Linear Algebra
Professor K.C Shivakumar
Department of Mathematics
Indian Institute of Technology, Madras
Module 14-Self-Adjoint, Normal and Unitary Operators
Lecture 51
Normal Operators, Spectral Theorem

See in the last lecture we had discussed the properties of unitary operators on a finite dimensional inner product space in todays lecture I will discuss the normal operator, the case of normal operators and prove what is called as the spectral theorem for normal operators on a finite dimensional inner product space. We must compare this spectral theorem for a normal operator with the one that I proved a couple of lectures ago for the self adjoint operator ok. Will make this comparison just before the statement or perhaps after the statement of the spectral theorem for a normal operator.

Let's look at some properties that we will require for a normal operator. So first of all what is a normal operator? Let me we have seen this before let me quickly recall the definition and prove some properties of normal operators.

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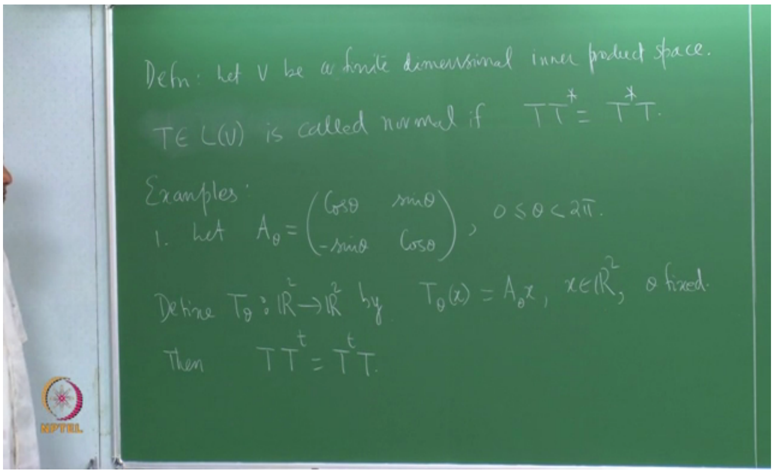

So definition let V be an inner product space the finite dimensional inner product space, let T be your linear operator on V be a normal operator I am sorry, I want to define a normal operator so let me say I want to define a normal operator, an operator T element of L V is called normal if is called normal if T-T star equals T star T. T star at the adjoint operator of T. Ok that is the definition.

We have seen this before let's look at a couple of examples one for the real inner product space and another one for the complex inner product space. Just a couple of examples, we will define the linear transformations through matrices, so the first one let's take A, actually A theta to be the rotation operator, we have come across this operator before so I am writing down the matrix first and then I will define the operator through this matrix.

A theta is Cos theta Sin theta minus Sin theta Cos theta, for theta lying in 0 to Pie, define a linear transformation I will call it T theta from R2 to R2 by the formula that T theta of x is equals A theta x, x in R2 for a fixed theta. Then you can verify that T theta is a normal operator that is see this a real inner product space so instead of star we consider the transpose T. So this equation T-T star equal to T star T in this case is T-T transpose equals T transpose T that is T is a normal operator ok so this is an example of a normal operator on a real inner product space let me give an example for a complex inner product space.

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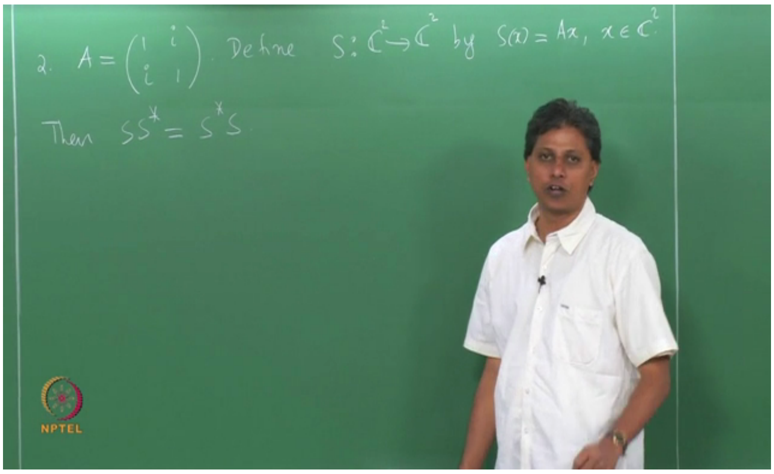

Example 2, again I take a matrix and then define the linear transformation through the matrix, say 1 i, (())(04:42) define let's call it S perhaps S from C2 to C2 by S of x equals A x, x belongs to C the I leave it as an exercise for you to verify that S is normal that is S-S star equals S star S. if I remember right it turns out to be identity so these are this is a unitary operator this is an orthogonal operator ok, so these are some of the examples of normal operators. Let's now look at some properties a couple of properties really for a normal operator that will be required in the rest of the discussion.

