Real Analysis
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Lecture - 19
Closed Sets

So, we were discussing the properties of closed sets in the last class, let us just recall a few properties which we discussed.

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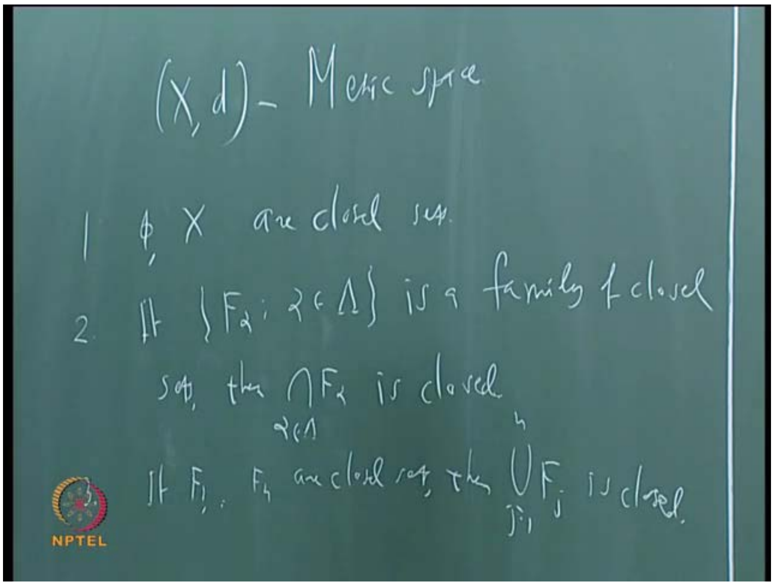

Let us say that X d is a matrix space then the first thing that we saw was that empty set and full space X are close sets and then another thing was that if we take any arbitrary family of close sets then its intersection is also closed. So, if F alpha say alpha in some indexing set lambda is a family of close sets then intersection F alpha in lambda is closed and lastly for I will this is not true of unions in case of unions you have to take only finite family.

So, if F 1, F 2, F n are close sets then union F j, j equal from 1 to n is closed and we have seen that as far as proofs of these are concern and one can proceed in several ways. But, one way will be that use the fact that any set A is closed if and only it is compliment is open and while using that we can say that since each F alpha is close compliment of F alpha is open.

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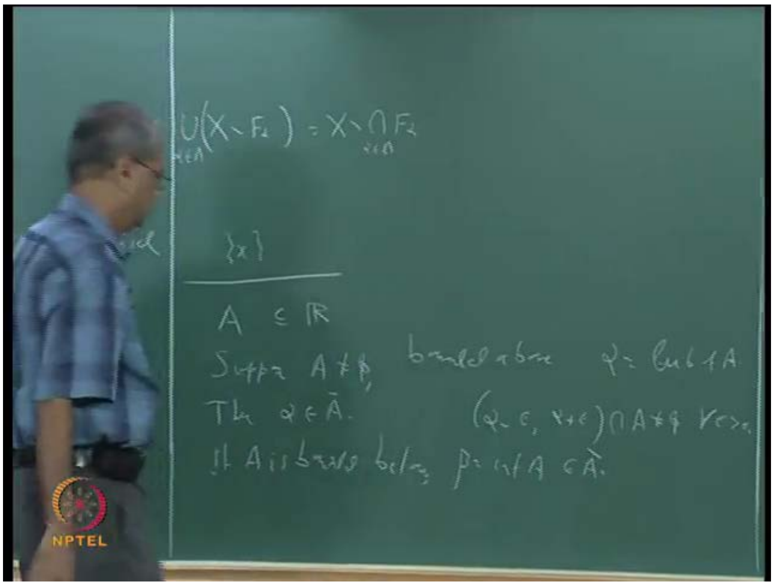

That is X minus F alpha is open and we have seen that in case of, in case of open sets union any arbitrary family of open sets is open. So, union of these were alpha lambda this will be open and this you need this nothing but X minus intersection F of and this must be opened and this is open means this must be closed, similarly you can prove about the intersections also.

But, of course that is not the only way you can prove this directly also, for example how does one prove that any set is close you just take any point in the closure I am sure that point is in the, in the sets. For example, suppose you want to do it that way take a point in closure of this x is a closure of intersection of what does it mean, if you take any open ball with centre at x then it must intersect this set it is the intersection. But, if it is something intersects intersection then it must intersect each of the set which is same as saying that if it is the closure of F alpha for each F alpha, but, F alpha is closed, so x belongs to F alpha.

Now, if it happens for each alpha it belongs to intersection, so whichever way either you use the compliments or prove directly the proofs are fairly straight forward. Then next thing that we will also see for example in the discrete matrix space we have seen that every sub set is open. It also means that every sub set is closed because U compliment will be open, now in case of the real line we have seen that every open set every non empty open set is a union of a countable I believe of disjoint open intervals. Now, does it say that how to get close sets in the real life in because all that you do is that you take us any countable family of open intervals and drop that family.

So, whatever remains must be a closed set, so that use again several examples of close sets we also seen that close ball is a close set. Let me just say couple of things about that in any matrix space singleton x is always a closed set suppose you take a singleton x is it said that it must be closed. Is it obvious that it is compliment must be in open set because you take the compliment x minus singleton x can you easily find a ball contended with disjoint from this set singleton x. So, compliment is open, so this must be closed it again it means it will if you combine this with this last term sortation, here it will mean that every finite set is closed because every finite set is a finite mean of single term sets.

So, it is closed of course as we all said seen in the case of open sets we cannot replace this by arbitrary family, you cannot say that an arbitrary family of close, the union of the arbitrary family of close sets is close because what you can do that you can take any arbitrary set then that will be the union of singleton sets. So, if that way true it will bring that every set is close, so which is on the case, so this last and if otherwise also you can easily find counter examples to show that you cannot replace the finiteness, you cannot dispense with the assumption of finiteness here.

