**Real Analysis**
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**Lecture - 40**
**Integrable Functions**

We have seen that the one of the relatively easy way of showing that a function is integral either Riemann integrable or Riemann Stieltjes integrable is the one criteria which we discussed last time and that is the following.

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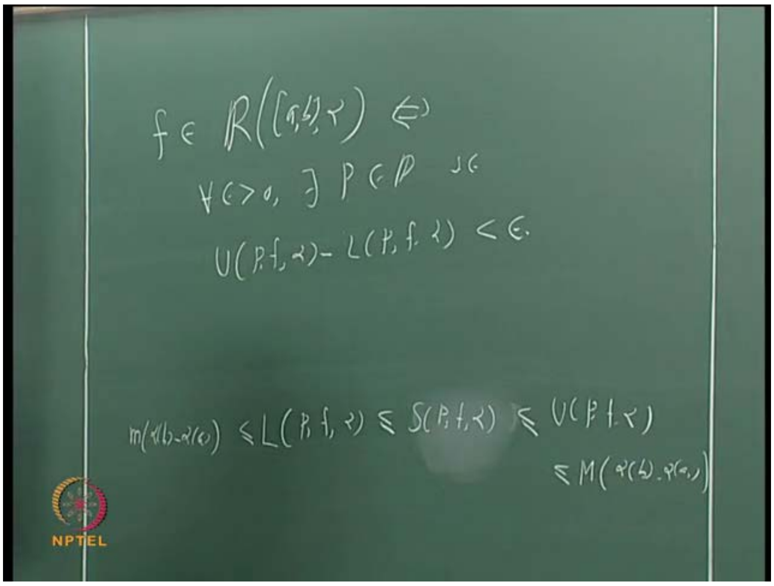

Let me write if f is integrable that is R a b alpha if and only if for every epsilon bigger than 0, there exist a partition P in this script P, such that for this particular partition U P f alpha minus l P f alpha is less than epsilon. Using this, we have shown that every continuous function is Riemann integrable and also with a slight modification of the same proof.

We have observed that every continuous whenever alpha is monotonically increasing function every continuous function is also Riemann Stieltjes integrable. The main tool used there was if you remember that every continuous function defined on such a closed end interval is uniformly continuous. Using that, we subdivided the partition in a in a particular way for continuous functions, we can also notice one more thing in fact we have noticed that whatever be the partition this part is always true that is l P f alpha.

This is always less than or equal to U P f alpha, then we also observed let me keep this is this is less than or equal to any Riemann Stieltjes sum and this is less than or equal to upper sum and this upper sum is less than or equal to M into alpha b minus alpha a. This lower sum is always bigger than or equal to m into alpha b minus alpha a where as usual this m and M have the same meaning which we have discussed earlier namely that M is the supremum of f x over the interval and m is the infimum. What we also know is that whenever the function is integrable the upper and lower integrals coincide and upper integral is the infimum of upper sums.

So, it will always be less than or equal to any upper sum, similarly lower integral is the supremum of the lower sums, so it is bigger than or equal to any lower sum. So, both these numbers lower integral as well as the upper integral they lie between any of the lower sum and the upper sum. In particular, when the function is integrable, the number integral is nothing but the common value because we said that the function integrable is same as we say that the upper integral is same as the lower integral. So, that common number will also lie between these two always, not these two this and this, this and this.

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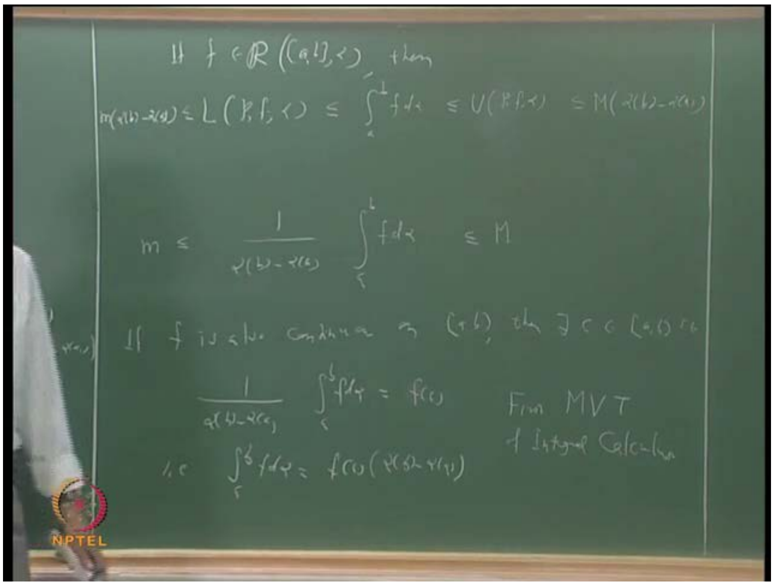

So, what we can observe is that here let me write it once again here if f is integrable that is if f is this R a b alpha is then we can say this this is less than or equal to integral a to b f d alpha and this is less than or equal to integral. This is less than or equal to U P f alpha and we have already seen that this is less than or equal to M into alpha b minus alpha a.

This is bigger than or equal to m into alpha b minus alpha a and usually this function alpha will be strictly monotonically increasing function. So, alpha b minus alpha a will be strictly bigger than 0 in most cases unless in some very trivial examples.