So the first property is the following, this first property that I am going to write down should remind you of a result for a self adjoint operator ok.

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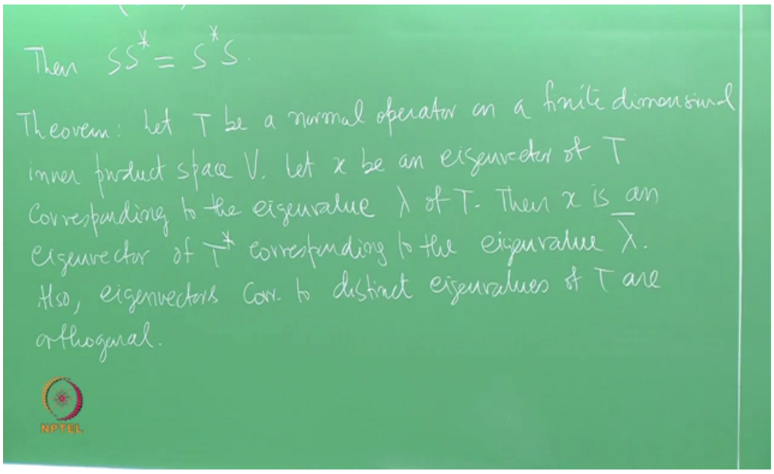

So the first one is let me call it as a theorem let T be a normal operator on a finite dimensional inner product space V. We don't really need the space to be finite dimensional but still I will assume that V is finite dimensional. Let x be an eigenvector of T corresponding to the eigenvalue lambda of T then we will show that x is an eigenvector of T star corresponding to the eigenvalue lambda bar.

We will require this result in later discussion ok so let's prove this result a I told you, you, you must make a comparison of this result with the case of self adjoint operators. In the case of self adjoint operators we have shown that the eigenvalues of (())(08:07) operator are real numbers we have also shown that eigenvectors corresponding to distinct eigenvalues of a self adjoint operator are orthogonal ok. Let me include the second part, also eigenvectors so we have a all that I am saying is that we have a similar result for a normal operator also.

Also eigenvectors corresponding to distinct eigenvalues of T are not just linearly independent they are orthogonal also are orthogonal.

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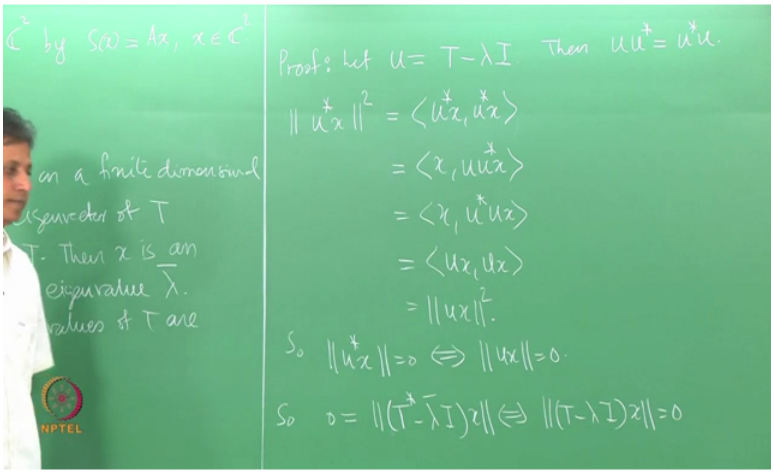

Ok so let's see the proof of this result, T is given to be a normal operator let us now define an operator U by the formula T minus lambda i, lambda is any complex number in particular we take lambda to be the number the eigenvalue that we started for example. So take U to be this then it is easy to see that U is normal I leave that part for you to verify then u-U star equals U star U.

Ok so let's now look at so U is normal let's now look at norm of U star x the whole square this by definition inner product U star x with itself I use a property of the adjoint operator this operator U star when it comes to this argument it goes as U so this is x U-U star x I will now use the equation U-U star equals U star U so I can write this as x, U star U x again I bring this to this argument it will comes as U so that is U x inner product of U x with itself which is now norm of U x square.