But, this whole process of removing countable family of open intervals from real line can lead to fairly complicated sets we shall see one example shortly. But, before that let me also make one more common, so suppose A is a subset of a real line, suppose A is the subset of real line and suppose and suppose A is non empty and bounded above, it has say suppose A is non empty and bounded above. Then below that by l u b, x n, if a subset and bounded above it must have at least upper bound, so let us say alpha, let us say alpha is l u b of A then what I want to say that alpha must be in the closure A, suppose alpha is supreme of A.

Then alpha must be in the closure of A then alpha belongs to A closure, how does this follow, suppose you take any open ball with center at alpha. Now, open ball in the real line is nothing but we open interval it is for alpha minus R to alpha plus R or we can say in of the form alpha minus epsilon to alpha plus epsilon. Now, such an interval must contain a point from A, we have already seen that if you take any is it positive to epsilon alpha minus epsilon is not an upper bound there must exist X in A such that X is strictly bigger that alpha minus epsilon and it is obviously less nor equal to alpha B because alpha is an upper bound.

So, such an x belongs to all this interval alpha minus epsilon alpha plus epsilon, so in intersection of this with A will be always non empty for every epsilon bigger than this that is same as saying that alpha belongs to A closure. In particular, it means that if a set is closed then it contains at least upper bound, can we say similar thing about the greatest lower bound also, of then give a similar proof to show that if it is, if a set is bounded below then it must, then it is greatest lower bound also belongs to the closure of A.

So, similarly we can say that if A is bounded below is bounded below and we ties the let us say infimum of A then this beta belongs to A closure, so supreme of A and infimum of A they are always in the closure of A. So, in particular if A is a close set it will contain its supremum as well as infimum, now let us move to the example which I was saying that dropping this countable family of open sets can lead to fairly complicated examples of closed sets.

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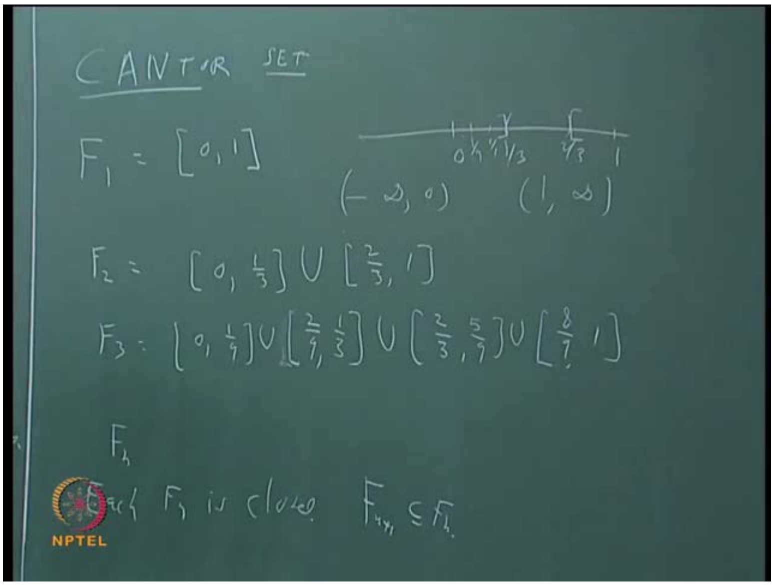

One of the most famous example of this take is what is called cantor set, to begin with we take the interval close interval let us say suppose I called x at F 1 I will take F 1 as the interval 0 to 1, it is it is an example of a close ball. So, it is a close, so it is a close so we already know F 1 is close and we can also say that F 1 is obtained from real line, so this is let us say close interval 0 to 1. So, it is obtained from the realigned by dropping the interval minus infinity is 0 and 1, 2 infinity these two are open intervals, so if you draw those to intervals from the realigned what remains is this set F 1.

So, that is a close set, now what we do in the next case is we draw the middle one-forth of this interval that is you take this interval 1 by 3 to 2 by 3, take this open interval 1 by 3 to 2 by 3 and drop this from, this from this drop that also, what will remain is it will be close interval 0 to 1 by 3, then close interval 2 by 3 to 1. So, what we get by this is this, so let us take the union of these two, so suppose I call that as F 2, suppose I call that is F 2 then that is also close set, that is also close set all.

So, it is obtained from the realignment dropping these two and close this one let the dropping feel open intervals then we continue this process further, now what we do is that, so 1 is F 2, F 2 is 0 to 1 by 3 and then 2 by 3, 2, 1, 0 to 1 by 3 n 2 by 3 to 1. Then for each of these two intervals I again drop middle one-third open interval, so what is that in this case it will main 1 by it is 0 to 1 by 3, so you can say, I can say I can write this. So, it will be 1 by 9, 2, 2 by 9 and from, here also it will be something this is 3 minus it is 4 by 9 to 5 by 9, so suppose I drop that, so what will remain is this F 3 will be say 0 to 1 by 9 union 2 by 9 to 3 by 9 that is nothing but 1 by 3.

So, 2 by 9 to 1 by 3 and then from, here it is 2 by 3 that is 2 by 3 is 4 by X 2 by 3 to 5 by 9 union say 8 by 9, 2, 10 by 9 that is 1, 2 by 3 is same as 4 by 9, so 4 by 9 to 5 by 9, so 6 by 9 to say 1 by 9 is top remaining is 8 by 9 to 1. So, 4 union of four close sets or you can say this same thing as dropping something like five open intervals or more, so if 3 is also close and then continue this way, so in each stage whatever are the each, so counting is waste. Suppose I call the n say line that is F n what is F n, F n is each F n is a finite union of close intervals, each F n is a finite union of closed intervals and each F n is obtained from F n minus 1, F n minus 1 in continuous order finite number of closed intervals.