So, what one can say further is the following that is I can say that this m is less than or equal one by alpha b minus alpha a into integral a to b f d alpha. This is less than or equal to M this much is true for any integrable function; if the function is continuous we can say something more. What is this, M is supremum and we know that if the function is continuous on this interval, then this value is attained that means there is some point let us say the point x 1 such that f at x 1 is 1 is equal to m.

Similarly, there is some point let us say x 2 such that f at x 2 is m that means that these values m and M are not taken actually taken by the function and if the function is continuous, we know that if it takes any two values. It will take all the values which lie in between those two the famous intermediate value theorem. So, that means what that means there should be some point at which the function takes this value also. That will happen a function is continuous, so let us let us just see what is the consequence of that is so if f is also continuous, continuous on a b.

Then, there exists a point c in this interval a b such that such that what should happen that is one by alpha b minus alpha a into this this is same as c that is 1 by alpha b minus alpha a into integral a to b f d alpha this is equal to c. This is equal to f x because there exists some point c at which the function f takes the value given by this. It is f of c equal to this number what is same as we say suppose we rewrite the same in a different manner that is integral a to b f d alpha that is same as f at c multiplied by alpha b minus alpha a and this statement. Whenever f is continuous, there exists c in a b satisfying this last that is called the first mean value theorem of integral calculus.

That is called the first mean value theorem of integral calculus what is theorem, if f is continuous, once it is continuous it is automatically integrable. That is what we have already shown then there exists c such that integral a to b f d alpha is equal to f c into alpha b minus alpha a that is called first mean value theorem. We will use the standard short form for mean value theorem first mean value theorem of integral calculus mean value theorem. It looks very similar to mean value theorems of differential calculus; it talks about some point c inside the interval a b such that something happens at that point.

First, obviously it means something is going to follow something somewhat are going to follow, but that is why, but anyway it will not happen immediately after sometime. So, that is about these continuous functions, there is one more class of functions in case of which we can prove in a straight forward manner. Those functions are integrable and those are monotonic functions, so there is another theorem that you want to see.

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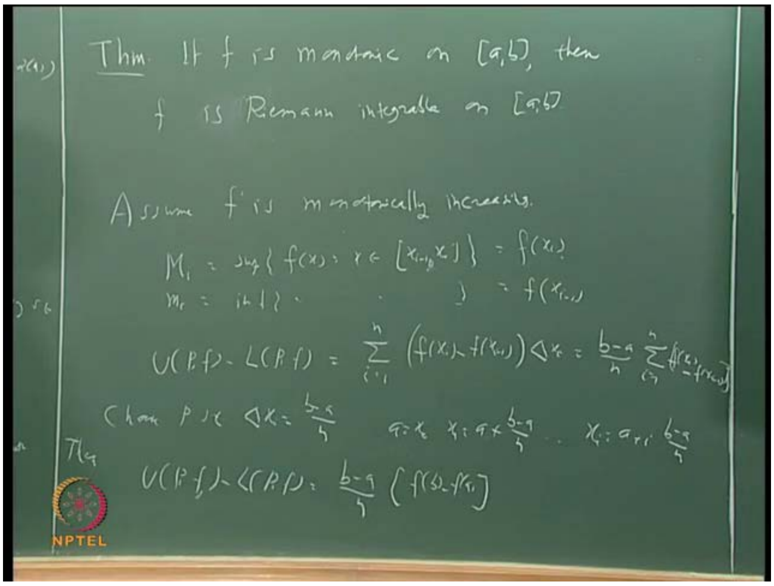


Let us as usual just as we have done in case of continuous functions, we shall first prove for Riemann integrals and then we will go to this arbitrary Riemann Stieltjes integral. So, if f is monotonic on a b then f is of course, we are always assuming that this function alpha is for time being I am not talking about alpha and f is Riemann integral. Let me write in words f is Riemann integrable, now what happens if f is monotonic if it is monotonic, what it means, it means either it is monotonically increasing on either monotonically decreasing. Let us prove in one of the cases proof in other cases will be similar assume f is monotonically increasing in order to know what is happening.

In this case, see the first thing that we can observe here is that you have defined, let us recall the definition is from the M i and M i. This s what was this M i, M i was nothing but supremum of f x where x is belongs to this i th sum interval x i minus 1, 2, x i, but we know that f is monotonically increasing. So, what can we say about this M i for an increasing function on an interval what will be the maximum value that it takes at the right hand end point.

This is nothing but f of x, f of x, similarly M i that is nothing but infimum of the same. That will be nothing but f at x i minus 1 f at x i minus 1 for any such partition suppose we look at the difference between the upper sum and lower sums, since we are talking about the Riemann sums that us just look at U P f minus l P f. This will be the thing, but sigma i going from 1 to n, sigma i going from 1 to n would nothing but f x i minus f x i minus 1 and multiplied by delta x i, delta x i is the thing, but x i minus x i minus 1. Now, suppose I choose a partition in such a way that this delta x i is constant, suppose a chooser partition in such a way that this delta x i is constant if the delta x i is constant.

