suppose E is compact suppose e is compact then then this c of E is then saying that every continuous function is bounded is same as saying that C of E is a subset of B of E. Then, c of E is a subset of b of E do you agree this is nothing but saying that every continuous function on compact set is bounded. This is nothing but saying in the words since C is compact every continuous real valued function is bounded.

Now, what does this theorem say about C of E look at this last statement if each we have already seen that uniform convergence is nothing but convergence in this metric uniform. Convergence is nothing but convergence in this metric, so what does this theorem say that suppose you take a sequence f n in C E and if f n then obviously all each of this f n must be in B E. If f n convergence to f in this metric than that f also belongs to C of E what does that mean? This C of E is a closed subset of this metric space, but suppose you know that a metric space is complete and if a subset is closed then what can you say about that subset that is also complete.

You cannot say it is compact, but it is it is complete you can always say it is complete, so a closed subset of a complete metric space is again complete. So, what this theorem says or what this corollary says is that C of E is a closed sub space or closed subset of C. In fact, we can also say closed sub space because the C of E, if you regard this B of E as a vector space C of E is also a sub space because c of e is also a vector space it is a sub space and it is a closed subset. If you want you can say sub space of b of e and hence complete and hence complete that means, what we have shown is that hence complete that means you look at this space.

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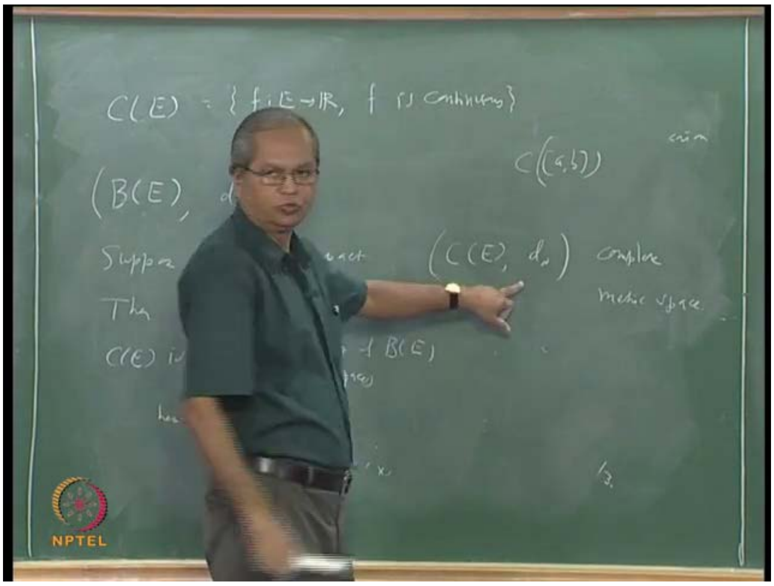

Whenever E is compact, if you look at this space C of E with respect to this metric, this is a complete metric space this is a complete metric space. In particular spaces like C of a b with this metric supremum given by this are a complete metric space. Now, sometimes we are also interested in knowing see we have seen an example that every uniformly convergence sequence is point wise convergent, but a converse is false.

In general, the converse is false sometimes we also want to know what additional conditions can be put on a point wise convergence sequence so as to make it uniformly convergent. There is one very well known theorem in that direction that is called Dini's theorem what it says is that if the sequence is monotonically increasing or monotonically decreasing. If it is point wise convergent, then it is also uniformly convergent, but this is something we shall prove in the in the next class.