So, from each of those intervals you drop the middle one-third open interval, so it will contain, so whatever we means will be F 1 whatever we minds will be F n, so each F n is closed, also F 1 contains F 2, F 2 contains F 3. So, in general F 9 contains F n plus 1, so what we get is each F n is closed each F n is closed and also F n or I say F n plus 1 is contend is F n, such F n we leave of 6 is called decreasing family. When F n plus 1 is containing F n, it is called a decreasing family of sets, so it is a decreasing family of close sets and what is the cantor set C, cantor set C is intersection of all such F n.

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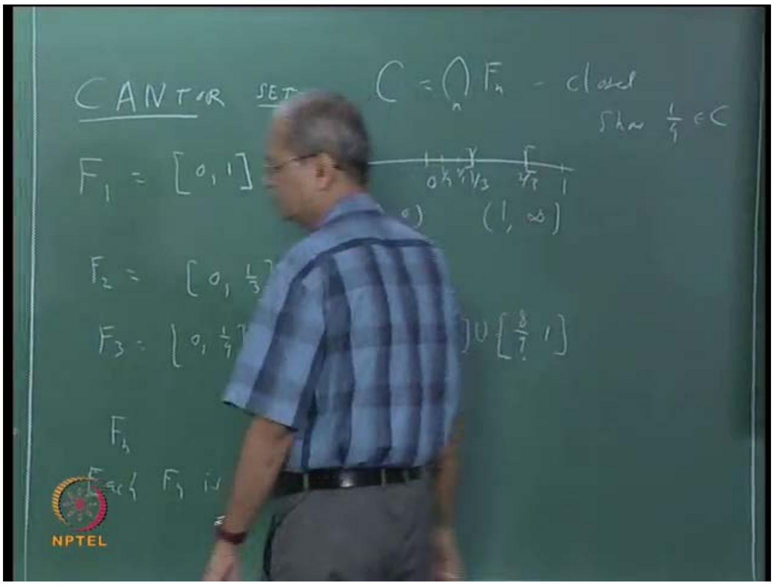

So, let me just write the suppose I call cantor set C it is intersection of since each F n is closed and we have already seen the intersection of any family of closed sets is closed. See itself is a closed set C is an example of closed set, now the question is whenever the cantor set contains any points at all because we have dropped so many intervals. But, you can see that at most stage 0 or 1 gets dropped, so cantor set obviously contains any points at all because you have dropped so many intervals, but you can see that at most is 0 or 1 gets dropped.

So, cantor set obviously contains 0 as well as 1 similarly it will contains all the end points 1 by 3, 2 by 3 this end points of the intervals not getting interrupt, so there will be several such points this 1 by 9, 2 by 9 etcetera, so many sets are, so many are there. I will give an exercise show that 1 by 4 belongs to 6, show that 1 by 4 belong 6 will require some work, but this cantor set has several interesting properties and you will come across this set again and again. There is one way of deciding whether a particular number belonging to cantor set or not and to do that we will look at what is called.

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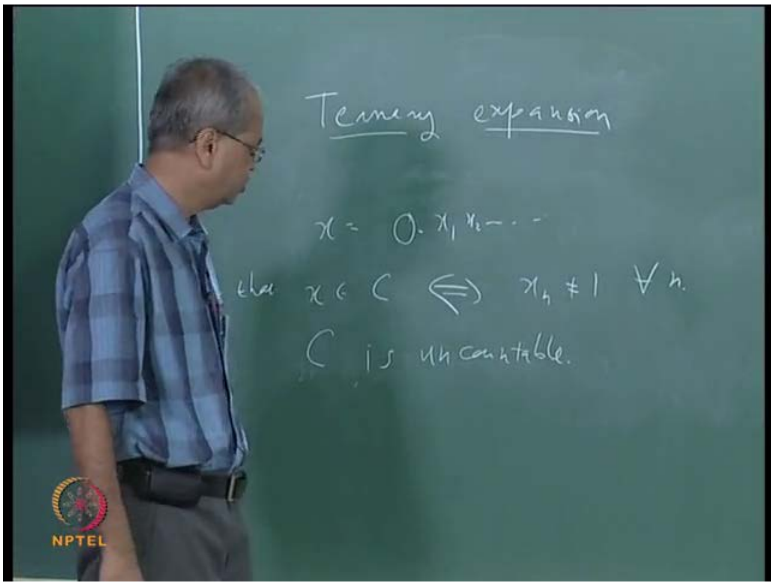

Ternery expansion, ternery expansion is similar to decimal expansion or boundary expansion decimal expansion is what it is nothing but what you write each real number be as a, as a decimal number in a decimal expansion. But, the digits occurring at 0 to 9, digits occurring 0 to 9 in binary expansion you take only 0 and 1, so similarly in ternery expansion you take 0, 1 and 2. So, each number since the number are lying between 0 to 1 there will be no integer part, so every number will be of this point say x 1, x 2 etcetera very each of this x n will be either 0 ,1 or 2 that is ternery expansion.

So, every number in the interval 0 to 1 you can write using it is ternery expansion, so whatever to say is that suppose if the number is x what about say at x belongs to C if none of these x ns are 1, if the ternery, if the no digit in the ternery expansion is equal to 1 then that number is in C. So, x belongs to C if an only if that is x, x is equal to this point x for which is the ternery expansion if x and if and only x n is not equal to 1 for all n.

In simple language, if this ternery expansion does not contain the digit 1 it will contain only 0 or 2, so this once you show this that will be easy to show that 1 by 4 belongs to C will be easy. Secondly once you show this you will also be able to show the following that C is uncountable, this follows that C is uncountable you have to show that set of all points that is ternery expansion contains only 0 and 2, that set is uncountable.

The proof will be again similar, so called diagonal procedure, so called diagonal procedure of cantor using that you can get C is uncountable, so this cantor set is uncountable it contains uncountable depending points. But, another interesting facts is suppose you look at the length of the intervals which are dropped from this intervals 0 to 1, so in the first stage you had dropped the intervals 1 by 3 to 2 by 3.