Then, there is only one is possible if there are n sub intervals each must be of length 1 by n. So, choose a partition in such a way the n we shall choose later, but first will say that if there are choose P such that choose P such that delta x i. There are n sub intervals and if each interval is has to be of the same length and the total length is b minus a, so each of the delta x i must be b minus a by n. So, each of the delta must be b minus a by n and so what will the partition that is also easy to say we know a is x naught.

Then, x 1 will be nothing but a plus a plus b minus a by n that is x 1 x 2 will be plus 2 times b minus a by n etcetera. So, in general one can say that x i is equal to a plus i times b minus a by n choose this partition so far this partition what would happen to this U P f minus l P f this delta x i b minus a by n. So, that will come outside the summation sign, so they will come outside summation sign this is nothing but b minus a by n and multiplied by this sigma i going from 1 to n if x i minus f x i minus 1. So, once this delta x i has come outside, what will remain, suppose we take this sum for example, first element will be f x i 1 minus f x 0, second will be f x 2 minus f x 1 etcetera.

So, it will be simply f of x is nothing but f b minus f a f b minus f a, so you can put that here f b minus f a, so you do that simplification. So, what we get this is U P f minus l P f this is the thing, but this b minus a divided by n and multiplied by f b minus f a. So, then now come comeback this criteria, so what we said that f is integrable if and only if for every epsilon, there exists a partition P such that u P in our case this is alpha is just x. So, U P f minus l P f is less than epsilon, so in our case what is U P f minus l P f it is b minus a etcetera, but there is only one thing that depends on this partition that is n.

Everything else is fixed, so the only question is this given epsilon can we choose n such that this whole thing becomes less than epsilon. That is obvious because we can make n sufficiently large so that this is b minus a into this number. Suppose, this number is fixed suppose this number is k this whole thing is k by n, then choose n such that n is bigger than epsilon by k n is bigger than epsilon by k.

Then, this difference will be less than epsilon and that will means that that means that f is Riemann integrable, f is Riemann integrable. Now, coming back to Riemann Stieltjes integrable Riemann Stieltjes integrability, one can do all these calculations up to this. This will be replaced by U P f alpha minus l P f alpha let me carry out here, suppose instead of Riemann integrability, suppose I look at Riemann Stieltjes integrability, then this will not change M i minus m i, those numbers only depend upon f.

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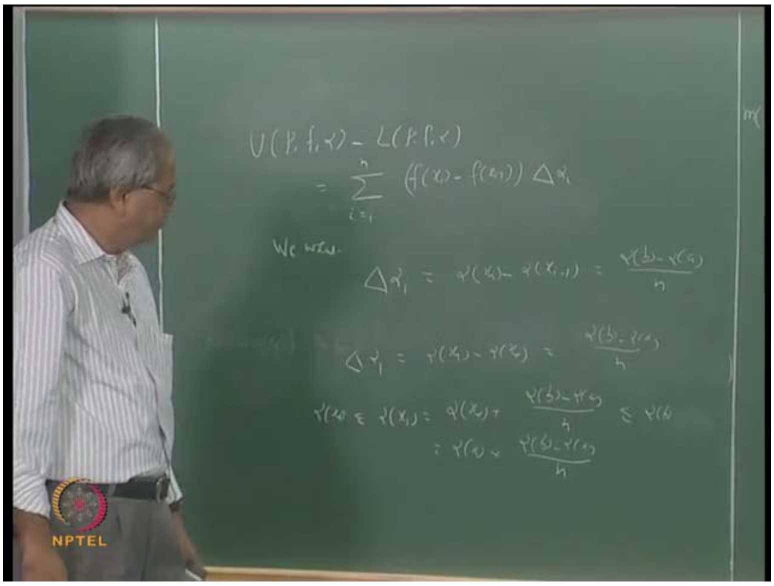

So, if I look at this difference U P f alpha minus l P f alpha this will be sigma i going from 1 to n and then multiplied by f sorry f x i minus f x i minus 1 and this instead of delta x i, it will be delta alpha i. It will be delta alpha i and what is delta alpha, I recall delta alpha i is the thing, but alpha x I minus alpha x i minus 1. So, instead of delta x i over there, we have delta alpha i here, we have delta alpha i here and what is our next argument.

We have said that can choose a partition in such a way such that the delta x i is constant the delta x i and the delta x i turned out to be if it has to be constant it has to be b minus a by n. So, what will be the change here we will low on this delta alpha i to be alpha b minus alpha a divide by n delta alpha delta alpha should be alpha b minus alpha a divided by n. Now, the whole question is whether that is if we want to imitate that we want this we want delta alpha i is equal to alpha b minus alpha a divided by n. For example, suppose I want let us say suppose I look at delta alpha 1 delta alpha 1 that is delta alpha 1 is nothing but alpha at x 1 minus alpha at x naught.

This should be same as delta alpha b minus alpha a alpha b minus alpha a divided by n or what it means it means alpha x 1 should be same as alpha x naught plus alpha b alpha by n. Now, if you compare this situation here, it was easy to say that chose x 1 such that x 1 is a plus b minus a by n choose x 1 is nothing but a plus b minus a by n, but here what is the case, we want to choose x 1 such that alpha x 1 is equal to alpha x naught plus alpha b minus alpha a by n. So, the question is whether x 1 x naught exists or not, whether the function alpha takes this value or not right that is the question and in general the only thing we know about the function alpha is that it is monotonically increasing.

