Lecture - 11
Infinite Series of Real Numbers

We will continue our discussion about the infinite series, which we started in the last class.

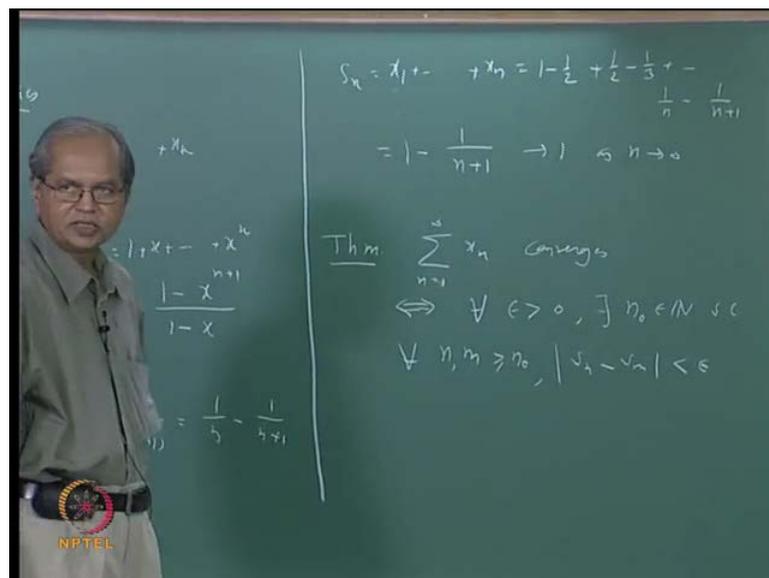(Refer Slide Time: 00:23)
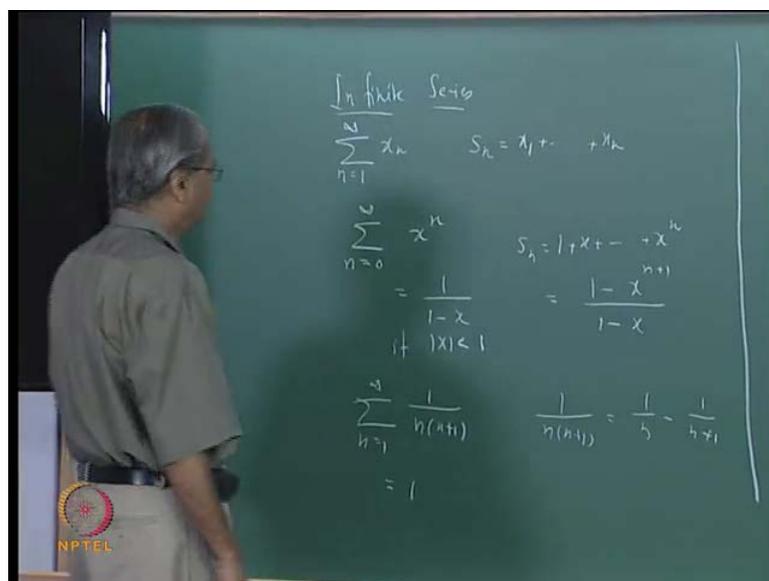
Let us recall that infinite series is an expression of the following form sigma x n, n going from either 1 or 0 or some such number. And we have seen that along with every such infinite series we are set a sequence S n, which we call a sequence of partial sum, it is sum of the first n elements in that sequence. Depending on whatever happens to S n, we say that if S n converges we say that series converges or sum able and whatever is the limit the of S n, we say that is the sum of the series.

And similarly, if S n diverges we say that the series diverges. So, we can easily see 1 or 2 standard examples of this before proceeding. So, let us now this is as you know it is called geometric series sigma x to the power n and it is customary to start this for n going from 0 to infinity. So, in this case if you look at S n, S n will be 1 plus x plus etcetera up to x to the power n. So, this is nothing but since this is geometric progression so, you can

sum this as 1 minus x to the power n plus 1 divided by 1 minus x. So, whether the series converges or not, depends on whether the limit of this exists as n goes to infinity.