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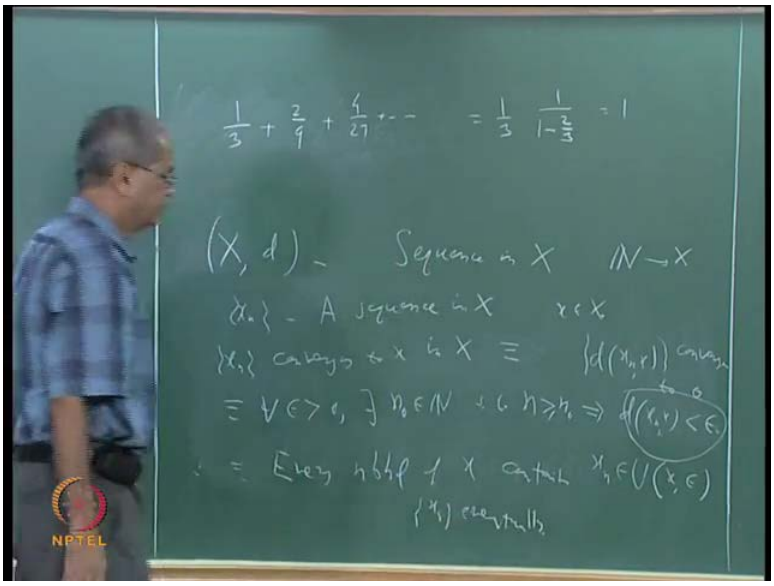

So, that length of interval is 1 by 3 in the next case, next case which intervals we have dropped we have dropped two intervals and the length of each of those intervals is 1 by 9. So, the total length that is dropped is 2 by 9 of course remember those intervals are disjoint from this 1 by 3 to 2 by 3 because they are coming from here and life of each of that is 1 by 9 there are two such intervals. In the next stage, what will happen you will draft for the intervals and the length of each of them will be 1 by 27, so the total length dropped will be 4 by 27 etcetera. So, if you know want to look at the sum of the all intervals dropped some of the lengths of all the intervals dropped that will be given by this infinite series, that will be given by this infinite sets.

Now, the obvious question is whether this series converges, now what is the answer it is, it is a geometric series geometric series with the common ratio 2 by 3 and the first term is 1 by 3. So, obviously converges and what is it is some the first term is 1 by 3 and divided by 1 divided by 1 minus 2 by 3 that will be the sum, now what is that it is the nothing but 1 that means the sum of the intervals dropped. That is 1, that is 1 or same as saying that it will lock if you had some notation or some notion of the length of any set then the length to the cantor set is 0 because from the interval 0 to 1, you have dropped the intervals whose total length is 1.

So, what remains is of length is 0 of course this divide more precise when you have, when you defined what is called a major of a set which you will do in the next semester. Where you will learn as course on, whenever you learn a course on major integration you will make this a idea, so cantor set is an example of a set whose major is 0, but it is an uncountable set x n all the time we will conclude this discussion about the close sets and talk about these of closed sets. Let us go to the next topic and suppose if I take a matrix space X and what we want to now say is what is meant by a sequence in X of course we have already do define sequence, we have already know what is meant by sequence.

Sequence is a function whose domain is set of all natural numbers so sequence in X is nothing but a function from n to x functions from n to x, so what we want to say, now what is meant by saying that a sequence in x is converges, what a sequence in x is converges to a point in x. So, let us say x n is a sequence in x where, now x n arbitrary matrix space x n is a sequence in X and suppose you take a point x in X, so what is the meaning of saying that x n converges to x that is what we want to find.

So, x n converges to x to X in this X, so the by definition this is the following of course one can define in several ways and all those definitions are equivalent and define the definition will be used in different context. So, we should just see all those things shortly but the first is this given any sequence x n like this and a point x n, x we can follow this the d x n x this should be a sequence of real numbers distance between x n and x that is the real number and it is sequence of dominating real numbers. So, we can always talk of what is meant by what is meant by saying that this sequence of real numbers converge something that is something that we already defined.

So, if this sequence converges to 0 we say that x n converges to X, so let me again repeat x n converges to X it is same as 8 that the sequence d x n, x converges to 0 and sequence d x n, x is sequence of real numbers and that is something we have already defined. So, this is the definition for convergent of a sequence in, but only thing is that this is a sequence in real numbers whereas this is the sequence in this matrix space it is the sequence in the in the matrix space. Now, we have definition for what is meant by saying that a sequence in any matrix space converges to some point in that matrix space, now let us write this in a more elaborate form and, so that we will we able to prove the various properties of convergent sequences.

Now, what is the meaning of this suppose I want, suppose I write the definition in the full form if there given any epsilon positive I can find some natural number n 0 etcetera. So, let us let us write that in the full form, so this means the following for every epsilon bigger than 0 there exists n 0 in n such that N not equal to n 0, N not equal to n 0 should imply what distance between x and X, and 0 that is what the x and x panel the 0 that should be less than x term d x n and is already not negative. So, what d n and x 0 is nothing but d x n x, so this means distance between x n and x is less than X, so what it means it given any epsilon bigger than 0 we should exist some n 0 such that for all N not equal to n 0 distance between x n and x is less than 0.

Now, what I will do is I shall just re write this last thing distance between x n and x is less than epsilon, is it same as saying that x n belongs to an open ball which center at x n radius epsilon. So, that means this last thing is same as same that x n belongs to open ball with center at x n radius x, so what does it mean that sequence x n converges just to x means every open ball which centre x whatever the radius every open ball might centre at x should contain all points of the sequence after n bigger after some state n 0.