Only thing we know that it is monotonically increasing, but nothing further, but if we impose some additional conditions see is it is it clear to you that alpha x 1 that is anywhere bigger than or equal to alpha, see this is alpha is non negative number. So, this is alpha x naught is less than this alpha x 1 and this is obviously is less than or equal to alpha b minus alpha a not alpha b minus alpha a. This is I can say less than or equal to alpha b that is clear alright because I can say that alpha b alpha x naught is the thing, but alpha a alpha a plus alpha b minus alpha a divided by n.

One can say that for any positive n this will be less than or equal to alpha b minus alpha a and hence this whole thing will be less than or equal to alpha b. So, if we knew that alpha is continuous if we knew that alpha is continuous, then we can say that it takes this value it takes this value. So, it will take all the intermediate values, so I can find x 1 such that alpha at x 1 is equal to this, otherwise we cannot say this. So, if we add the hypothesis of continuity to alpha, then I can say that the choose alpha x 1 in this fashion having chosen alpha x 1 use the same technique and i can choose alpha x 2 also in the same fashion. So, if alpha is continuous we can choose the partition x naught in such a way that each delta alpha is satisfied in this condition.

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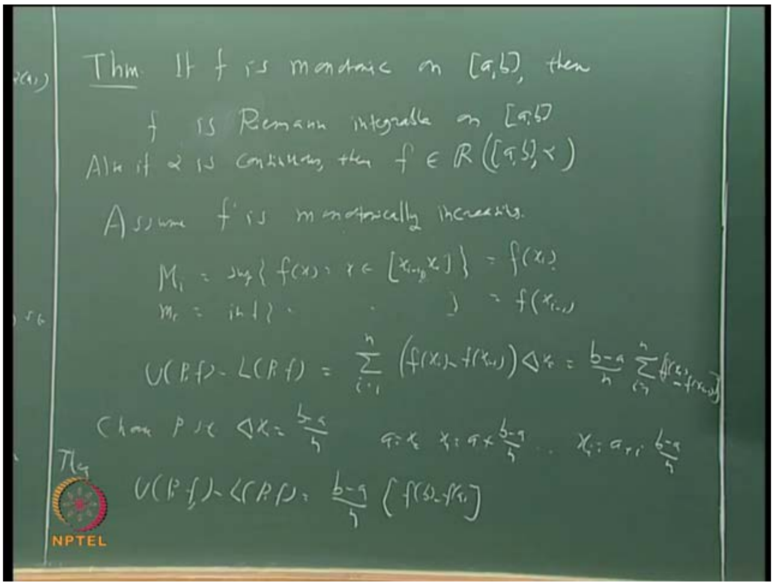

Otherwise, you may or may not be able to do this, so what it means is that this will work if you assume that f is monotonic with alpha should not only be only monotonic, but also continuous. Remember, that is true for Riemann integration because in case of Riemann integral alpha x is nothing but x that is a continuous function, so that is automatically satisfied. So, we can say that f is monotonic on a b then f is Riemann integrable, also if alpha is continuous, then it means all it is assumed that f is monotonic alpha is also monotonic increasing and continuous then f belongs to this R a b alpha.

Now, we go to another task just as in case of differential, if you know that a function is differentiable, and then what it can be. For example, suppose we know that f 1 and f 2 are differentiable, then f 1 plus f 2 are differentiable and etcetera and similar thing we want to do in case of integration. Suppose, that we know some functions are integrable, then constructive function using those functions and whether those new functions are also integrable or not. That is the question that we are going to deal with now, so let me start with this.

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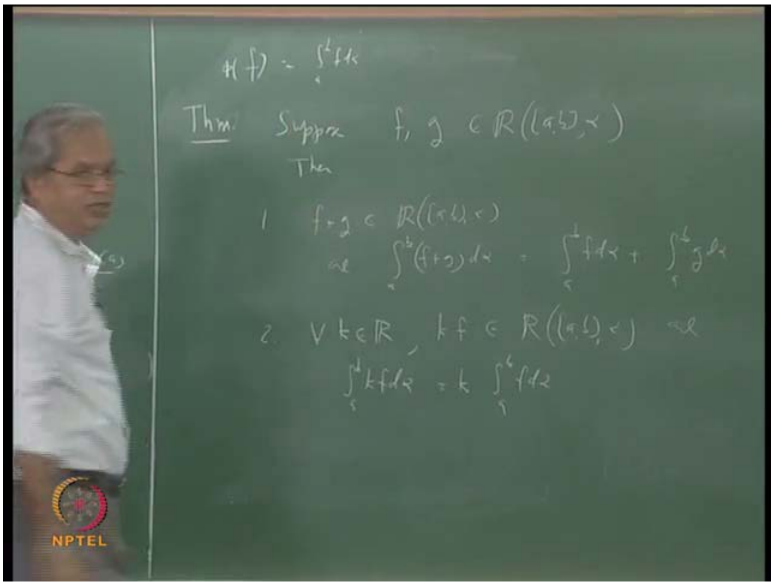

Suppose, this is a theorem, suppose f and g are that, this means f and g in words are Riemann integrable respect to some monotonically increasing function alpha. Then, there are several statements we want to make, I shall write one by one and then shall we shall discuss the corresponding proofs. Also, all these prove depend basically on the definition of integrability, so the first thing that we want to say is this if f and g are also, sorry f plus g f plus g is also integrable.