So in particular what this means is that norm of U star x equal to zero if and only if norm of U x equal to zero. Now look at U star, what is U star? U is T minus lambda i then U star is T star minus lambda bar i, so zero equals norm of I am writing T star minus lambda bar i of x if and only if norm of T minus lambda i x equals zero. So I have just written U star here and U here from this it follows that if lambda ok look at this equation. If lambda is eigenvalue x is a corresponding eigenvector for the transformation T for the normal operator T then it follows that x is an eigenvector the same x is an eigenvector for T star corresponding to the eigenvalue lambda bar ok.

Now this is what we wanted to show, that is the first part. Let's show the second part, second part is to show that eigenvectors corresponding to distinct eigenvalues of a normal matrix R orthogonal. Again the this part of the proof should remind you of what we did for the case of a self adjoint operator so let me go through this quickly.

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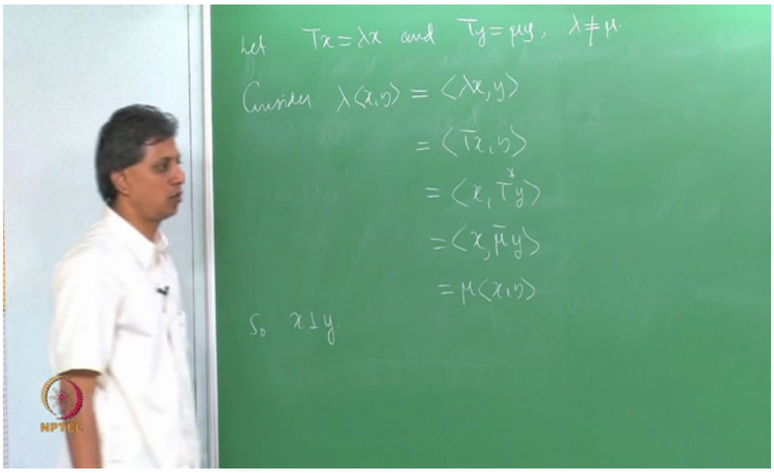

Now second part let T x equals lambda x and T y equals Mu y where I know that Mu and lambda are distinct, I must show that x is orthogonal to y. So I must show that inner product x y is zero.

So let's start with consider lambda times inner product x y I can bring this lambda to the first argument lambda x, y and then substitute lambda x equal to T x, this is T x, y apply the definition of the adjoint that is x T star y now use the previous result y is an eigenvalue sorry y is an eigenvector corresponding to the eigenvalue Mu of the operator T then we have seen that it must follow that y is an eigenvector corresponding to the eigenvalue Mu bar for the operator T star that is t star y equals Mu bar y. So this is using the first part and Mu bar is in the second argument it comes out with a conjugate and so conjugate double conjugate (()) (13:20) Mu times x y.

Now the result follows, if lambda is not equal to Mu then inner product x y must be zero and so x and y are orthogonal ok. So x is perpendicular to y that is what we wanted to show. Let's go back and see every unitary operator is a normal operator ok but a unitary operator the entry is atleast when you treat the columns or rows of unitary operator they are very much tractable in the following sense.

If U is a unitary operator then without writing on the board let me just explain this, if U is a unitary operator then it satisfies U-U star equals U star U equals identity ok. What this means is the following (just) let's look at the equation U-U star equal to identity, it means that if you look at the inner product of the first row of U into the inner product of the first column of U star ok that is the inner product of the first row of U with itself then by looking at the right hand side it follows that this is equal to one whereas the inner product of the first row with the inner product of the second column of U star, third column of U star, etc the last column of U star, these inner products are all zero.

This is the same as saying that the inner product of the first row with the second row the third row etc with the nth row that inner product is zero, is that clear? So every row of the matrix U if U is unitary has the property that every row if you take the dot product of the row with itself then it is a vector norm 1 whereas the dot product of the vector with every other row is zero. This means what? The rows of a unitary matrix form an orthonormal basis for C n. A similar argument can be given by looking at U star U equal to the identity to make the following statement that the columns of a unitary matrix form an orthonormal basis for C n.

A similar statement can be given for the case of a orthogonal matrix that is a real unitary operator. So for an orthogonal matrix the rows and columns form an orthonormal basis for R n for instance. But the case of a normal operators much more general, the product is not equal to identity it not so it not need be invertible a normal operator need not be invertible. So the statement about the entries of a normal operator is not easy to make but we will look at a particular case that is the essence of the next theorem that is if you know that an operator a normal operator T has the property that the matrix of T relative to a particular orthonormal basis is triangular that is upper triangular or lower triangular then we can say something.