Let us again go back to our terminology what we were study eventually we say that something some property of a sequence of the different elements of a sequence force. Eventually if there exist some n 0 such that for n bigger nor equal to n 0 that property is true, so what we can say is that given any open ball centre at x. The sequence x n lies eventually in that open ball sequence x n lies eventually in that open ball, can we also replace open ball by any open set, can I also say that every open set containing x will contain the sequence eventually.

So, x n converges to x means every open set containing x will contain the elements of the sequence eventually, there is some other equivalent formulation does it also mean that every number moved of x contains the sequence x elements eventually. So, let me just write at the last thing again that is this is F n only every number move of x every number moved of x contains x n eventually we also describe this same thing saying that x n converges to x.

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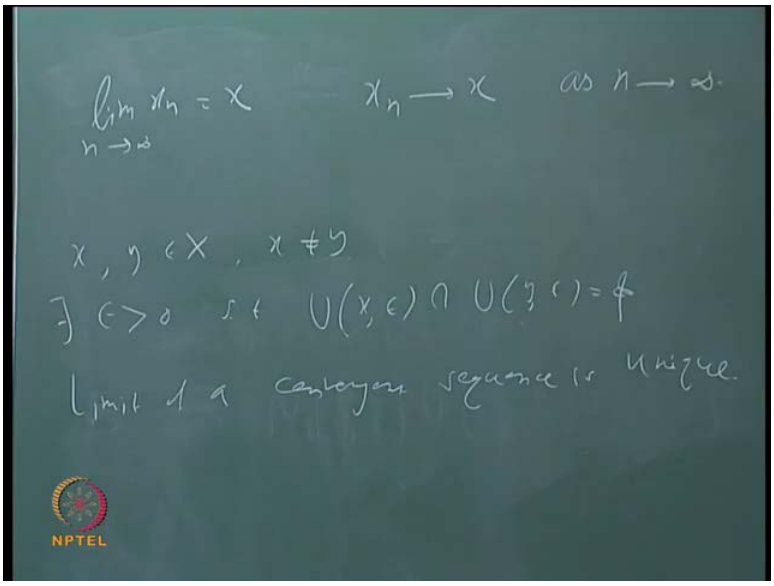

By also this same symbol limit of x n as n tends to infinity is equal to x, this is same as this is just one more notation for this same thing x n converges to x and also this notation x n converges to x, x n converges to x, x n tends to x as n tends to infinity. Sometimes we drop this also and simply say x n tends to x all this is meaning in the same the sequence x n converges to x. Now, we have a notion of what is meant by saying that a sequence in any arbitrary space x converges, now there is one obvious thing to follow from here that if a sequence converge of course a general sequence may or may not converge.

But, whenever it converges it has to converge to a unique point it cannot converge to two different elements in x because suppose given any clue points let us say x and y in x with x not equal to y what we have seen is that if you take any two different I think I had given that to you as an exercise you can always find a number. Such that open ball will send centre at x radius R and open ball will centre at y n radius R are disjoint, so this see F x is not equal to one then we can always find. Let us say suppose I call epsilon, suppose I call it epsilon there insists epsilon is bigger than 0 such that that is U x epsilon and you y epsilon these two are disjoint.

Now, suppose it is so happens that x n converges to x also and x n converges to y also then all points of x n must lie in this ball eventually and they also must lie in that eventually. Obviously that cannot happen or we just make this full argument to size we can say that there will exist some end one such that when n is bigger not equal to n 1 x n you accepts epsilon. So, in early when m is bigger nor equal to n 2 x n is in U y epsilon and, so you can take the maximum of n, n 1 and, so whenever n is bigger not equal to both then x n must lie in both U x y also, sorry U x epsilon also and U y epsilon also and that cannot happen because these were disjoint open points.

So, in other words if a sequence cannot converge to two distinct points, so let us, so we say that limit of a sequence limit of a convergent sequence is unique limit of a convergent sequence is given. Now, we have already seen several examples of the sequences that converges in real number with the usual matrix, we can also see a few examples in other matrix spaces before going to some more general concepts.

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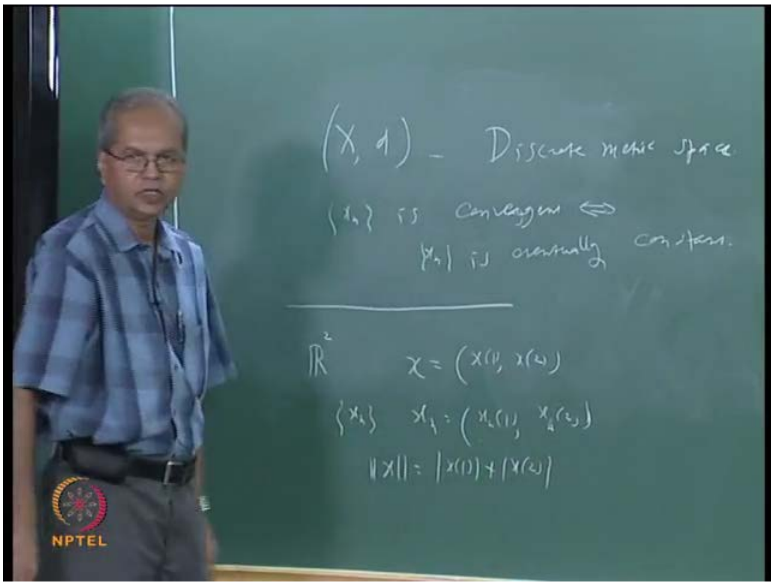

So, let us take say X d suppose discrete matrix then what will be what will be the convergent sequences in this discrete matrix space of course constant sequence will be common and this converge the same. But, are there any other examples that is sequence cannot be constant from the beginning, but it has to be eventually constant because if you can take this suppose this epsilon is less than 1. Then this ball with centre at x n is nothing but single term x and that means x n has to be equal to x or n bigger than or equal to n 0, which is same as saying that after n bigger or equal to n 0, which is same as saying that after n bigger or equal to n 0 the terms in the sequence become constant.

