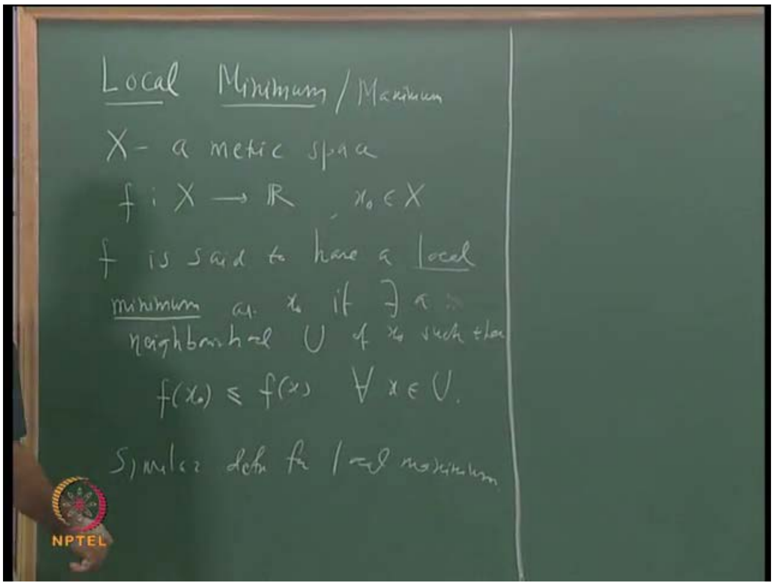
Real Analysis
Prof. S. H. Kulkarni
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 34
Mean Value Theorems

So, let me recall that we have studied or discussed the continuous functions on some subsets basically intervals on the real line. We discuss what are the types of discontinuity namely jumped discontinuity or discontinuity of first kind and the second kind. Then, going to the derivatives, we define what is meant by derivative of a function at a point and then we want to show something similar, but in an opposite direction namely we want to show that if a function is differentiable.

There is derivative, the derivative cannot have any jumped derivative also cannot have any jumped discontinuity, I am sorry about the monotonic functions. We have proved that monotonic functions are discontinuities or jumped discontinuities it cannot have discontinuities of the second type whereas in the case of derivatives it is opposite. In order to get to that result, we need some preparation, of course some of those things you already know, but still for the sake of completeness we shall recall, so to begin with let me recall the notion of what is call a local minimum.

(Refer Slide Time: 01:23)



Local minimum or similarly, local maximum as far as defining a local minimum or a maximum is concern, we really do not need any really speaking any differentiability or any such continuity or any such conditions. That just depends on the ordered structure of the real numbers, so this is something which we can define at in any set or for that matter we do, so any metric space. Suppose, x is a metric space and f is a function from X to R. We take x naught in X, and then we say that f has a local minimum at x naught, so that is what we want to define f is said to have a local minimum.

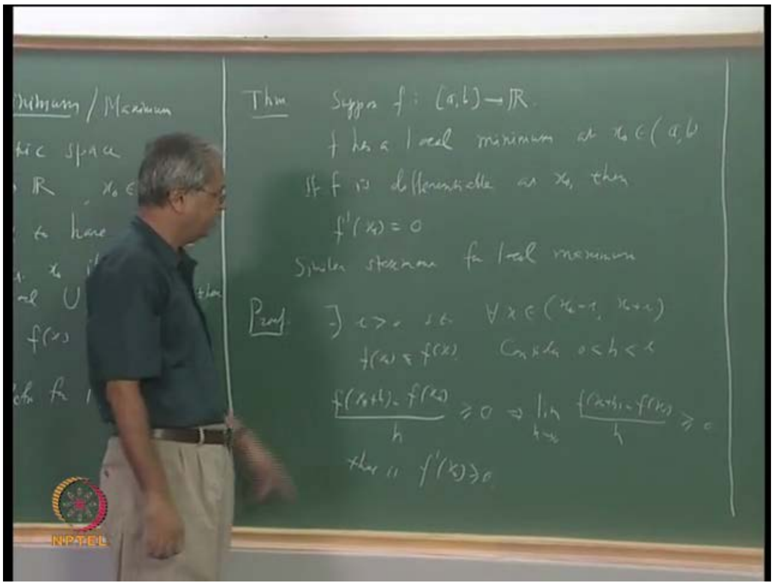
Remember, this term local is very important in this definition, you all know what is meant by a neighborhood of a point, that is something that is defined. Let me also mention here that there is a slight difference in the way in which neighborhood is defined in different books. If we take the definition of Rudin, it is any open set containing the point x naught is called the neighborhood of that point, whereas if you take Siemens, he says that any set which contains an open set containing x naught is call the neighborhood of that point.

So, that is the slight difference there and so what Siemens will call open neighborhood is what neighborhood according to Rudin, but in most of the applications that we consider about the neighborhoods we usually deal with open neighborhoods only. So, we shall follow Rudin's notation by neighborhood, we will mean open neighborhood that means an open set containing x naught. In most cases, it will be an open ball containing x naught or open ball with radius x naught, so will say that f is said to have local minimum at x naught if there exist neighborhood.