And out of you can see only term, which depends on n here is this, x to the power n plus 1, all the term other terms are independent of n. Here we have seen that these converges to 0, if mod x is less than 1, so it means that this series converges to 1 by 1 minus x, if mod x is less than 1. So, we can say that it is some of the series is 1 by 1 minus x if mod x is less than 1. If mod x is bigger than or equal to 1, this will not exist limit of this will not exist and this never just that is also fairly easy to see all. Let us one more example, suppose you take sigma 1 by n into n plus 1, n going from 1 to infinity. Now, here only one fine thing to notice is the following that 1 by n into n plus 1 is nothing but 1 by n minus 1 by n plus 1.
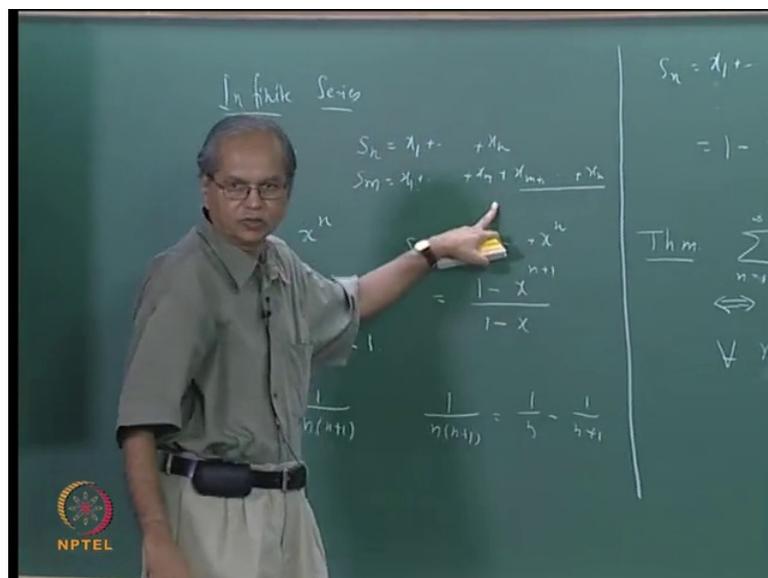
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Let us say S n as x 1 plus etcetera x n and so that is x 1 is 1 minus half that is 1 by n minus 1 by n plus 1. So, x 2 is 1 by 2 minus 1 by 3 etcetera, etcetera. So finally, x n is 1 by n minus 1 by n plus 1. And it is clear that middle terms will cancel and final sum will be nothing but 1 minus 1 by n plus 1 and this tends to 1 as n tends to infinity. So, this series converges and it is sum is 1. So, in general this is how you decide whether a series converges or not look at the partial sum and take the limit of the partial sum as n goes to infinity.

Next what we want to see is that apply the various theorems that we have known about a sequences to get the corresponding results about a series. Let us look at that the last thing that we saw about sequences first. We have seen that sequence of real numbers converges if and only if it is Cauchy sequence. So, what how does it translate in terms of infinite series, we can say that this let us write this as a theorem, sigma x n, n going 1 to infinity converges if and only if this S_n is a Cauchy sequence. Let us write this in the full form, what is meant by Cauchy sequence that for every epsilon bigger than 0, there exist n 0 etcetera.

So, that means if and only if for every epsilon bigger than 0, there exist n 0 in n, such that for all n and m bigger naught equal to n 0, we must have mod S n minus S m less than epsilon. Only thing is I want to make a slight change in the way of writing this mod S n minus S m, n and m both are bigger naught equal to n 0 so, let say one of them is bigger than the other. So, suppose let say n is bigger than m and m is bigger naught equal to n 0 then, S n is x 1 plus x 2 etcetera plus x n and what will be S m.

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S m will be x 1 plus x 2 plus x m then if since m is smaller plus x m plus 1 etcetera, etcetera up to x n. So, what is S n minus S m, it is this part x m plus 1 etcetera up to x n. So, I can say that instead of writing like this I shall say this is same as sigma x j, j going from m plus 1 to n.

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So, if you take any two numbers m and n bigger than n 0 and suppose m is strictly less than n. And if you take all the sums of all the terms going from x m plus 1 to x n, then that should be less than epsilon. In other words given any epsilon for large values of n, some of that part whatever large numbers m and n you take, you take the sum from x m plus 1 to x n. It is substitute value should be less than epsilon. There is one very important and immediate, but simple consequence of this. You can see that there is nothing to prove here.

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This simply follows by observing that we have just re-written the same thing that S n converges, if and only if S n is a Cauchy sequence in a different language. One obvious corollary of this is the following, suppose I take m is equal to n then, there will be only 1 term here and that will be it mod x n less than epsilon. So, what is the meaning of that for every epsilon there exist n 0, such that whenever n is bigger naught equal to n 0 mod x n is less than epsilon. What is the meaning of that it means x n tends to 0. So, if the series if sigma x n converges then, x n go goes to 0.

So, if sigma x n converges and that follows simply by taking n is equal to m here because that is how we define saying that x n converges to 0. So, if sigma x n converges then, x n goes to that let us write it like this, limit of x n as n tends to infinity, that is equal to 0. Now, there is a very obvious question here and that is what about the converse. Again it is well known that the converse is false, but the only way to say something is false, you need to give a counter example. So, that is something we shall see little bit later. So, that support these this is sometimes called Cauchy criteria for the convergence of the series. There is one more thing that follows from this and that is following.

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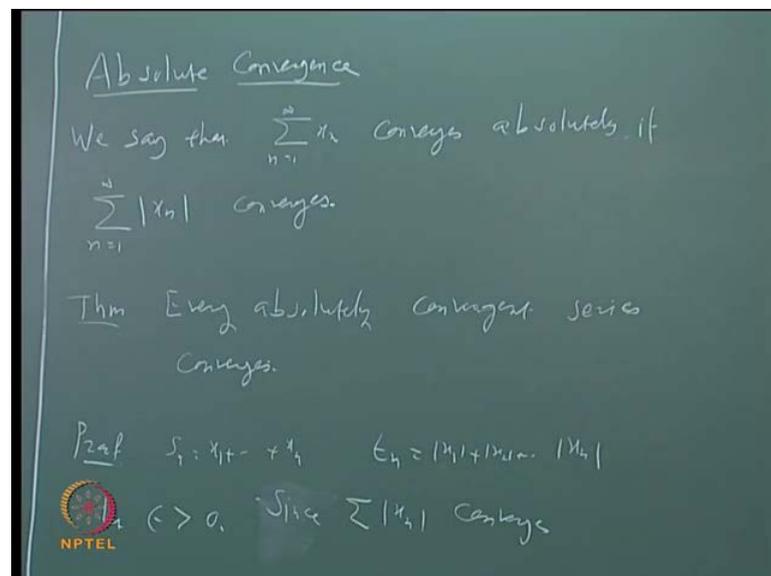
Let us define what is meant by absolute convergence. Now, we say that the series sigma x n converges absolute n going from 1 to infinity converges absolutely. If you take the series of absolute value sigma mod x n, n going from 1 to infinity that converges. And so, again the obvious question is what the relationship between the convergence and