Then it follows that the matrix must be a diagonal matrix ok, so this is what we will show, so let me make this statement precise.

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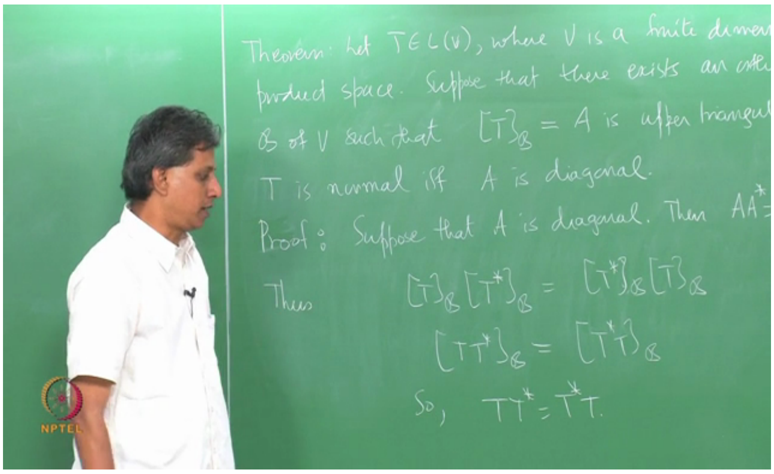

So this is what I want to prove I will state this as a theorem. Let T belong to L V where V is the finite dimensional inner product space. Suppose that there exists an orthonormal basis I will call it script B orthonormal basis B of V such that the matrix of T relative to this basis equals A is let us say upper triangular, till now I have not made any assumption on T, T us just any operator with the property that there exists an orthonormal basis with a extra property that the matrix of T relative to the orthonormal basis is upper triangular or lower triangular it does not matter.

Then T is then we can characterize normality of T, necessary sufficient condition for T to be normal is that A is diagonal ok, when T is normal if and only if A is diagonal. As I told you this say something less about the matrix A that is you need to make some assumption on the matrix of T to an orthonormal basis but nevertheless this will be useful for us in proving the spectral theorem for a normal operator ok, let's see the proof of this. So proof, let's perhaps prove the easy part here. Suppose that A is diagonal by the way all the spaces are complex spaces.

Suppose that A is diagonal then any two diagonal operators commute in two diagonal matrices commute so I can say this much that A-A star equals A star A this is always true any two see(())(19:28) is diagonal A star is also diagonal so this happens this means what? Thus, what is A? A is matrix of T relative to B and A star is the matrix of A star is conjugate of the matrix of T relative to B but then we know we have proved this before that T B star is T star B so I have this equation and on the right hand side I will write A star A that is T star B into the matrix of T relative to B have this equation and there is a property that we have proved for the product you can replace that by the composition, composition I have not used the circles I will simply say that T P star relative to B is T star T relative to B, now this a calculation that we have done a number of times before.

What follows from this is that the operators must be the same that is action of the operators so I have two operators let us say S and T the action of S on a basis equal to the action of T on that basis then we have seen that this defines S uniquely so from this it follows that T-T star equals T star T. So is it clear? So T is normal if A is diagonal then we have shown the T is normal, let's prove the converse part, that is not straight forward. The converse part is to show that if T is normal then A is diagonal will do will consider one column of the matrix A at a time and show that all elements except the principle diagonal element must be zero.

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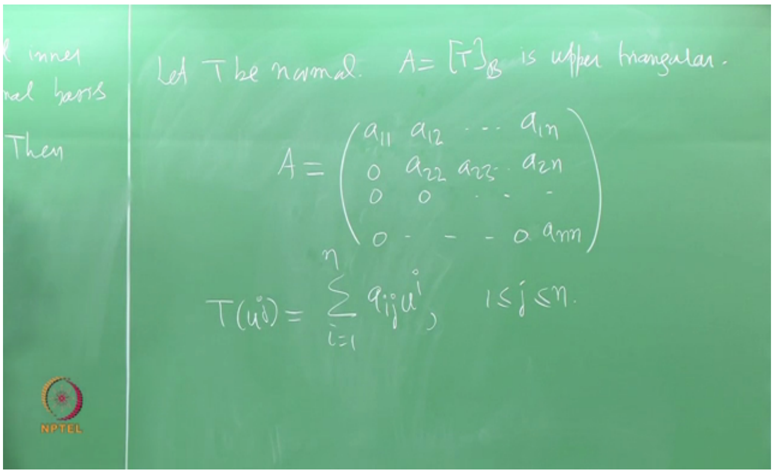