Suppose, I call at neighborhood U, U of x naught such that what should happen is that in this neighborhood for every x in this U f x should be bigger than or equal to f x naught. Then, we say that f has a local minimum at x naught, so such that f x naught is less, not equal to f x for every x in U is this clear. For defining this, we do not need any additional property for of f, f has to just define and it should be a real valued function. Now, similar definition can be given for local maximum, similar definition is that clear similar definition can be given for local maximum, so we just make it here local maximum, what will be change here?

Instead of this, suppose I say maximum, this inequality will be reversed f x naught will be bigger not equal to f x for every x in U. So, let me simply say similar definition for local maximum, so this is for any metric space next what we want to observe is that. If this is happening, in particular if x is a real line or some subset of a real line.

(Refer Slide Time: 06:26)



Then, suppose for example in particular if f is an interval and suppose f has a local minimum or local maximum at x naught, then if also if f is differentiable at that point x naught, then the derivative should be 0. We already know, but let us recall the proof because that is something we shall need in proving what I have, proving the theorem which I have mentioned in the beginning.

So, by the way let me also mention what this minimum and maximum, sometimes together are referred as extreme values or extremes. So, let us now come to this observation or theorem, suppose f, I defined from a b to r and f has a local minimum at x naught in the open interval a b x naught in the f, f is differentiable at x naught then derivative at that f prime derivative at that point x naught must be 0. Similar statement for local maximum similar statement for local maximum, now let us just see how we can prove this, so let us go to the proof of this we can see these things little bit quickly because most of these things you have already suppose to know.

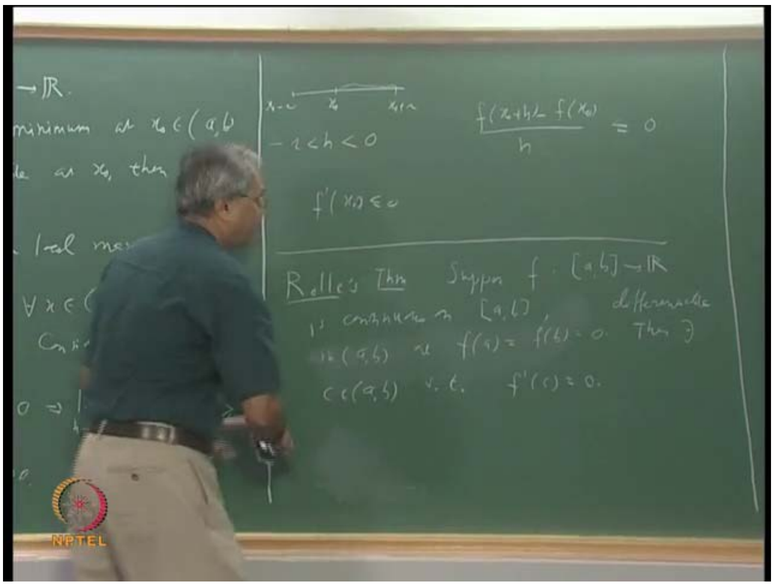
We are just revising because we need it in proving that theorem about the derivatives, so coming back to this when since f has a local minimum at a. Let us take this case f has a local minimum if f has a local minimum t, there exist a neighborhood such that in that neighborhood this happens and in case of x naught neighborhood means open set and every open set will contain an open interval. We call that interval as x naught minus R to x naught plus R for some procedure.

So, we can say that we can say that there exist R bigger than 0 such that for every x in X naught minus R to x naught plus R what should happen f of x naught should be less not than equal to f x f of x naught should be less not than equal to f x. Now, suppose you take any h which is positive and less than R suppose we take any h, so consider 0 less than h less than R and also consider f of x naught plus h minus f at x naught divided by h.

Then, what can we say about this since f of x naught is less not see once this h is less than R x naught plus h lies in this interval x naught plus h lies in this interval, so we must have that f of x naught must be less not equal to f of x naught plus h. That means this is a non negative quantity, you are dividing it by again a non negative quantity, so this must be bigger not equal to 0 and so since this is true for every h bigger than 0 and what we know is that derivative is the limit derivative is the limit of this as h goes to 0.

So, this must be not bigger equal to 0, this implies that with in limit of this f of x naught plus h minus f of x naught divided by h. Limit of this as h tends to 0, this must be bigger not equal to 0 or which is same as saying that f prime at x naught is bigger not equal to 0 that is correct.

(Refer Slide Time: 11:19)



Now, in a similar way see what we have done now is that we have taken h between 0 to R, similarly we can take h between minus r to 0 that is see when you take h from 0 to R. This means, let us say this is x naught this is x naught plus R, this is x naught minus R where you take h between 0 to R, that means x, x naught plus h lies in this part. So, we can similarly, consider minus R less than h less than 0 in which case it means that x naught plus h lies here x naught plus h lies here. We can say that in let again consider this f of x naught plus h minus f x not divided by h this time what is the argument.

We can again say that the numerator is bigger than or equal to 0, but this time denominator is negative, so this quotient will be less not equal to 0, so this less not equal to 0 for all h. Then, again you can follow the same steps as we have done here, so taking limit as h tends to 0 that will give that f prime x naught is less not equal to 0. So, we will show that f prime at x naught is less not equal to 0 and then combine these two things, here we have shown that f prime at x naught is less not equal to 0.