absolute convergence is? And then again the well known thing here is that every absolutely convergent series is convergent. So, let us write in that every absolutely convergent series is convergent.

Proof follows from this Cauchy's criteria. So, let us let us see how does it that follow. Now, since we are dealing with two series here sigma x n and sigma mod x n, let us give some notations for the partial sums of both. For this any way we have already denoted by S n. So, let us say S n as x 1 plus etcetera up to x n and suppose partial sum of this series I call t n, t n is let us say mod x 1 plus mod x 2 etcetera plus mod x n. All we want to show that these series converges and using this theorem it is enough to show that this happens. So, let us say that let epsilon be bigger than 0 then, we know that since this series converges, we know that this series satisfies this criteria. So, for that series there exist n 0, such that all these things happen.

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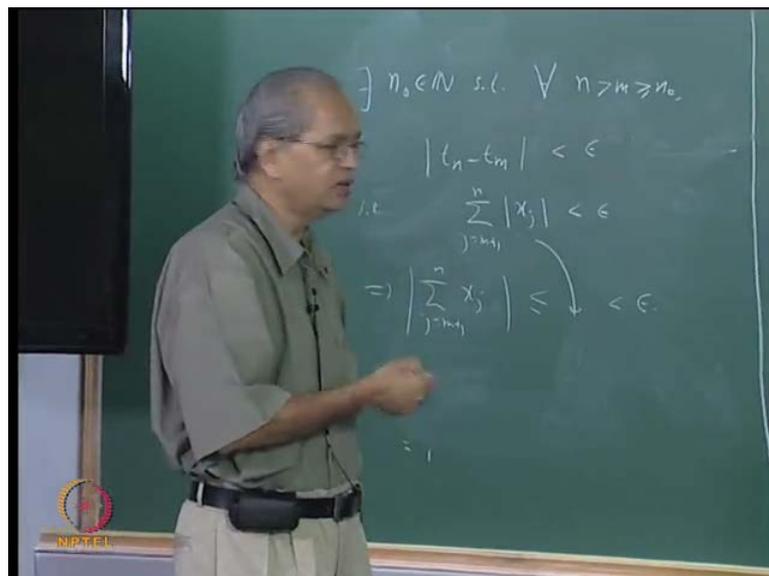
So, there exist n 0 since sigma mod x n converges there exists n 0 in n such that, for all n bigger than m bigger naught equal to n 0 mod t n minus t m is less than epsilon. But what is t n minus t m, t n is mod x 1 plus mod x 2 plus mod x etcetera. So, similarly, we can say that t n minus t m should be same as sigma mod x j, j is going from m plus 1 to n. So, this is nothing but that is we can say that sigma mod x j, j going from m plus 1 to n less than epsilon.

Now, what is the relationship between this term and this term? This is less not equal to this so, this implies that sigma x j, j going from m plus 1 to n, absolute value of this that is less not equals to this and that is less than epsilon. So, this is a very important theorem again let us see what exactly this depends on every absolutely convergent series converges.

How did we prove this, we proved this using Cauchy's criteria. How was Cauchy's criteria proved by using the fact that every Cauchy's sequence is convergent and how was that proved by using Lube axiom order completeness of r. So, the fact that every absolutely convergent series, converges depends in fact on the completeness or order completeness of the real number system or actually the so called Lube axiom. Now, since again here also the question what can we say about a converse that, is it true that whenever a series converges, it is also converges absolutely that is false.

And again to show that it is false, we need to give a counter example, but that also we shall discuss little later that is, when we come across some more methods of checking convergence and divergence. Now, there is one very good thing here is that this gives us a class of convergent series. Let us whatever the series is given what you can do first is to check whether it is absolutely convergent. And to check absolutely convergent you just need to look at sigma mod x n and terms of this series are always non negative. So, if we define some methods of checking the convergence or divergence of the series of non negative terms that will good for a checking absolute convergence.

And since we know that every absolutely convergent series such series will converge. So, let us first confine our attention for a sometime to what are called series of non negative terms. Obviously what does this mean that say we will say that the series sigma x n, n going from 1 to infinity is the series of non negative term. This simply means that each of x n is bigger nor equal to 0 for all n. Of course sometime these are also called series of positive terms, which would mean that each x n is bigger than 0, but you can see that that is a minor difference.