We can say something more integral a to b f plus g d alpha. This is same as integral a to b f d alpha plus integral a to b g d and second thing is f for g. Suppose, we take any k for real number k in r for every real number k k times f is also Riemann Stieltjes integrable and integral a to b k f d alpha. This is same as k times integral a to b f d alpha, what does it mean in words that if you take two functions in in this if f and g are Riemann Stieltjes integrable. Then, there sum as well as product with a constant is also integrable, in other words since you are also doing a course in linear algebra simultaneously. It is basically same as saying that this is a real vector space set of all Riemann Stieltjes integrable.

Suppose, alpha is kept fixed then set of all Riemann Stieltjes integrable functions is a vector space over real numbers. If you take all functions on a b that is also a vector space you can say this is a subspace of that and what can we say about, can we also describe this part in terms of linear algebra using terminology of linear algebra. Have you studied what is meant by linear functional have you studied what is meant by linear functional.

So, what does this say integral a to b f plus g d alpha is same as integral f d alpha plus integral g d alpha. Second things is I say that integral k f d alpha is same as k times integral a to b f d alpha.

So, in other words if you take this map k going to integral a to b f d alpha or suppose we define phi f is equal to integral a to b f d alpha. Then, phi is a linear functional on this vector space, so basically proving this properties of this integral is basically showing that integration is a linear functional on this vector space of all integrable functions. Now, since there are three functions in f g and f plus g and whenever we want to talk about the integration, obviously upper sums lower sums are going to be come to picture. So, we have discussed the upper sums lower sums of three different functions, so we need some notation for this M i and M i etcetera.

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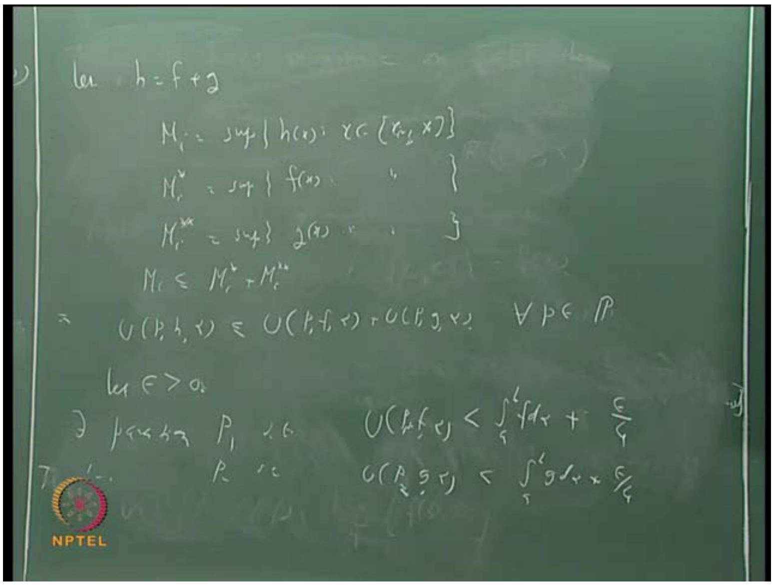

So, let us start with this, let us say suppose I will call this is function h let h be equal to f plus g. Then, we will have to talk about the upper sums corresponding to the function h corresponding to the function f and corresponding to the function g, already suppose let us say that we will have this. Suppose, I call this M i as the supreme of let us say h x h x from x c h x for x c x i minus 1 to x i, I will lead similar numbers for corresponding to f and corresponding g. So, let us say suppose i denote this by M i star as supremum of f x same thing x is in x i minus 1 x i and M i double star as supremum of f x in not f x g x for same thing.

Again, x is varying with the interval x i minus 1 to x i can be seen obvious relationship between this m i, M i star and M i double star, what is that M i must be less than or equal to M i m i star plus mi double star. So, we can say that this M i must be less than or equal to M i star plus M i double star, what does it says about the upper sums we can say that from this it follows that upper sum of h must be less than or equal to upper sum of f because all that you are doing is multiplying each of this number by delta alpha i.

Then, we are taking the sum, so we can say that this implies that u P h alpha is less than or equal to U P f alpha plus u P g remember we are going to use same method to prove that it is integrable, what is the method that given epsilon. We will produce a partition P such that u P h alpha minus l P h alpha is less than epsilon that is the idea. So, what we can what we can do now is that and now this is true for every partition whatever be the partition that upper sum of h is less than or equal to the upper sum of f plus upper sum of g. That is true for every partition, but we know that f and g integrable, we know that f and g are integrable.

So, what we can say is that given any epsilon given any epsilon, I can find a partition suppose I call it partition P 1, let us let us start with epsilon let us start with epsilon bigger that 0, what is the aim? Our aim is to produce a partition P such that for that particular partition u of h u P h alpha minus l P f alpha that should be less than epsilon. That is the aim, but we know that such a thing can be done for f and g, such partitions can be found for f and g, so we shall use we shall use that fact first. Let us take the upper sums one can say that there exists partition suppose I call it partition P 1, there exists partition.