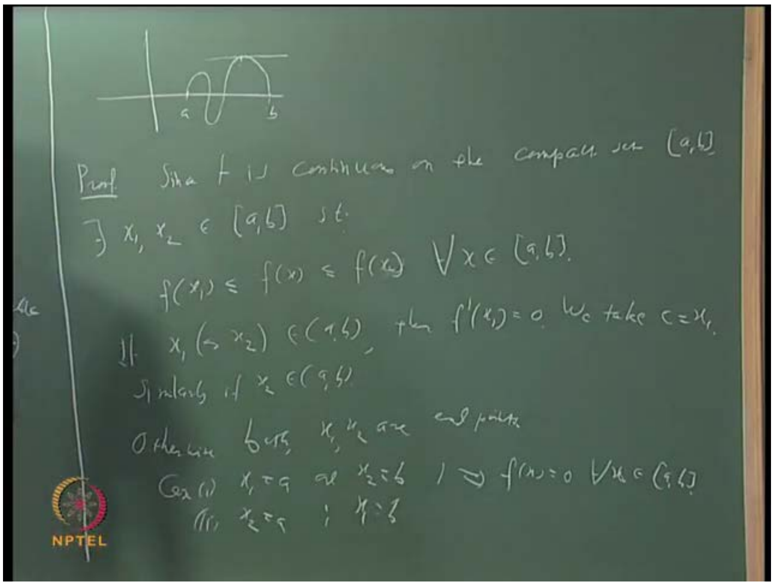
Here, we have shown that the same thing is bigger not equal to 0, so combining these two things we obtain that f prime at x naught is equal to 0 that is clear. Now, in a similar way you can prove if it has a local maximum, only thing is that these inequalities will be reversed or other way of saying that see the same thing is that if f has a local maximum at x naught, then minus f has a local minimum at x naught.

So, the derivative of minus f at x naught is 0 which is basically same thing as saying that f prime at x naught is to 0, so whatever we do with the conclusion will be the same. Now, next thing that we will obtain from here are the various versions of this mean value theorems, now in case of mean value theorem, basically we use the fact that you have already discuss the properties of the continuous functions on compact sets in particular. We know that if a continuous function on a compact set is always bounded and it attains its maximum or minimum in a compact set.

Every closed and bounded interval all the real line is compact this, what we shall use in in proving the next result, what is again well know what is called Rolle's theorem. Again, this is something you would have, you would have seen, but perhaps without realizing that, you are using the compactness of the interval in getting the main idea of the proof. So, let us say what does Rolle's theorem say, suppose we construct a function suppose f from a b to R is continuous on this closed interval a b and differentiable in open interval a b.

Then, there exist a point c in the open interval a, b, wait one more thing differentiable in a b and values of f at a f at a is equal to f of b and for the sake of convenience we can assume that value to be 0, f of a equal to f of b. This part is strictly not necessary though, then there exist there exist a number c in a b such that I have said already that I shall be using this short form such for such that such that f prime c is equal to 0. Now, before going to the proof of this theorem, let us just see what this theorem says geometrically that is you have a function, suppose you just look at the graph of the function, this is let me draw that let us say this is a and that is b.

(Refer Slide Time: 16:41)



The function is 0 here as well as here, so it may start either go up or go down, but whatever it is, ultimately it has to come down and meet this point b, so what this says is that at some point c its tangent should be parallel to x axis. It can happen at several points it may happens at several points, but there should exist at least one point at which the slope of the tangent is f prime c. As you know f prime c represent the slope of the tangent slope of the tangent is same as 0 as saying that the tangent becomes parallel to x axis.

Now, where does this all maximum, minimum etcetera coming into picture in this it is what I said before starting this theorem that f is function which is continuous on a b and a b closed interval a to b is a compact set. So, the function must attain its maximum or minimum some maximum as well as its minimum at some point in a b. Let us let us say at some point that is the beginning of the proof inside the interval a b. So, suppose I call those points as x 1 or x 2, so we can say that since f is continuous on the compact set on the compact set a b at some point, it has to attain its minimum.

At some point, it has to attain its maximum, so we can say that I will call those points namely x 1 and x 2, we can say that there exist x 1 and x 2 in a b such that suppose it is minimum here and maximum there. It is minimum here and maximum there such that f at x 1 is less not equal to f at x is less not equal to f at x 2 for every x in a b that is it is minimum at x 1 and maximum at x 2. Suppose, one of these point is an interior point, let us take that as x, suppose one of the let us say if x 1 or x 2 belong to a b, let me first take the case x 2 I shall suppose x 1 belongs to a b then it an interior point.

It is a minimum, in fact if this happens for every x in a b and if it is an interior point, you can always find a open interval containing that point inside which this happens, so it is also a local minimum. Since, we have assume that f is differentiable in the open interval a b if x 1 is in the open interval a b f prime at x 1 must be 0, so you take c as x 1, so if x one belongs to a b then f prime x 1 is 0. We take c as x 1 similar argument if x 2 is in a b similar argument if x 2 is in a b if x 2 is in open interval.