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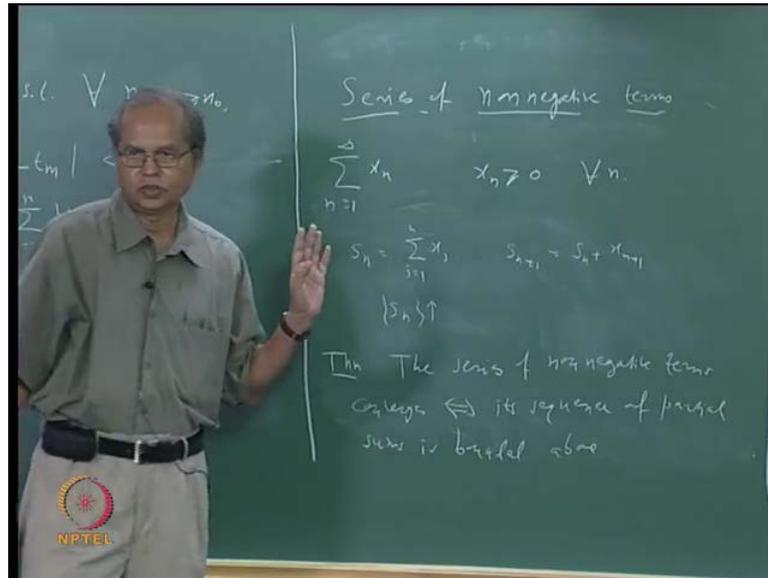
If some of the x n is 0, you can simply remove them from the series; it will not make any difference to the sum of the series. Now, what is the major advantage over the other series, it is again if you look at a sequence of partial sums let us again look a sequence of partial sums S n. Remember S n is sigma x j, j going from 1 to n and S n plus 1 is this plus x n, x n plus 1 is a is S n plus x n plus 1. And we know that each x n is neither bigger nor equal to 0, so what does it say about a sequence S n.

It is a botanically increasing sequence, S n is the series of non negative terms then, and the sequence of partial sums is a monotonically increasing sequence. So, we know this that S n is a monotonically increasing sequence. And in case of monotonically increasing sequences we know that that converges if and only if it is bounded above. So, that immediately gives us one method of checking convergence of this series, what is that? The series of positive term non negative terms converges if and only if, it is sequence of partial sums is bounded above. So, we can just write that as theorem, the series of non negative terms converges if and only if it is sequence of partial sums is bounded above. So, this gives the method of checking whether a series of non negative terms is convergent or not. All that you need to do is look a sequence of partial sums and simply show that it is bounded above, if you show that, it is enough to show that a series is a convergent. Similarly, if you show that it is unbounded then, you immediately confirm that it is it is divergent.

Now, there is one more thing, the sequence is monotonically increasing, so even if you are able to show this for some sub sequence it is enough. Suppose instead of showing that S n is bounded above, I simply show that a sequence s 1 s 3 s 5 s 7 that is bounded above. Will that be sufficient? Because s 4 is less than s 5, s 6 is not equal to 7, if s 1 s 3 s 5 that is bounded that is enough similarly, if that is unbounded again that is enough. So, remember when the sequence is monotonic to show that it is bounded above, it just sufficient to show that any sub sequence is bounded above.

And off course to show that it is un bounded that is always there, even if you show that some sub because if the sequence is bounded all its sub sequences also must be bounded, that is true for any sequences whether monotonic or not. So, to show that sequence is unbounded it is enough to show that some sub sequences is un bounded, but in case of monotonically increasing sequences, to show that sequences bounded above for that also it is sufficient to show that some sub sequences is bounded above. Now, this particular thing that is this observation leads to again very important observation, this so called comparison test.

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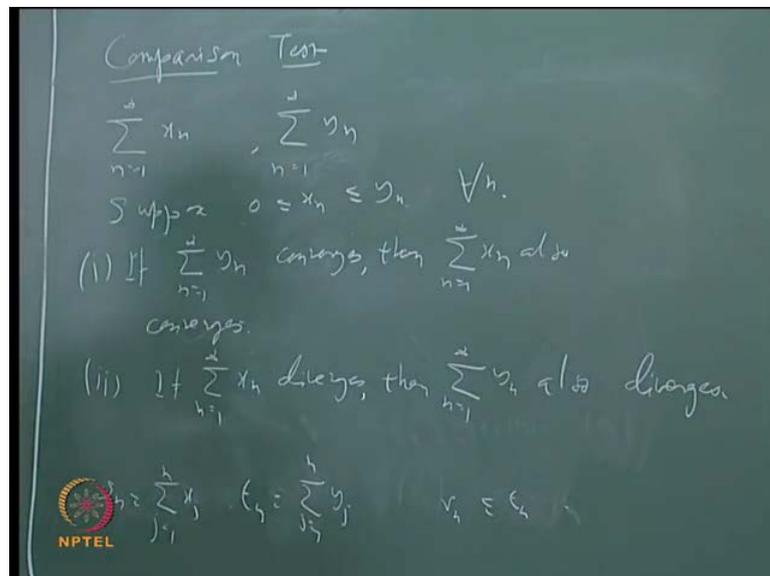
 About the idea of comparison test is basically just this that, see basically here we want to check that a sequence S n, once there are two series and if the terms of both are non negative. Then, suppose let us say there are two series x n and y n and suppose for each n x n is less not equal y n then, suppose you denote the corresponding partial sums by S n

and t n then obviously S n is less nor equal to t n. So, let us just write this suppose there two series sigma x n, n going from 1 to infinity and sigma by n, n going from 1 to infinity and suppose this happens suppose 0 less nor equal to x n less nor equal to y.