Here, we are going to have several partitions and we are going to use the thing that we have shown earlier that whenever you take a refinement the upper sums decrease and lower sum increase. So, we will find a partition corresponding to f corresponding to g and then take something which is common refinement of both. So, we can say that there exists a partition P 1 such that we know that f and g are integrable, so integral is nothing but infimum of this upper sums in fact that what we have called the upper integral, but when the functions are integrable.

Those two things are the same, so I can say that I find some partition such that the difference between the integral and the upper sum is less than epsilon or epsilon by 2 epsilon by 4 whatever we want. This is what we can do, so I can say that there exist a partition P 1 such that see u P 1 f alpha u P 1 f alpha minus integral a to b f d alpha this is less. I do not remember, let us start with epsilon by 4, if require, we shall change that epsilon by 4 or epsilon by 3 or 6, whatever we require. This can be done then one can say that, similarly there exists a partition P 2 such that upper sum u P 1 g alpha is less than integral.

I can rewrite this inequality, I can bring this to the other side that is convenient instead of writing like this. I shall say this upper sum is less than this integral plus epsilon by 4 and similarly this upper sum is less integral g d alpha plus epsilon by 4. This one was P 2, now consider the partition P 1 union P 2, and consider the partition P 1 union P 2, for that partition both of these inequalities should be satisfied, suppose I call that partition P 3.

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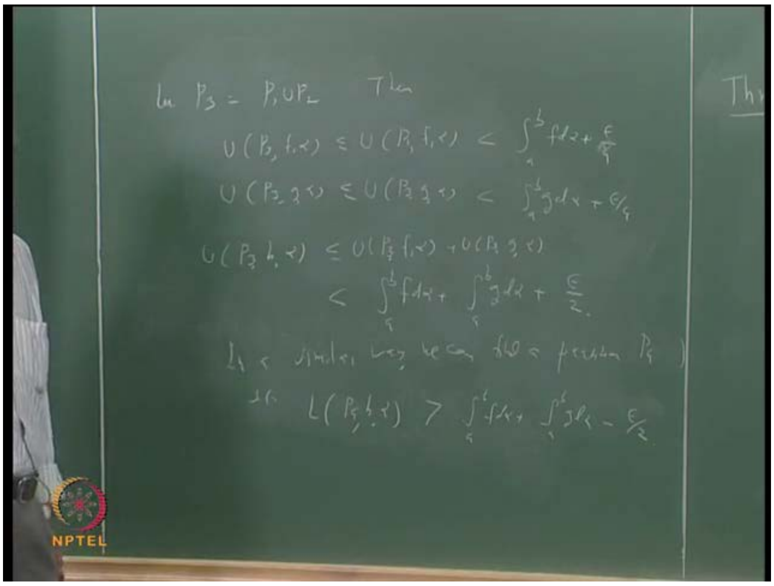

Let P 3 equal to P 1 union P 2, then what we want to say is that then u P 3 f alpha, this is less than or equal to u P 1 f alpha. That is less than integral a to b f d alpha plus epsilon by 4 and similarly, also u P 3 g alpha, this is less than or equal to this is P 3 is also refinement of P 2. This is less than or equal to u P 3 g alpha and that is less than integral a to b g d alpha plus epsilon by 4. Now, you look at u P 3 h alpha, we have seen earlier that for any partition u P h alpha is less than or equal to U P f alpha plus u P g alpha. So, u P 3 h alpha that should be less than or equal to u P 3 f alpha plus u P 3 g alpha and this is less than integral a to b f d alpha plus epsilon by 4.

This is less than integral a to b g d alpha plus epsilon by 4, so there sum should be less than this plus this plus epsilon by 2, so this should be less than integral a to b f d alpha plus integral a to b g d alpha plus epsilon by 2. So, we agree to this, now whatever we have done for the upper sums. We can do a similar thing for the lower sums, what is the obvious changes here, you will take M i, m i star and M i double star and inequalities will be reversed M i s will be bigger than or equal to mi star is that. In other words, lower sum of h alpha will be bigger than or equal to lower sum of f alpha plus lower sum of g alpha and all.

So, everything you have done in similar way you can you can do everywhere inequalities will be reversed for example, this l P 1 alpha will be bigger than integral a to b f d alpha minus epsilon by 4. So, one can say that following a similar method, you can find a partition you can find a partition such that for that partition lower sum is bigger than this number minus epsilon by 2. So, I shall skip those details do that on your own to convince yourself, I will say that in the similar way, we can we can find a partition, we can find a partition. Now, suppose I call that partition P 4, P 4 such that such that lower sum with respect to this P 4 h alpha lower sum with respect to P 4 h alpha is bigger than integral a to b f d alpha plus integral a to b g d alpha minus epsilon by 2.

So, again we go back to the same thing, here we partition P 3 for which u P 3 is less than this number plus epsilon by 2 and here we have partition P 4 such that this is bigger than this number minus epsilon. If this were the same partition, we could have immediately said that this minus this is less than epsilon, but that is not a that is that is not the case. So, what we know how to deal with such things just take the union of the two that will be common partition for both common refinement of both and so both the inequalities are satisfied.