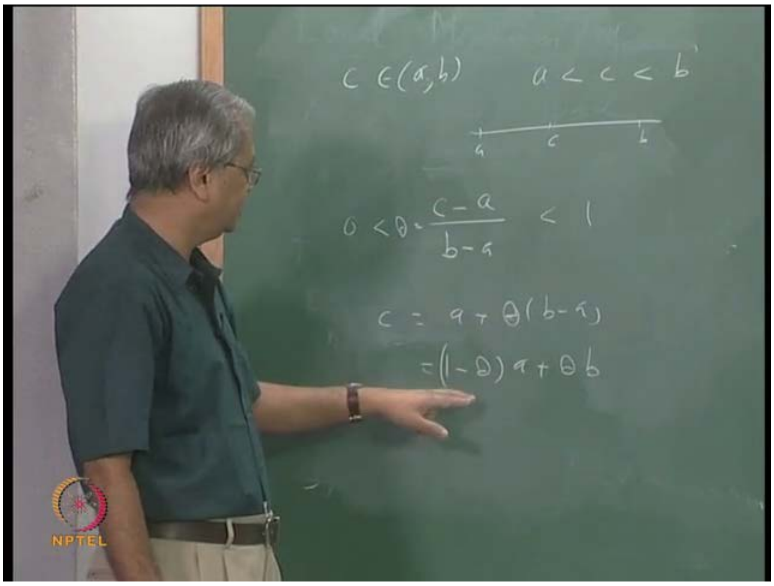
Similarly, if argument here since x 2 is maximum, you can consider in fact this open interval a b to be its neighborhood, so in that open neighborhood it is f of x 2 is bigger not equal to f x. So, some maximum and since f is differentiable at x 2 f prime at x 2 must be 0 and in which case you can take c is equal to x 2, so what is left if both of these things are not possible, in what way it can happen both x 1 and x 2 must be end points. Either x 1 equal to a and x 2 equal to b or x 2 equal to a and x 1 equal to b in fact argument in both cases is similar, let us just look at that, so this is one case if one of the points is in interior.

Otherwise, both x 1 x 2 are end points, so what is the possibly that is we can say case one is x 1 equal to a and x 2 equal to b, of course either case the proof will be the same and case two is x 2 equal to a and x 1 equal to b. In either case, you let us look at case one suppose x 1 is equal to a and x 2 equal to b what do we know about f at a and f at b both of them are 0.

Now, look at this suppose x 1 is a and x 2 is b, this will mean that 0 less not equal to f x less not equal to 0 for all x. So, this will mean that f is a constant function 0, so this will for example, imply that f x is equal to 0 for all x in a b, that means it is a constant function. So, its derivative is wherever it is differentiable, its derivative is 0, so you can take c as any point in the in the interval in the open interval.

Similarly, in this case even if x 2 is a and x 1 is b still it will mean that f x is 0 for all x, so it is a constant function and its derivative is 0 everywhere, wherever it is differentiable its derivative is 0. So, you can take again c as any point in the open interval is that clear, so that completes of course, I am sure you would have seen this theorem earlier and used it also, but perhaps you have not realize this step in the proof. The proof depends crucially on the compactness of the of the interval a b, now before proceeding further let us just say something about notation that is something that is used fairly often. While discussing this mean value theorems, we have said that c is a point in the open interval a b.

(Refer Slide Time: 24:48)

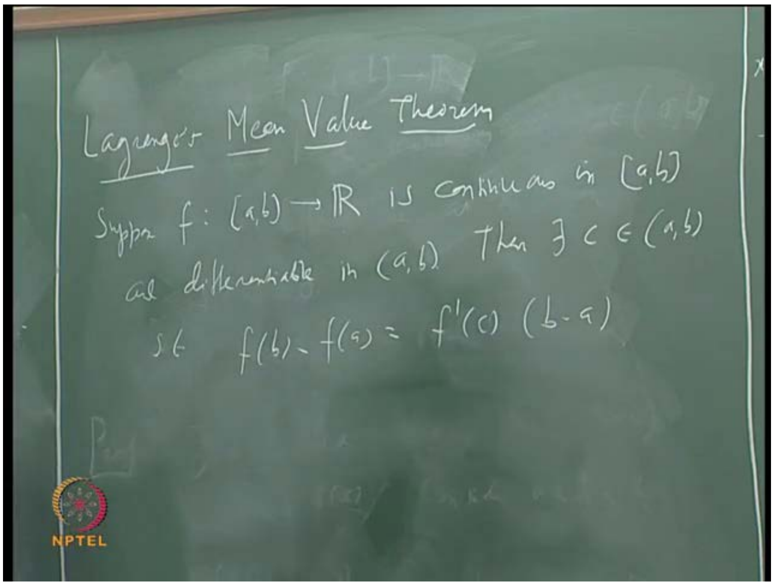


Here, c is a point in the open interval a b which see which as same as saying that a less than c less than b, that is if you draw the diagram a b in fact it cannot be b or a, it has to be some point other than that. So, if you look at this what is customary is that look at this c minus a divided by b minus a, this is a number which is strict c minus a is strictly less than b minus a.

So, this is the number which is strictly less than 1, also strictly bigger than 0 strictly bigger than 0, it is fairly customary to call this number theta theta is is equal to c minus a. Suppose, you call this as theta, then you write this c as a plus theta times b minus a, a plus theta times b minus a or another way of writing the same thing is this is same as saying that this is one minus theta times a plus theta times b. This last thing is what is called the convex combination of a and b, you may have come across this what convex combination of a and b.