Now, so what a comparison test says is that, if sigma y n converges then, sigma x n also converges and what is the proof, proof is what I said just now. So, suppose you take S n as a sequence of partial sums of this x n, so let us say S n is sigma x j, j is going from 1 to n and t n has sigma y j, j going from 1 to n. We know that x n is neither less nor equal y n for each n that is in other words x j is less not equal y j for each j. So, what does it say about S n and t n so, it means S n is less not equal to t n.

Remember both are monotonically increasing sequences, so all that we need to once we have a series of non negative terms, all that we need to check is that a sequence of partial sums is bounded above. Now, if sigma y n converges, it means t n is bounded above if sigma y n converges it means t n is bounded above. Does it say immediately that S n also must be bounded above? And well sometimes this whole thing is also said in our slight, I mean let us write that is because it is fairly common.

If sigma x n diverges then sigma y n also diverges can you see that, this is basically the same thing written in a different language because if this were false, what must happen that a sigma x n converges and sigma y n diverges so that is not possible. Sometimes this series when this happens, we say that the series y n terminates the series x n. And if the series which dominates converges when the series, which is dominated also converges all and if the series, which is dominated diverges, this series also diverges.

After defining the convergence and divergence of a series, I have said that if you remember that convergence or divergence of a series does not depend on what happens to a finite number of terms. So, in finite number of terms are added or removed it does not matter for the convergence or divergence. Now, using that what we can say here is that see generally, we say that this happens for all n, but even if this happens for all n bigger nor equal to some n 0, because what happens to the first finite number of terms does not matter.

So, in comparison test that is in general we use this condition 0 less nor equal to x n less nor equal to y n for all n, but since it does not matter of what happens to a finite number of terms, even if this condition is satisfied after some n. Let us say n neither bigger nor

equal to 100, still the same conclusion will be valid. Now, this is a very useful test to check given the series of positive terms converges or diverges.

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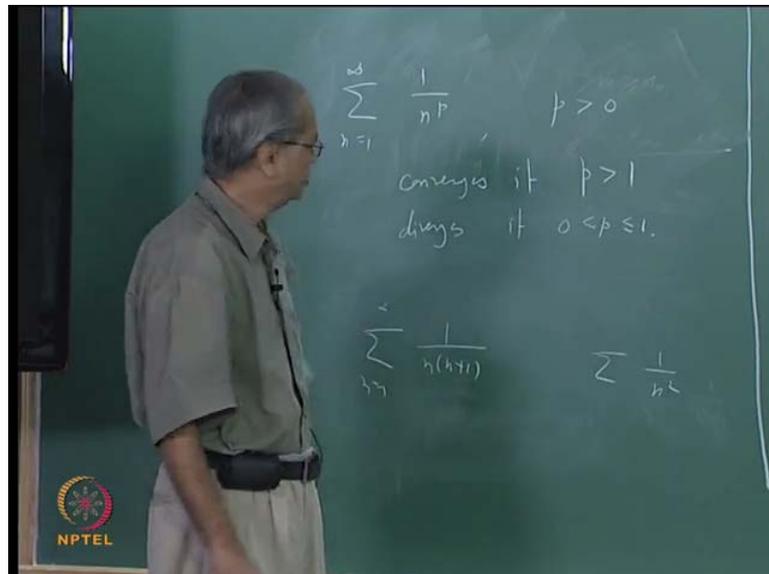
To make effective use of this test, we should have some standard list of series, which converge or diverge and that list is given by this sigma 1 by n to the power p, n going from 1 to infinity and p is bigger than 0. And again this is a well-known reason that this will converge if p is bigger than 1 and this diverges if p is. So, this converges if p is strictly bigger than 1 and diverges if 0 less than p less not equal to 1.

So, once we have given a series of positive terms, if it is possible to compare that series with one of this series then, we can conclude about the convergence or divergence of any of those series. Let us first look at the proof of this, by the way given a series of or any series not only series of positive terms, any series in general there can be several ways of showing that the series converges or diverges. Now, the unique method for example, we have seen that this series sigma 1 by n into n plus 1, we have seen that this series converges.

Now, I can say that we have seen that by different method, but this also series of non negative terms. So, we can also compare this with the series sigma 1 by n, n square, can you see that one. This each of this term is less not equal to each of this term here and by suppose we proved this then, by this it will mean that this is a convergent series. So, that will give this is also a convergent series, of course this comparison test will not say

anything about this sum. Again that is also not very correct to say, it will say that the sum of this is less not equal to sum of this.

Suppose in this case suppose sigma by n converges and then sigma x n also converges, the sum of this series will be less not equal to some of this series, but beyond that we cannot say anything. Now, let us see how this can be proved what I will do is that, I shall just take one sample case of each step, one for the convergence and one for divergence.