So, in discrete matrix space we only, we only convergent sequence are the once which are eventually constant, so we can say that, then we can say that x n is convergent if and only if x and is eventually constant. Let me just write it this way, eventually constant and this is the first few term may be different, but after some stage it must become a constant sequence.

So, these are the only sequences which converge these are the only sequences which converge, let us take some other space in case of R we already know in case of R a we already know what are the converging sequences etcetera we have, we have done a complete thorough discussion on that. So, let us take some of the space, so let me take the space R 2, now to discuss the sequences, so till now we have been using this suppose I take a point x and denote that point x as x 1, x 2.

Now, I will slightly change this notation because this notation is not very convenient when I want to discuss the sequences in R 2 because then I will not take say suppose say this is x n then that will confuse with this x 1 and x 2. So, instead of this I shall use this notation x 1 and x 2, x 1 and x 2 that is the first coordinate is x of 1, so can coordinate x of 2 what is the idea this if I take a sequence x and do U, if I take a sequence x and do U, I can say that this x and that will be x n 1 and x n 2.

So, in other words every sequence x and in R 2 will give raise to two sequences of real numbers, so x and 1 and x n 2, x n 1 and x n 2 of course you talk about the convergence in R 2 we have to start with some matrix in R 2 some matrix. Let us take this, so called one matrix given by, given by one norm, what is a one matrix like this norm of x is mod x 1 plus mod x 2, plus mod x 2 then what about say is the following.

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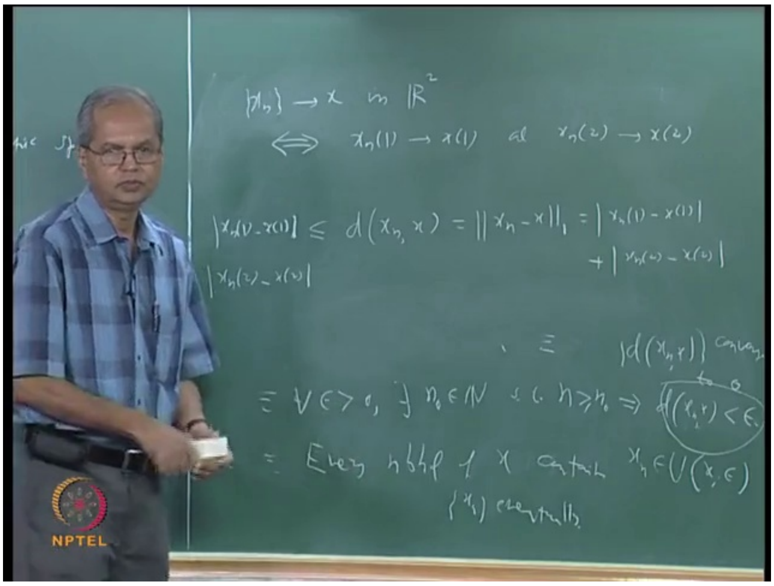

The sequence x n converges to x in R 2, converges to x in R 2 if n only if n only what should happened.

Student: Both should converge.

That is both the sequence x n 1 and x n 2 both should converge and converge to what x n 1 should converge to x 1 and x n 2 should converge to x 2, x n 1 should converge to x 1. So, if and only I will say x n 1 converges to x 1 and x n 2 converges to x 2 everywhere you can write as intense, now how does that follow how do, how does to prove this. We have to look at the way in this is what we had called do U suffix 1 this is what we have called norm suffix 1, so basically we have to look at the way in this norm suffix 1 is defined, so suppose we look at say distance.

But, see what is distance between x n and x what we have to show is that the sequence x n converges to x means given any epsilon you can find some n 0, so that whenever n is bigger nor equal to n 0 this is less that epsilon. But, what will be the distance between x n minus x suffix 1, norm of x n minus 1 suffix 1 and what is this by that definition it is nothing but mod x n 1 minus x 1 plus mod x n 2 minus x 2. That is definition is it also correct to say that this will be always bigger than or equal to mod x n 1 minus x 1, will it also be bigger than or equal to this mod x n 2 minus x 2, now does it follow from here that if this is less than epsilon these two also be less than epsilon.

So, this way if x n converges to x then x n 1 must converge to x 1 and x n 2 must converge to x 2, what about the converse suppose you know that x n 1 converges to x 1 and x n 2 converges to x 2, how will I say that x n converges to x and again the usual that x norm by two proofs. We can say that given any epsilon you find n 1 such that this part becomes less than epsilon by 2 there exists some n 2 because, so that whenever n is bigger nor equal to n 2 this becomes less than epsilon by 2. Then this whole thing will become less than epsilon for n bigger not equal to both n minus n 2, nothing new idea usual way of proofs, so what that means is that if you take a sequence x n in R 2 that is that converges to some point x in R 2 if and only x n 1 converges to x 1 and x n 2 converges to x.

Is there anything particular about this converge to a can I replace that by say R, so what you can we see that the same thing will be 1 if I take R 3, R 4, R 5 anything arguments should be similar. So, we can say that in general x n converges to x in R k, R k of course R k with this norm, R k with this norm what will the norm take it should be mod x 1 plus mod x 2 etcetera up to mod x k. So, instead of this I will say n and 1 by this 2 x j for j equal to 1, 2 etcetera up to k what will be the obvious thing argument obviously x n converges to x, each of this number mod x each of this number mod x n j minus x j, each of them are going to be less nor equal to distance between x and x.

So, this made it is trivial and then for the reverse argument we have to take something like epsilon by k and then take n 1, n 2, n 3 etcetera for each of those that is that is rate for. In other words, what we can say is that in this space R k, a sequence x n converges to x if every sequence x n will give rise to k sequences followed by this different context. If all those k sequences converge then x converges or vice versa.