Ok so let T be normal and what is given I am just (())(21:36) and I am calling that as a matrix A right, what is given is the following, A equal to the matrix of T relative to B is upper triangular so let me write the matrix the form of the matrix it is upper triangular so it is something like this, a 11, a 12 etc a1 n this entry must be zero a 22, a23 etc a2 n this entry must be zero this must be zero etc let me write the last row 0, etc 0 this is a n n. So this is form of the matrix A if T is normal I must now show that all these entries are also zero, this is the principle diagonal primarily we must show that all these entries are zero ok so will do it one row at a time ok.

Now let's look at the relationship between the entries of A and the matrix of T relative to B. In other words I will just use the formula for this equation, so this says T of U j equals summation i equals 1 to n A I j Ui this is true for all j, 1 less or equal to j, less and or equal to n. Now this equation we have encountered a number of times earlier. If A is a matrix of T relative to B then this is what this equation must hold.

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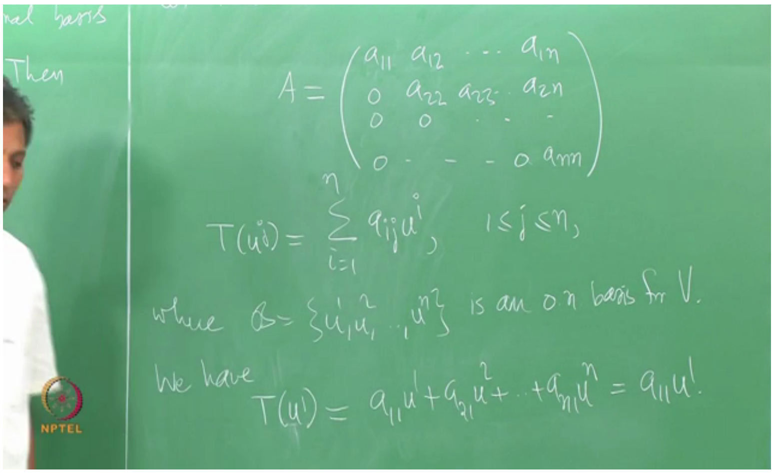

Where see we are making the standard assumption that what are these U j’s? U j’s form a basis that is U1, U2 etc U n this is an orthonormal basis where B is an orthonormal basis for V, this is standard terminology.

Ok let’s now look at T U1, that is j equals 1, we have T U1 to be so j equals 1 so it is ai1 u1 so T u1 is let me there just write this a11 u1 plus a21 u2 plus etc an 1 u n but see (a21) a23 etc a21 a31 etc an 1 they are all zero so this is just a11 u1 ok, T u1 equals a11 u1 I will appeal to one of the properties that we have proved just now. By the way what this means is that u1 is an eigenvector corresponding to the eigenvalue a11 for the operator T what follows is that u1 must also be an eigenvector for the conjugate a11 bar corresponding to T star so let me write that equation.

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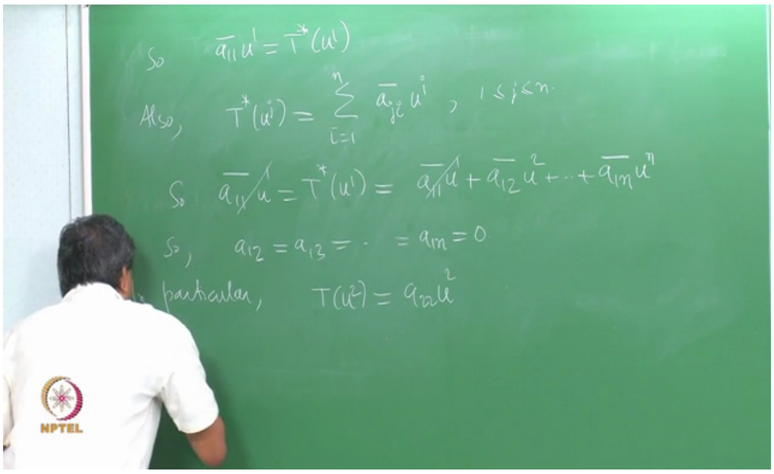

So let me say that a11 bar u1 equals T star of u1 and then look at T star of u1, what is the formula for t star of u1? Similar to what I wrote down for T U j I will write down a similar equation I will write down an equation for T star U j and then look at the particular cases. Also T star of U j is summation i equals 1 to n this time it is a j i bar T star U j it is a j i bar U y so where I have made use of the fact that the matrix of the conjugate of the matrix of T relative to B is the matrix of the conjugate T star relative to the same basis B.