So, what we are saying that you if it is an point lying in the interval of course with this understanding, we can even take c as a or b if for example, you take c as a, then you can take theta as 0 and if theta is 1, then c will coincide with b. So, this is something that is used commonly instead of saying that there exist c in a b, one says that exist theta lying strictly between 0 and 1 such that f prime. Then, whatever you write this c in terms of theta a plus theta into b minus a theta is 0, then let us again go to the next theorem in this same.

(Refer Slide Time: 27:12)

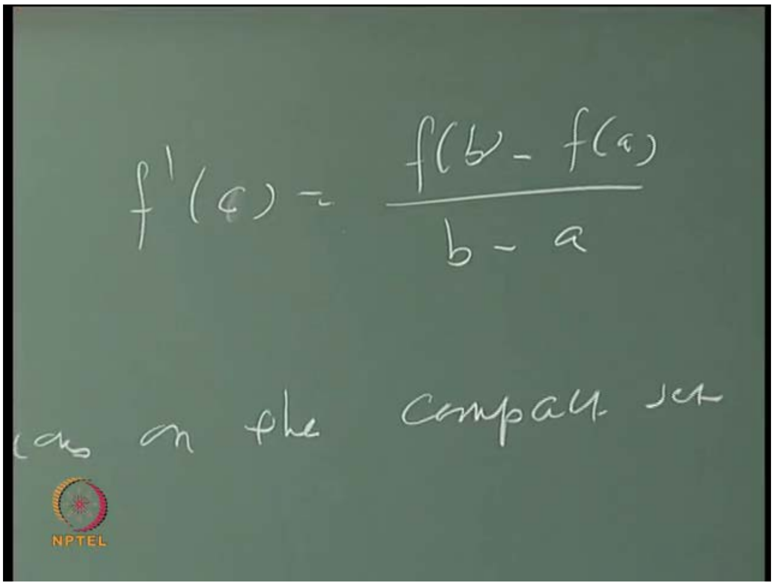


That is called Lagrange's mean value theorem, what is the difference we will have this both of this assumptions f is continuous on a b and differentiable in the open interval a b. Only thing is, we skip this last assumption, we do not assume anything about the values of f a and f b, so suppose f from a b to R is continuous in a b and differentiable in open interval a b. As I said, we are not assuming this last thing, but conclusion is again similar it says that there exists a point c in a b satisfying something not this equation something else. Then, there exist c in the open interval a b such that f b minus f a is equal to f prime c into b minus a noise.

Now, before proceeding further with the proof of this theorem, let me first observe a few things is it is it clear to all you that this Rolle's theorem is a special case of mean value theorem. If you get mean value theorem, you can prove Rolle's theorem also because suppose this is true, then in in addition if a equal to f b, then this left hand side become 0. So, this f prime c must be 0, so one could have said that you just prove this mean value theorem and then get Rolle's theorem from that the only problem with that.

That is the standard proof of mean value theorem uses Rolle's theorem, that is why we proved Rolle's theorem first and what is the proof we construct a function. This satisfies all the properties of this Rolle's theorem and get a point c and then show that point c satisfies whatever is written here. So, that is the proof, now again just as we have seen this geometric interpretation of Rolle's theorem, it is again convenient to look at the geometric interpretation of this theorem. Also before proceeding any further, now let me again go back to this this graph here, one can write this equation in a slightly different form I can say f prime c equal to whatever remaining here.

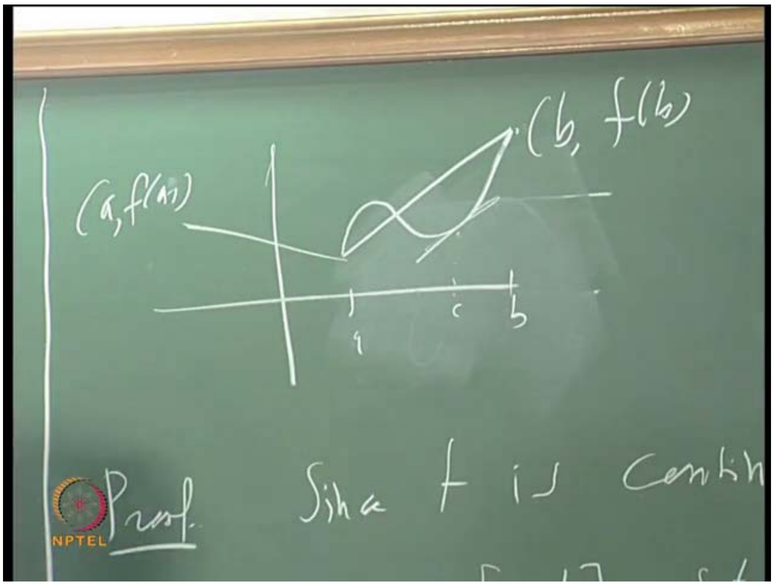
(Refer Slide Time: 30:45)



So, what does this mean, that here it will mean that f prime c is equal to f b minus f a divided by b minus a, f prime c is equal f b minus f a divided by b minus a. Now, we have already noticed that f prime c represents the slope of the tangent at the point c. That is something we know what about this what about this f b minus f a divided by b minus a, what does that represent?