Now, let us take a slightly different question, instead of this norm suppose I use some other norm suppose I look at what let us say norm suffix infinity, what is norm suffix infinity. It is it is the maximum of mod x 1 mod x 2 etcetera we need the same similar not the same, will the similar proof work this one is true if because this is the maximum, so each of this on x j minus x j will be less nor equal to this, here you will have to give some other argument. So, what is important is that it does not matter even if you take similar norm this thing will be still true whatever norm you take, here it will be still be true if x n converge to x if and only x n j minus to x for x j for listing.

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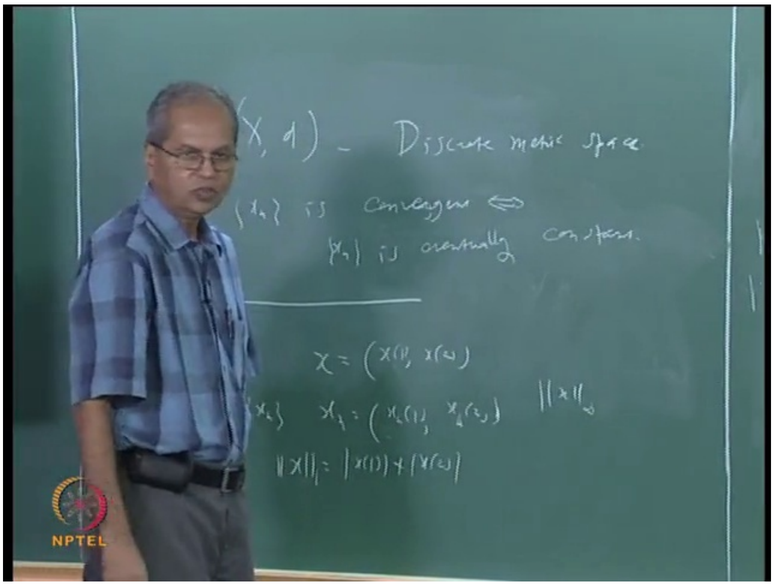

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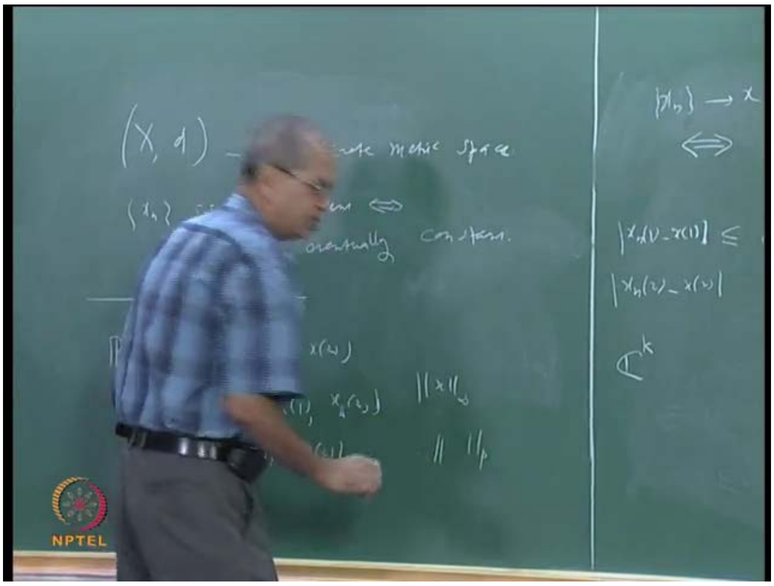

A similar thing you can say about this space also C k because there also norms that define in similar manner and again not only if this norm 1 or norm suffix infinity. But, even though so many norm suffix p that we have defined for that also for those norms belongs to also this will still be true. We shall not repeat the argument because the arguments are essentially similar, now let us go back to again a general matrix space and we consider similar concept of what is known as a, let me write, here. In case of real numbers we have defined what is meant by a Cauchy sequence we can, we can take the same similar concept in arbitrary matrix space.

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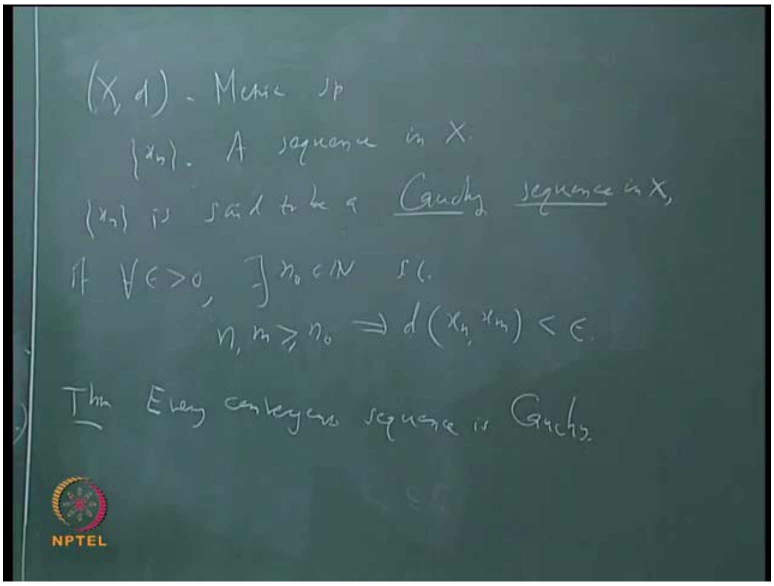

So, suppose X d is a matrix space and x n is a sequence in x, sequence in x then we say that x n is said to be a Cauchy sequence in x, if, this is Cauchy sequence, what should happen for every epsilon bigger than 0. Every epsilon bigger that 0 there exists n 0 in n, there exists n 0 in n such that if you take any two n and n and m because not equal to n 0 then the distance between x n and x m should be less than epsilon. So, what we select n and m bigger not equal to n 0, this implies distance between x n and x m is less than epsilon these are the rest questions.