Ok so I have this again 1 less and or equal to j less and or equal to n in particular look at j equals 1 and so let me go back to this equation a11 bar u1 equals T star u1 which can now be computed as j is 1 so it is a11 bar u1 plus a12 bar u2 plus etc plus a1 n is that ok j is 1 and i varies from 1 to n. So a1 n bar u n. Now a11 bar u1 that can be cancelled look at what remains, what remains is a12 bar u2 etc p plus a1 bar u1 equal to 0.

Now u1 u2 etc u n form an orthonormal basis in particular they are linearly independent so subset is also linearly independent so u2 etc u n is a linearly independent subset so this is an equation like let us say alpha to u2 etc plus alpha n Un equal to 0 so each of the scalar must be zero so what follows is that a12 bar so see the conjugate are zero so the complex number also must be zero so a12 equals a13 etc equals a1 n all this must be zero. Observe what this are, this are precisely the first row entries except the first entry.

So the first row entries have been shown to be zero, what we have shown is that a12, a13 etc a1 n they are all zero in particular they ok (in particular) see will appeal to induction but we need to make one observation, in particular a12 0 so if you look at the formula for T u j when j equals 2 what happens? The formula for T u j when j equals 2 gives me a22 u2 and that is like this equation I am sorry yeah that is like this equation T u1 equals a11 u1 so once I have a guarantee for that then we can appeal to induction, that is I will say that in particular what follows is that T u2 equals a22 u2, because a12 is zero and so I am looking at the second column, all entries below a22 are already zero because this is upper triangular matrix.

Now what follows is that u2 is an eigenvector corresponding to the eigenvalue a22 for the operator T so u22 will be an eigenvector corresponding to eigenvalue a22 bar for the operator T star I proceed as before, there are only finitely many columns so it follows that a is diagonal. So I will simply say by induction it follows that a is diagonal ok, now this an important intermediate result in order for us to prove the spectral theorem for a normal operator ok. Now before going to the spectral theorem we will prove what is called as a Shur Triangularization theorem ok Shur Triangularization theorem.

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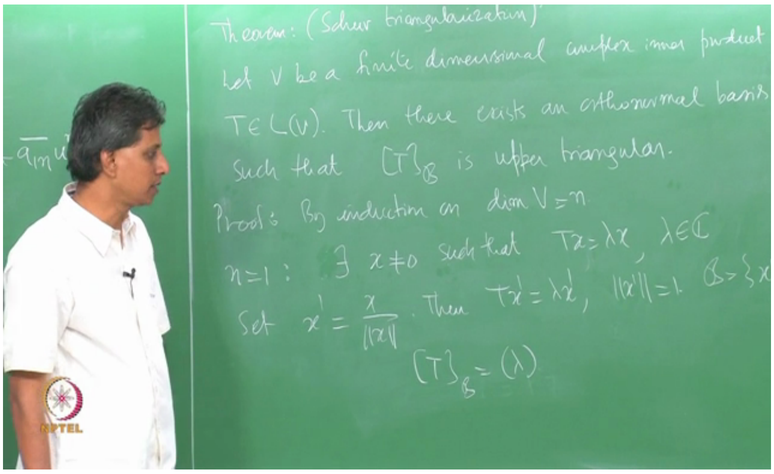

So let me write this theorem, this is called Shur Triangularization theorem, the statement is the following. Let V be a finite dimensional inner complex inner product space and T be any operator. Now this result is true for any operator there are no conditions on T normality is (()) (30:11) there is no such condition. So what is the statement? Finite dimensional complex inner product space T is a linear operator on V then it says that you can do a little less than diagonalization then there exists an orthonormal basis script B of the vector space V such that the matrix of T relative to B is upper triangular ok.

For any operator T on a complex inner product space what is important to observe is that the eigenvalues belong to the underline field because C is an algebraically closed field ok. So let’s understand that there is no counter part of this result for the case of a real inner product space. However if you know that T is an operator on a real inner product space with the property that all the eigenvalues of the operator T are real then we can write down a similar statement ok. So let's understand the importance of the fact that this is a underline field is a is the complex field means it is an algebraically closed field.