It represent the slope of the cord joining the point a f of course f a need not be 0, f a can be somewhere, it can be somewhere here also f a the point f a at a f a and b f b. Let us suppose I change this graph here, let us say at a it may be somewhere here it may be something like this. Also, this is b and then this is the point a f a that is the point b f b, so the cord joining slope of the cord joining these two points.

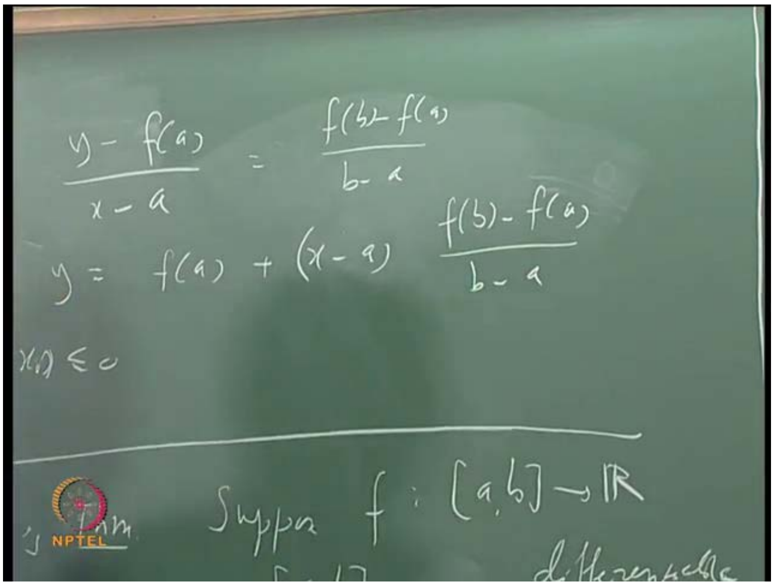
(Refer Slide Time: 31:36)



So, this is the point b f b point a f a, so that number on the right hand side is the slope of the cord and what does the theorem, the tangent is same as the slope of that cord. That means the tangent is parallel to that cord, the tangent is parallel to that cord joining the points a f b, that is you can say in this case some may be somewhere here, that can be c or it can be also somewhere here. Remember, we have not said anything about uniqueness even in Rolle's theorem or in mean value theorem we have not said that then point c is unique.

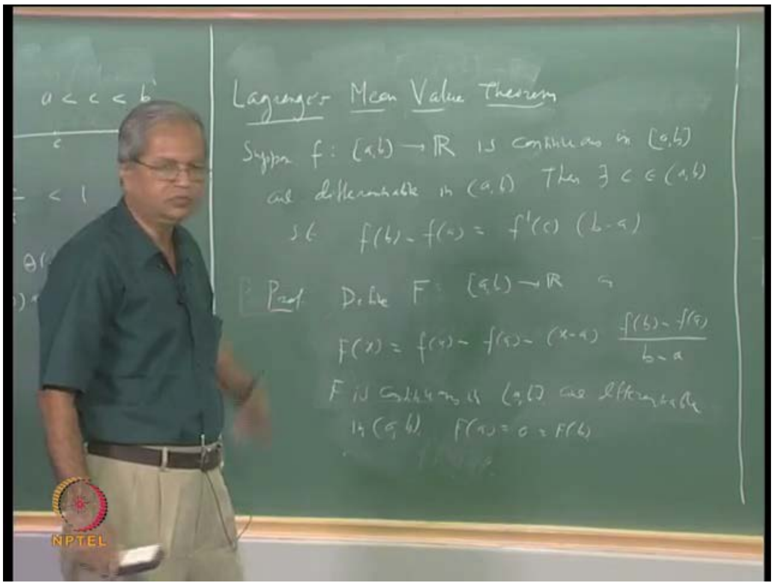
In fact in this second case, when we say that the function can be constant then in fact any point in the interval a b can be taken as the point c, so the point c given by conclusion either by Rolle's theorem or mean value theorem is never unique. So, what we do now is that we just consider the difference between the curve, this curve and this straight line just consider between this curve and that is the function which we call b, f that difference become 0 at the end two end points. So, at some interval its derivatives should be 0 and that will give the required thing, so what will be the equation of this straight line the point line straight line joining a f a and b f b, suppose arbitrary point, let us say here is x y.

(Refer Slide Time: 33:49)



For example, I can say that y minus a y minus f a divided by x minus a that should be same as one can say f b minus f a divided by x minus a or which is same as saying that y is b minus a. This is same as saying that y is equal to f a plus x minus a into this f b minus f a divided by b minus a, now what I do is that consider define the function.