So, the concepts which we generalize from the real numbers to any matrix what we have done is that we have just replaced definition of the usual Cauchy sequence by see in the usual C sequence of real numbers this would have been mod x n minus x. This is nothing but the distance between x n and x n in the real numbers, so what we have done is that the kind of thing that we defined in the case of real numbers we just generalized that to any matrix space.

Now, we can talk of what is meant by Cauchy sequence in any matrix space, now what do we now about the relation between a Cauchy sequence and convergent sequence. In real numbers, we know that every convergent sequence is Cauchy and we also know that every Cauchy sequence is convergent. Now, let us see which of this is that true in arbitrary matrix space, so first of all it is very easy to show that in any matrix space every convergent sequence is Cauchy. So, that is, that is the theorem every convergent sequence is Cauchy, every convergent sequence is Cauchy.

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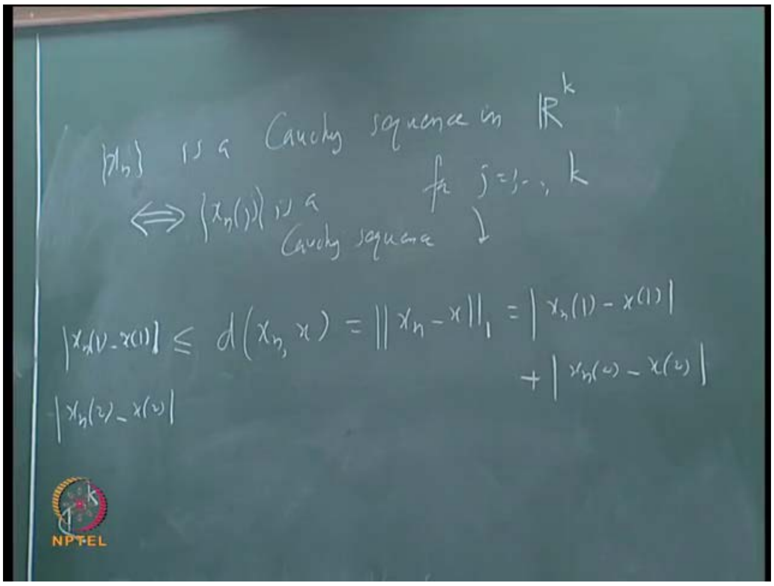

Now, before going to the proof of that theorem let me again make a comment on this spaces R 2, R 3, R k etcetera. Now, we have seen some description of convergent sequences in each of these spaces suppose I want a similar description about the Cauchy sequences, instead of saying that x n converges to x in R k.

Suppose I say that, suppose x n is a Cauchy sequence in R k, x n is a Cauchy sequence in R k what will this be replace by if and only what can I say can say if and only each of this x and j. We can say that this can replace by say that x n j is a Cauchy sequence for j equal to 1 to k, so what we saying is that in each of this spaces are R k if you take any sequence it leads to k sequences of real numbers by taking co ordinate face. So, if the sequence x n is convergent, each of those k sequences should be convergent, similarly if the sequence x n is Cauchy each of those k sequences also should be Cauchy and vice versa, let us also look at some other spaces we come to this theorem later.

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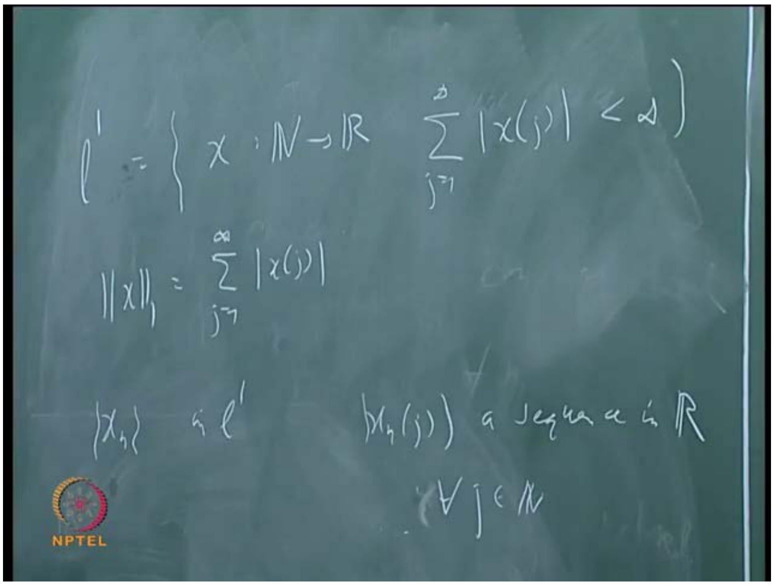

For example, let us take the space l 1 what was the space l 1 we have said directly this is space of all sequences such that sigma mod x n is convergent. So, this is the space of all sequence x such that x is from N to R such that sigma mod, using a similar to notation for immediate x n I shall call it x, x of let us say x of j, j equal from 1 to infinity, this is a convergent space that is description of x n.

We have seen that this is the matrix space in fact it is non linear space given by this norms suffix 1 is sigma mod x j, j going from 1 to infinity. Now, I can ask a similar questions in this space itself, so suppose now I take a sequence x n in l 1 in spaces like R k and C k what was happening that we should take one sequence in R k that lead to k sequences such of real numbers.

In this case what will happen that there be infinitely many sequence, if x n is sequence in l 1 each of this x n j this will be a sequence in R, each of this x n j will be for each j, so this is for each j in, for each j in n and then we can ask a similar questions. Here, also suppose x n tends to x n, l 1 suppose x n converges to x in l 1 then can we say that each of this x n j converges to x j as such sequence of real number. I think I will stop with this, we shall consider the proof of this theorem tomorrow.