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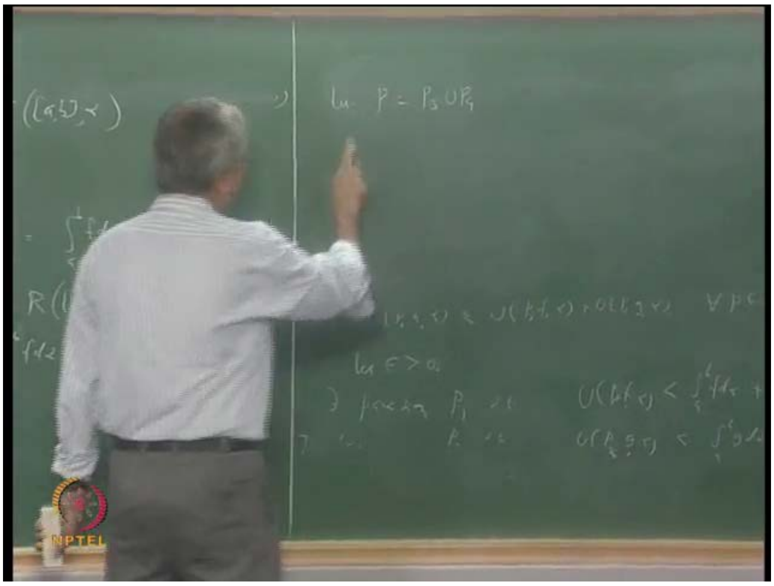

So, I will just take this statement I require, I will go back here, so let P what are what the partition we have are P 3 and P 4. Here, P 3 union P 4 and we want to show that this b is required partition P is refinement of both P 3 and P 4 and we know that for refinements upper sums decrease and lower sum is increased, so let me just write it here.

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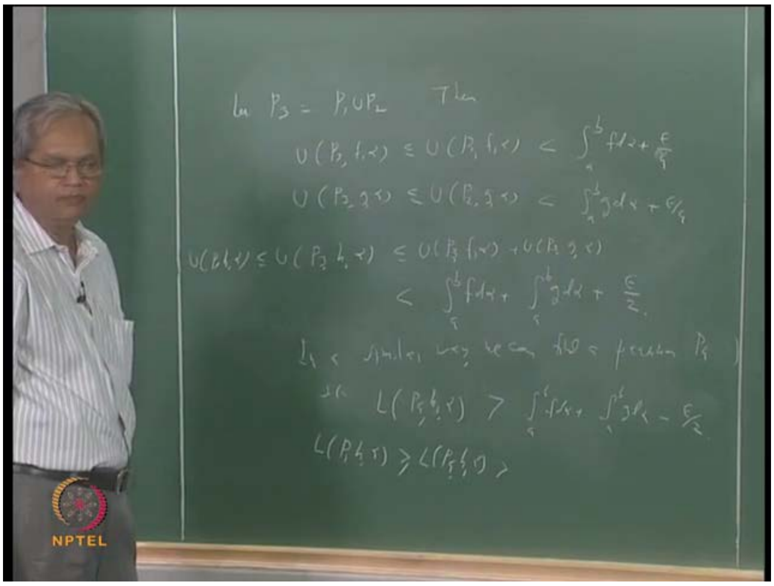

So, I can say that this is u P h alpha u P h alpha is less than or equal to this and hence less than less than this plus epsilon by 2, sorry this should have been everywhere, this should have been that is correct. This third line here P 2 in the second in the middle u P g alpha.

That is right because the use the fact that P 3 is a refinement, then where were we using that P is P 3 union P 4, so u P h alpha is less than or equal to u P 3 h alpha. Similarly, lower sums increase, so lower sum with respect to P should be bigger than or equal to lower sum with respect to P 4, so we can also say that l P h alpha is bigger than or equal to l P 4 h alpha.

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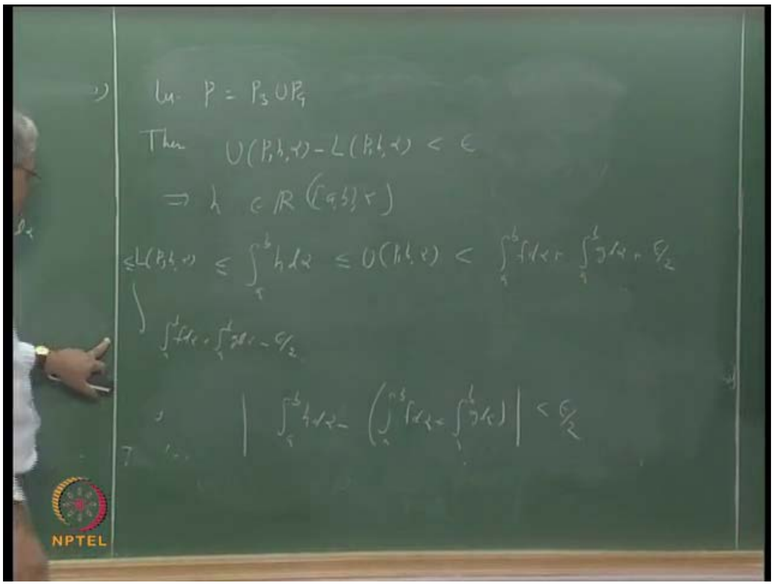

So, that is bigger than all these things, then u P h alpha minus l P h alpha is less than this. This is less than this, so when you subtract these two numbers, these will get cancelled, so this will be epsilon by 2 minus this is less than epsilon. So, what we so this shows that this implies that h is Riemann Stieltjes integrable, it shows this first thing and though it does not, so immediately this you can observe that we have done whatever is required for showing this. We have done whatever is required for showing this, once we show once we show that h is Riemann Stieltjes integrable, it means that integral h d alpha it means integral h d alpha exists and that is less than or equal to any upper sum.