(Refer Slide Time: 34:38)



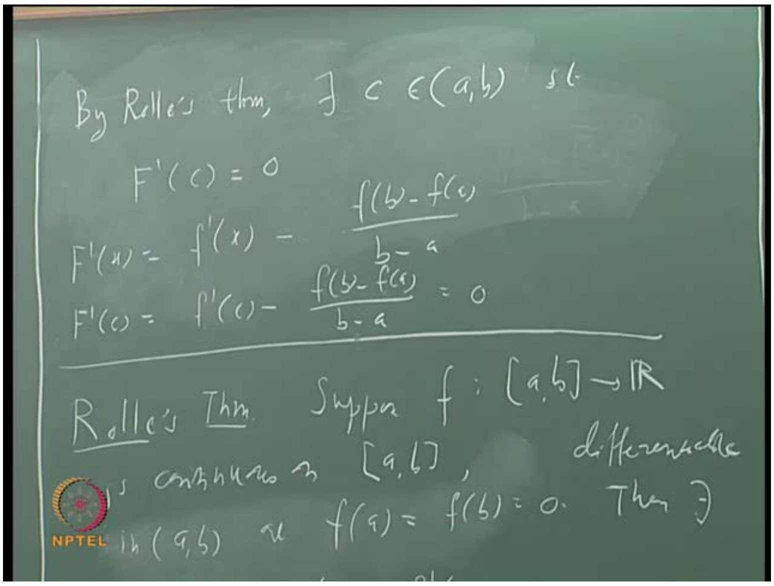
Suppose, I call that function F from a b to R as follows, as big f at x is equal to f x minus f a minus x minus a into f b minus f a divided by b minus a. So, what does this F represent geometrically, the first y equal to f x that is the given curve f x here and the second term that is this straight line. That is this straight line, so at any point, it represents this, for example suppose I take the point x here, it represent this difference, it represents this difference at any point. Now, the idea is to show that this function F satisfies all the above that is this Rolle's theorem and hence the conclusion. So, let us check one by one see what are this is small f that is continuous.

Anyway, this is a constant function x minus a is a continuous function this is multiplied by a constant, so that is continuous that is no problem. So, if is continuous in a what is the next thing we require, it is differentiable at a b, so what about that look at the right hand side. We already know that f is differentiable in a b, this is anyway constant that is differentiable everywhere, x minus a is again a polynomial differentiable everywhere and you can say that this whole thing is a polynomial it this this part is constant.

So, wherever small f is differentiable b f is also differentiable because this is the only term which decides and we know that small f is differentiable in the open interval and a b. So, we can say that hence by that it satisfies all the hypothesis of Rolle's theorem, hence by Rolle's theorem, we have to check that let us write what about F at a it is small f at a. So, this part is 0 a minus a that is 0, so fine so F at a is 0 what about F at b this will be f b minus f a, then minus b minus a multiplied by that b minus a should cancel with this and we will get f b. So, ultimately f b minus f a minus f b minus f a, so this is also 0.

In fact, you can go back to this diagram and see that we have observe that F represent a difference between this curve and this straight line and that difference become 0 here as well as at this point here. So, using Rolle's theorem, we must get a point c such that at that point this F prime at c becomes 0.

(Refer Slide Time: 39:36)

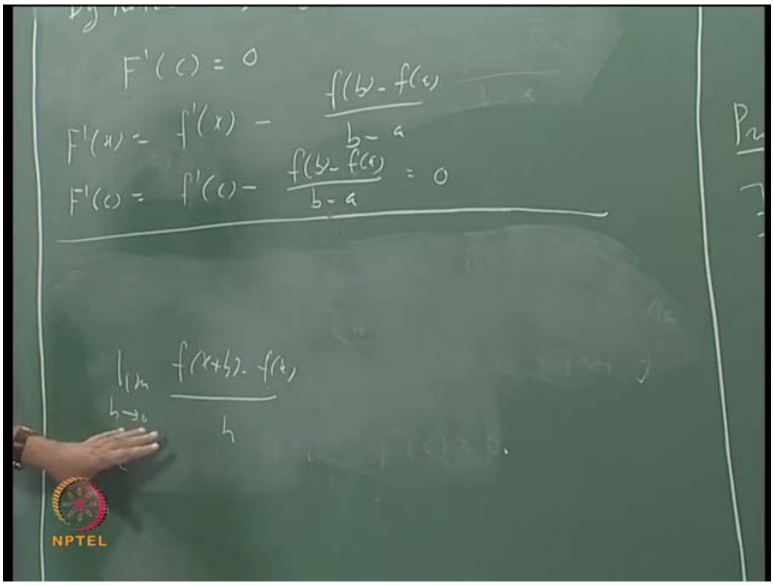


So, we can say that by Rolle's theorem there exist c in the open interval a b such that if f prime at c equal to 0, now let us rewrite this what is what is F prime. We know what is F at x, we know what is big f at x, so find out what is big, so differentiate this what we know is that f prime at x should be same as small f prime at x derivative of this is 0. Derivative of this x minus a will be 1, so it will remain as so f prime, so F prime is nothing but big prime x minus this f b minus f a divided by b minus a.

Now, put it back here, so F prime at c is equal to f prime at c minus this f b minus f a divided by b minus a divided by and we have shown that that is 0 that is 0. That is same as saying that small f prime at c is same as f b minus f a divided by b minus a, which is again same as what we wanted here f b minus f a is equal to f prime c multiplied by b minus a that is clear. Now, there are several uses of this mean value theorem many of things you would have used without perhaps realizing that what is involved here is the mean value theorem.

Let me just give one example, you know that if a function is monotonically increasing in an interval, then its derivative and if it is differentiable then the derivative must be bigger than or equal to 0. Simply follow from the definition because suppose we look at the definition again, it is limit as h tends to 0 as f of x plus h minus f x divided by h.