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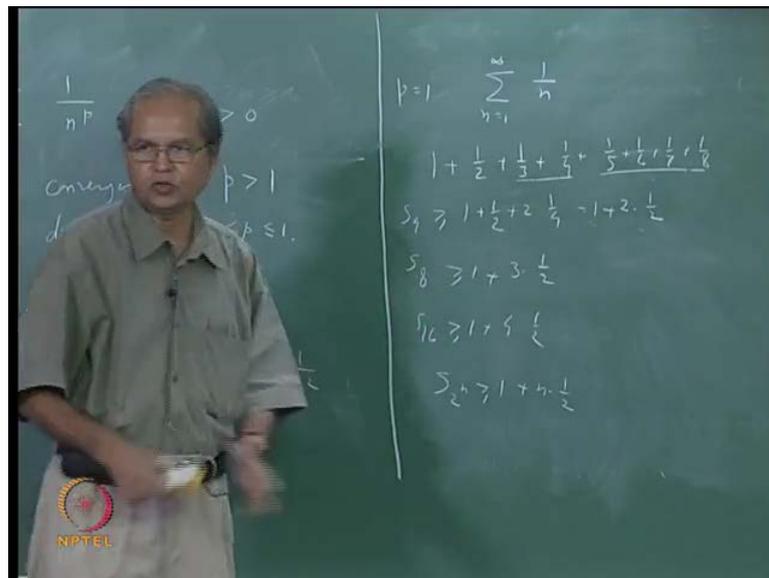
And the general case, I shall leave it to you as an exercise. So, let me take this case first suppose I take the case p equal to 1 that means we are looking at the series sigma 1 by n, n going from 1 to infinity. To show that this is divergent, we need to show that it is sequence of partial sums is unbounded. And I told you just some time ago that is sufficient to show that some sub sequence is unbounded. And the idea is as follows look at this partial sum 1 by 1 by 2 plus 1 by 3 plus 1 by 4 etcetera, etcetera.

Let us say there last point I will write little later, you can say that suppose I take the partial sum s 1 that is one partial sum s 2 is 1 plus half partial sum is 2. Suppose I look at s 4 then, I will say that s 4 is not bigger nor equal to 1 plus half plus these two terms 1 by 3 plus 1 by 4. Now, each of these terms are this 1 by 4 is smaller so, I can say that these is bigger than 2 into 1 by 4 so, which is again 1 by 4. So, s 4 is bigger not equal to 1 plus 2 into1 by 2 s 4 is bigger than 1.

Now, next I consider let us say s 8, that is I take next 4 terms that is 1 by 5 plus 1 by 6 plus 1 by 7 plus 1 by 8, take this group. In this group the smallest term is 1 by 8 and suppose since I have taken 4 terms, the sum of them must be bigger not equal to 4 into 1 by 8. So, they must belong 4 into 1 by 8. So, that is again 1 by 2 so, that gets adapt to this because if s 4 is all ready bigger than equal to 1 plus 2 into 1 by 2, s 8 should be bigger nor equal to 1 plus 3 into 1 by 2.

Now, can you predict something from here? Similarly, for example Suppose I take now s 16, s 16 that will be again 1 by 9 plus 1 by let us say up to 1 by 16, the smallest term there will 1 by 16, there be x such terms that is sum is again 1 by 2. So, s 16 will be bigger not equal to 1 plus 4 into 1 by 2. So, what is the general pattern?

Student: S n square.

What n square, 8 is not S n Square, yes s 2 power n yes that is right. So, s suffix 2 power n is bigger than equal to 1 plus n into 1 by 2. And does it follow from here that this s 2 power n is unbounded because it is bigger not equal to 1 plus n by 2 and this number I can make arbitrarily big.

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So, this implies that is s suffix 2 power n is unbounded and hence S n is also unbounded and so, this series diverges. And the same proof will work if you take instead of n here, if

you take n power p for p less not equal to 1. Small modification of this will work, to show that the series diverges for this case 0 less than p less not equal to 1.

Now, let us take one from this group t bigger than 1and let me take sigma 1 by n square. And again I say whatever we do for p equal to 2 with obvious small modifications will for work for any p bigger than 1, that part I shall leave it to you as an exercise So, let us look now, again the idea is basically similar, this make the group of the partial sums here. And here we have chosen the smallest among them and then used the in equality less not equal to.

Now, we want to show that S n is bounded, so we need say inequality less not equal so obviously we should choose the biggest among that terms. That is the idea, so in this case S n is 1 plus 1 by 2 square plus 1 by 3 square plus 1 by 4 square etcetera. Let me begin with s 3, s 3 is 1 plus now I will take the group of these two. Now, in these two 1 by 2 square is the bigger term, so the sum two must be less not equal 2 times 1 by 2 square.

So, this will be less not equal to say two times 2 square. Next what I will do is I will take again the next 4 terms that is 1 by 4 square plus 1 by 5 square plus 1 by 6 square plus 1 by 7 square. So, that is s 7 not equal to this is less not equal to. So, s 7 will be less not equals to we leave 1 plus this 2 by 2 by square, let me write as 1 by 2. Here the smallest term is 1 by 4 square and there are 4 such terms, so the sum should be less not equal to 4 by 4 square. So, less than 1 by 4, 4 by 4 square. Now, you can predict what is going to happen next.