That is less than or equal to any upper sum and also bigger than or equal to any lower sum, so what we already have one partition for which we not only prove this, in fact we have proved much more on this. In fact, we have proved that u P h alpha is less than or equal to this number plus epsilon by 2, so let us just recall it.

So, once we know that h is Riemann Stieltjes integrable, it means that this number exists integral a to b h d alpha exists. That is something we have already proved and moreover this number is less than or equal to any upper sum. This number must be less than or equal to any upper sum, so that means in particular this is less than or equal to this u P h alpha this must be less than or equal to u P h alpha. We already know that this P is the property that u P h alpha is less than or equal to integral a to b f d alpha plus integral a to b g d alpha plus epsilon by 2.

That is something that we know that so that means this is less than integral a to b f d alpha plus integral a to b g d alpha plus epsilon by 2, what is more we also know that this is integral must be bigger than equal to every lower sum must be bigger than or equal to any lower sum. So, this must be in particular bigger than or equal to l P h alpha and we know that l P h alpha is bigger than or equal to again. Let us write this is bigger than l continue this is again this into number f d alpha plus integral a to b g d alpha minus epsilon by 2.

So, what does it mean, is this number a to b h d alpha this lies between whatever is the sum of these two numbers plus epsilon by 2 and this minus epsilon by 2. So, what can we say or difference between this and the sum of these two, suppose I want to look at this integral a to b h d alpha minus integral a to b f d alpha plus integral a to b g d alpha. Suppose, I want to look at the difference between, this difference in fact we have shown that this difference is less than epsilon by 2 because this shows this minus this number is less than epsilon by 2 and bigger than minus epsilon by 2.

So, this difference is less than epsilon by 2, but epsilon was arbitrary, you remember that we started with any arbitrary epsilon. For that epsilon we have shown that if you take this number and whatever there that this is difference between left hand side and right side is less than epsilon by 2 for any arbitrary epsilon. So, the only way is this can happen is that those numbers must coincide those numbers must coincide and that gives this result that is this if the two functions are Riemann Stieltjes integrable their sum is also Riemann Stieltjes integrable.

Integral of f plus g is same as integral of f plus integral of g, then what I would say is that the proof of this is also exactly similar. There are no new ideas involved here in fact it is even simpler because what was the whole idea relating to the upper sum of this f plus g with upper sum of f and upper sum of g. Similarly, for the lower sums here you have to do only for the function k times f and that is very easy because you can say that upper sum of that is you can easily show for any partition P.

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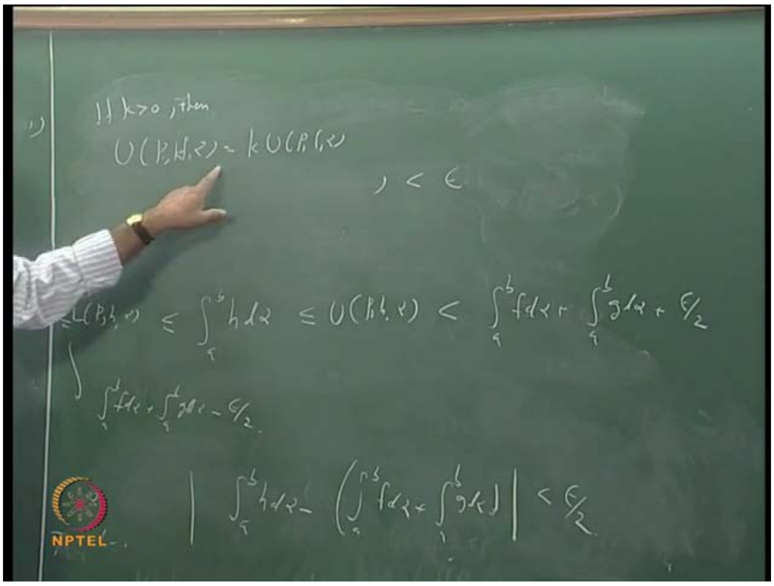


U P f alpha is say we want k times we want k times f of course, you will have to make a slight you have to be little bit careful. Here, you have cases k is bigger than or equal to 0 k less than or equal to 0 etcetera, if k is equal to 0, then there is nothing to be a proved if k is equal to 0. Then, this is nothing to be a proved because it is a constant function 0 and we know that is integrable if k is bigger than 0. Then, we can say this if k is bigger than 0, then then upper sum of k times this is same as k times U P f alpha. Similarly, one can say about the lower sums and so you can prove this at least when k is positive, you can prove when k is positive if k is negative.

This will be k times the lower sum of this if k is negative upper sum of P k f alpha will be nothing but k times lower sum. Similarly, for lower sums and again you can use basically use the same technique? So, no new ideas and involved to do prove this, so that is why I will skip this proof. Basically, all that you need to do is given epsilon, you have to produce a partition P such that for that partition difference between upper sum and lower sum is less than epsilon. Then, this part you can prove in the similar way using this, we will stop with that, there are many properties of this type and those we shall see in the next class.