Ok will show that it is upper triangular, the proof we be by induction ok. Proof by induction on the dimension let's call this n the case n equal to 1 there is nothing n equal to 1 it is trivial that is when n equal to 1 what is means is that we want to show that there is an orthonormal basis such that T B is upper triangular there is just one entry so 1 cross 1 matrix, now what is that matrix, see this is where we are using the fact this a complex inner product space there is an eigenvalue for the operator T. In other words let me say that there exists x not equal to zero such that T x equals lambda x for some complex number lambda.

We can call x1 as x by norm x, the Euclidean, x is not zero so this is well defined then it follows that T x1 equals lambda x1 with norm x1 V in 1 you take the basis consisting of just this single vector then the matrix of T relative to B is the number lambda which is trivially upper triangular. Let's assume that the result is true for finite dimensional inner product spaces of dimension n minus 1 and then prove it for spaces of dimension n. So in that it was the induction hypothesis is made will prove this result for all finite dimensional inner product spaces of dimension n.

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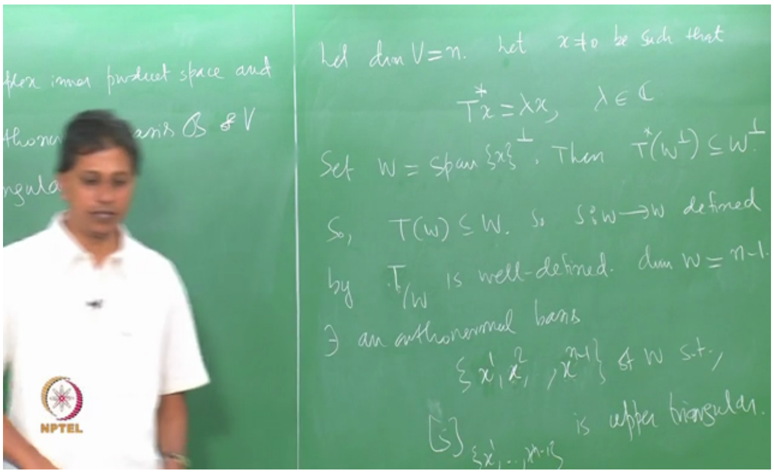

So let's take dimension of V to be n, let us now start with ok suppose dimension V is n see we made a statement that the operator T has an eigenvalue a similar statement holds because underline field is complex a similar statement can be made for T star so let me exploide that. Let x not equal to 0 be such that this time T star x equals lambda x for some complex number lambda. Existence of an eigenvalue and so there is by definition a non-zero vector x satisfying this equation. Let us now define W to be span of this vector x and take its perpendicular. So W (())(34:24) span x perpendicular, then what follows is that then obverse that x belongs to W perpendicular.

So what it means is that, T star of W perpendicular is contained in W perpendicular that the space W perpendicular is invariant under T star because x is an eigenvector ok it is a one dimensional W perpendicular is a one dimensional subspace spanned by the single vector x ok. Now T star W perpendicular is contained din W perpendicular we know that this is the same as saying that T of W is contained in W, this one of the result that we proved earlier. If T star is if W perpendicular is invariant under T star then W perpendicular, perpendicular must be invariant under T star-star that is W is invariant under T.

So T W contained in so it makes sense to so S from W to W defined by so I define a new operator S this is defined as the restriction of S to the restriction of S I am sorry the restriction of T to W this is well defined. So this operator is well defined S from W to W is well defined. See all that I need to know is that if you take an element W in W whether S of W belongs to W ok but by the definition S of W is T of that little W but T of W is contained in W so this S is well defined. This is another construction that we have used when we discuss the self adjoint operator case.

So S is well defined, what is the dimension of W? Dimension of W is n minus 1, see it is orthogonal compliment of a single vector so dimension W is n minus 1, I have an operator S on a finite dimensional inner product space of dimension n minus 1 so the induction hypothesis is applicable that is there exists an orthonormal basis so let me say there exists an orthonormal basis let me write down the elements as u1, u2 etc maybe x1, x2 etc x n minus 1 right, the space is of dimension n minus 1. Orthonormal basis of W such that the matrix of S relative to this x1 etc x n minus 1 is upper triangular. Ok so I have this by the induction hypothesis.