Next I will take next 8 terms that is starting 1 by 8 square and going up to 1 by 15 square, there are 8 terms each is less not equals to 1 by 8 square. So, the sum should be less not equal to 1 by 8 and that is s 15. So, s 15 is less not equals to 1 plus 1 by 2 plus 1 by 4 plus 1 by 8. So what follows from here, again you can predict what will general pattern this is number some 3 here we had got s 2 power n, it is not 2 power n, it is 2 power n minus 1.

And what that is neither less nor equal to you can say on the hand side you have geometric series with this common ratio 1 by 2 and we all ready showed that is a convergent series, so this partial sum are bounded. So, what it shows is that this s 2 power n minus 1, this is bounded above. See in general for an arbitrary sequence, if you

show that some sub sequence is bounded above will not imply that the whole sequence is bounded above.

But in case of monotonically increasing sequences that is true because what we have shown is that, this sub sequence that is s to the power n minus that is the sequence s 1 s 3 s 7 s 15, that sequence is bounded above. But what we know is that if you take for example, s 10 s 10 is less not equal to s 15 or similarly, if you take any n that n is less not equal to 2 to the power n minus 1. So, S n is less not equal to s 2 to the power n minus 1, so the whole sequence also is bounded above.

So, that shows that this series is convergent. Now, in a similar way you can take any p bigger than 1 and then this same idea will work, I shall leave that to you as an exercise. So, that support the comparison test and how one uses comparison test, here we have seen an example.
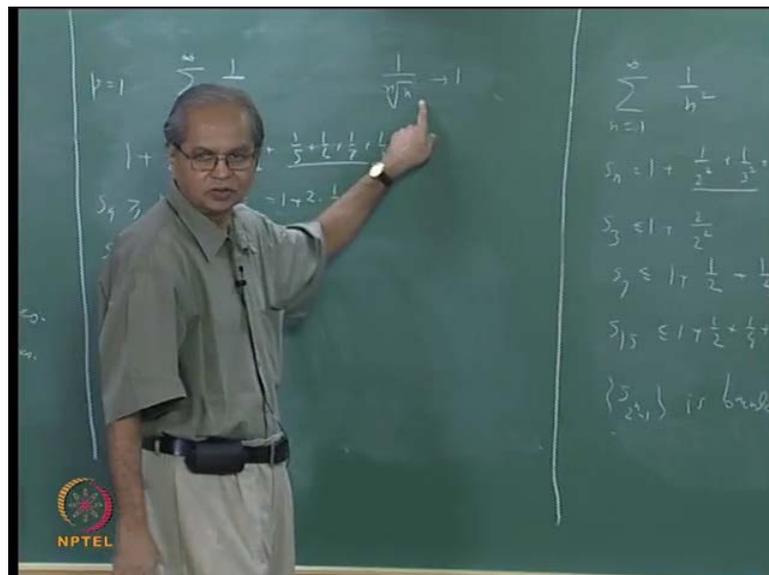
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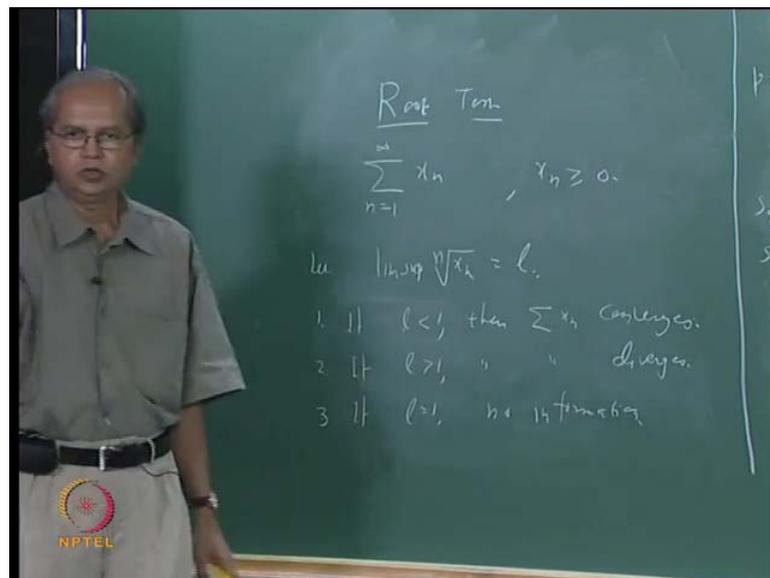
Let us Now, look at one more test called the checking the convergence of the series of non negative terms that is called Root test, very power full test this one. Of course, comparison test is quite powerful. Suppose, this is the series of the non negative terms once again, n going 1 to infinity each x n is bigger than 0. So, you look at n th root of x n, or x to the power whatever you call nth root of x n and consider the limits period of that.

Let see x n in general x t h root of x, n may or may not be convergent sequence. If it is bounded limit superior will always exist, if limit superior is does not exist as a finite real number, you take this as a infinity. So, let this number be l then, what does it test say, that test says if this is the first thing l is less than 1 then sigma x n converges. Second thing is that if l is bigger than 1 then, sigma x n diverges and if l is equal to 1, the test gives no information.

If l is equal to 1, the test you cannot conclude anything no conclusion, no information about convergence or divergence. As far this last thing is concerned what does it mean? If l is equal to 1 it is also possible it is convergence, it is also possible that series diverges. And you can say we already have examples of both these categories.