(Refer Slide Time: 41:50)

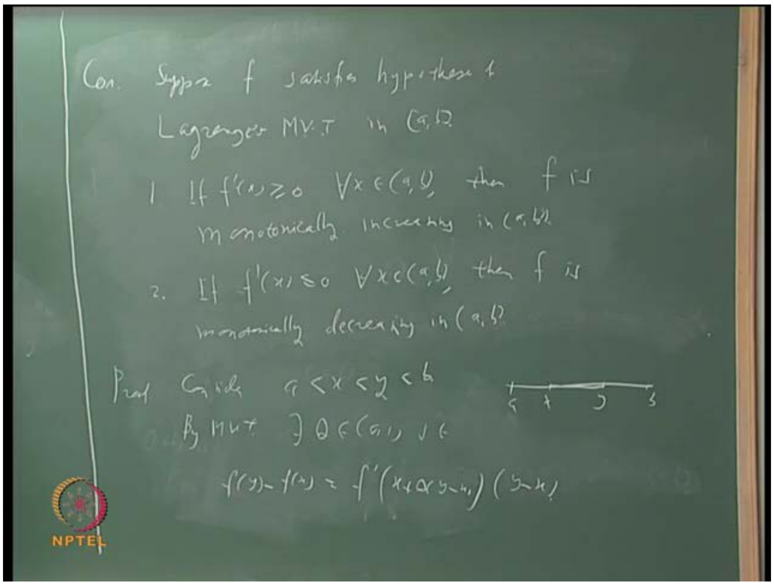


Suppose, it take some interval containing x and increasing then if h is positive f x will be not equal to f of x plus h that divided by h, so if the whole thing is a positive and so its limit will be bigger than or equal to 0. Similarly, if h is negative if h is negative, this will be less than that, but this h is also negative, so the ratio will be still be positive, so taking the limit you get that the derivative is bigger than the 1 equal to 0.

Similarly, one can show about the monotonically decreasing function the derivative must be less not equal to 0 at those points and again a special case of this is when the function is constant the derivative must be 0. Let us ask about the converse, suppose we know that suppose we that the derivative is bigger than or equal to 0 in an interval can we get from that fact that the function is pronominally increasing in that interval.

You have all used this you have all used this, but you can realize that it does not follow from this definition, it does not because from this definition. You can only be able to show that it a f of x less not f of x in some small interval containing x, but whether that happens throughout the given interval. That will not follow and that will follow by using mean value theorem, which will follow by using mean value theorem. Let us just see how, let us again assume that f satisfies this same hypothesis.

(Refer Slide Time: 43:54)



We can call this is a corollary of the mean value theorem suppose f satisfies hypothesis of Lagrange's mean value theorem in the interval a to b, that means it is continuous in the close interval, differentiable at the open interval. The first thing that we have to say is that if f prime x is bigger than or equal to 0 for all x in a b, then f is monotonically increasing in a b. Similar statement in support f prime plus less not equal to 0 f prime equal to 0 etcetera, if f prime x is less not equal to 0 for all x in a b, then f is monotonically decreasing and f prime x is 0. Then, it is monotonically decreasing as well as increasing, so it must be constant.

Let us just discuss the proof of this using mean value theorem, this will follow in a similar way now how does what is it what is required to show that a function is monotonically increasing. You take two points if x is x and y if x is less than that should be less than f of a, so let us consider the two points, consider a less not equal to x. Anyway, we want to show in the open interval only, so a less than x less than y less than b, now we have assumed that the functions satisfies the hypothesis of Lagrange's mean value theorem throughout the interval a to b.

So, in particular it does satisfy in the interval x to y, remember how we have taken x and y, this is let us say this is a x is some point here, y is some point here and that is b, just consider this interval x to y. So, suppose I apply mean value theorem to that interval x to y by mean value theorem, there should exist some point between x and y such that f y minus f x should be equal to derivative at that point multiplied by y minus x.

So, we can say that I will simply say by mean value theorem and for this I prefer to use this notation here because instead of saying it lies in the interval again discover use some more notations. The point z or some such things etcetera, so we will say that there exist theta by mean value theorem, there exist theta in the interval 0 to 1 such that f y minus f x i equal to f prime at x plus theta into y minus x.

This multiplied by y minus x, now let us take this first case, we know that f prime x is bigger than or equal to 0 for every x in a b, so in particular this is bigger than or equal to 0 and y minus x is also bigger than or equal to 0. So, f y minus f x is bigger than or equal to 0, so we have proved that whenever x is less than y whenever x is less than y f x is less than or equal to x y. That is same as saying that f is monotonically increasing, in a similar way you can proof in fact up to this the argument will be the same.

At this point, instead of using this you will be using this, of course this is something that you would have used earlier also without realizing that what is involved here is a mean value theorem. To convince yourself, you can try to prove this without using mean value theorem and you will you will see that it is it is not possible, but that you will not understand unless you make an effort to effort to prove.

So, this is about the mean value theorem and then we want to do after this there is a more general form of mean value theorem which is called Cauchy's mean value theorem. This deals with instead of dealing one function, it deals with two functions, but I think that is something we shall do in the in the next class, today we will stop with this.