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 If you look at the series 1 by n, in this case what will be n t h root of x n, it will be 1 n th root of n, already shown that n th root of n goes to 1 as n goes to infinity. So, this tends to 1. So, in this case l is equal to 1 and series diverges. On the other hand if you look at this series 1 by n square then, it will be 1 by n th to the power 2 to the power n, which limit also will be 1 and that series converges. If l is equal to you cannot conclude anything from the root test.

 Now, let us look at this one, remember that when we discuss this limit superior we have shown that if you take any epsilon then there exists some n 0, such that all elements of the sequence x n lie between less or less than that l plus. Suppose, I take any epsilon and

then if I look at number l plus epsilon then I can find number n 0, such that all the elements the sequence are less than that l plus epsilon.

Now, suppose l is less than 1 I can always choose an epsilon such a way that l plus epsilon is also less than 1. I can also l is a real number, l is less than 1, can I always choose a positive epsilon such that l plus epsilon are less than 1. Obviously 1 minus l by 2 that is a positive number, anyway it does not matter you choose some number, which is strictly bigger than l and less than 1.

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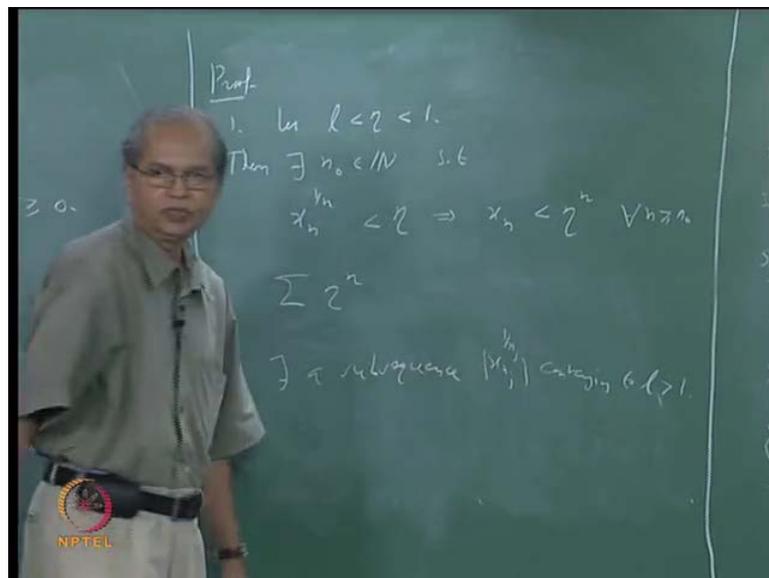
So, when the proof suppose I call that number eta, let the proof of this let l less than eta less than 1 then, take the difference between eta and l as epsilon. Then there should exist some n 0 such that for n neither bigger nor equal to n 0 x n is less than l plus epsilon, that is eta. So, I will simply say that there exists then, there exists n 0 8 n such that, n 0 in n such that it is not limit superior of x n remember, it is limit superior of n th route of x. So, let me x n to the power of 1 by n that is less than eta, l plus epsilon or which is same as that x n is less than eta to the power n.

Now, you just use the comparison test, see you have the series sigma x n and you take this says sigma eta power n. And this is where I said that this x n less than this is a not happening for all n, it is happening for all n bigger nor equal to n 0, but that is ok. This is something I commented after discussing the comparison test that is if x neither n less nor

equal to 1, even if it happens for some n neither bigger nor equal to n 0, still the conclusion of the comparison tests are valid.

Now, this geometric series and this eta is less than 1, so this is the convergent series. So, by comparison test this original series should also converge. See you may have learnt this route test even in your under graduate course, but it is highly unlikely you would have used this limit superior. In the under graduate books the way in which they explain, limit of this n th route of x n is equal to l, but this is more general because that limit may or may not exist, but limit superior will always exist.

So, this you can apply for more general series. Now, let us look at this second part that is also very similar basically for those by using the properties of limit superior. Remember what we have shown is that, see this is limit superior of this sequence. If you remember we have shown that there exists a subsequence of this, which converges to limit superior. So, let us suppose I call that subsequence let us say x n j, so there exists a subsequence x n j converging to n and l is bigger than 1 l is bigger than 1.

Now, let me just tell you rough argument here, if some sequence converges to something l bigger than 1, is it clear to you that infinitely many terms of that sequence must be bigger than 1. Now, if that is the case can that sequence tends to 0 not x n j, x n j to the power 1 by n j. So, inside limit terms are sequence x n j to the power 1 by n j, those are bigger than 1. If this is the case can this sequence x n j tends to 0, it cannot tend to 0.

Then can x n tend to 0, that cannot happen the sequence tends to 0 it cannot happen that infinitely terms the sequence are bigger than 1, but if the sequence x n does not tend to zero, what do you know about the convergence of the series. That was one of our first theorem that, if the sigma x n converges then the sequence x n must go to 0. That is not happening here, so the series sigma x n must be a divergent sequence and the third case we have already seen. So, that is information about the root test. There is one more test of this similar type, which is called the ratio test that we shall discuss in the next class. We will stop with this.