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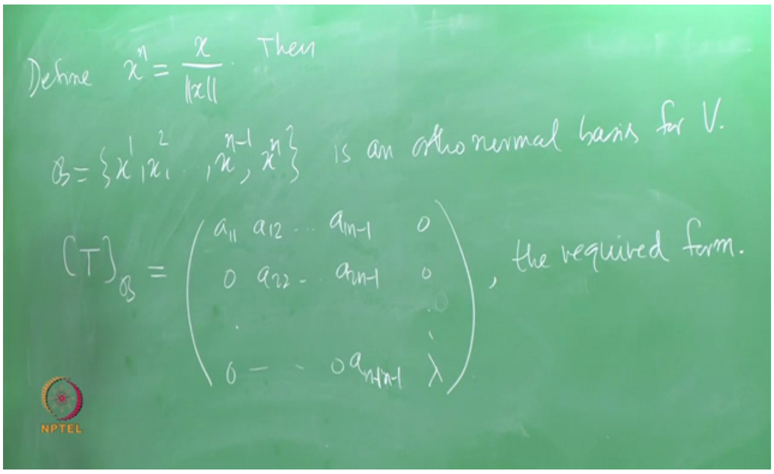

Let just write this full and see what it says ok but before that let's I will make use of this and then define a new vector x n as the first vector x that we started with divided by its norm. The vector x that we stared with need not be of norm 1 so I will take that divide it by its norm and call it x n then it is clear by the construction that the vector x1, x2 etc x n minus 1 which have been obtained before and the vector x n that we have defined newly through this vector x is an orthonormal basis this is an orthonormal basis for V that is we are writing V as W plus W perpendicular and then for W this is the basis orthonormal basis for W perpendicular x2 etc x n minus 1 is an orthonormal basis.

So come we know that when you combined this two it give rise to an orthonormal basis for V all that we need to do is to locate the matrix of T relative to this basis script B, which means what? You need to you take T apply it on x1 write it as a linear combination of x1, x2 etc x n minus 1 x n etc finally take x n look at the image of T look at the image of x n under T write it as a linear combination of x1, x2, etc x n. Now T x1, x2 etc x n minus 1 belong to W and the action of T is like the action of S and for S I have an orthonormal basis I am sorry the form of the matrix of S relative to x1, x n minus 1 is upper triangular ok so let's make use of that.

So all that I am saying is that the matrix of T relative to this basis B is of the form let's say a11 it is upper triangular for the first a12 etc a1 n minus 1 then it is upper triangular this entry is zero a22 etc a2 n minus 1 etc all this entries are zero a n minus 1 n minus 1 and then look at what happens so this purely comes from the matrix of S relative to x1, etc x n minus 1, what is T of x n? T of x n, see x n belongs to W I am sorry x n belong to W perpendicular and so I have I make use of this invariant subspace thing. So what follows is that the last entry will be 0, 0 etc lambda. Ok just check this steps, now it is now clear that this is a required form.

And so the matrix so if T is normal I am sorry if T is an operator on a finite dimensional complex inner product space then the matrix of T relative to B is upper triangular, this is the so called Shur Triangularization theorem.

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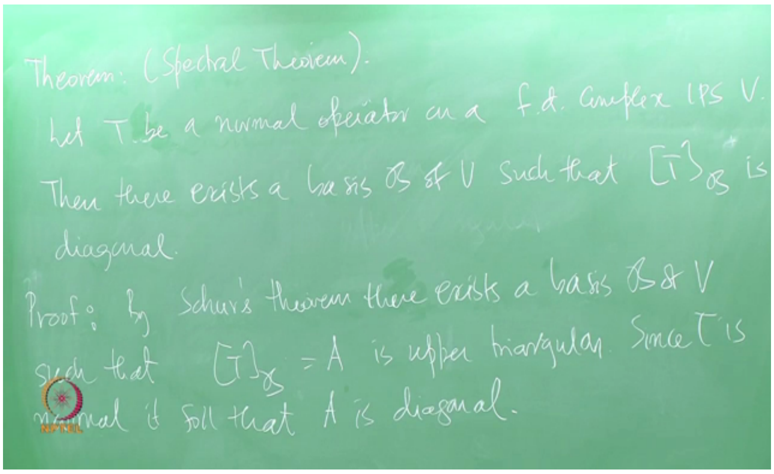

Let's now move on to prove the spectral theorem for a normal operator. Let T be a normal operator on a finite dimensional complex inner product space V then there exists a basis script B of V such that the matrix of T relative to B is diagonal. That is every diagonal you can say this is the same as saying that this is a matrix version of the statement that if you have a complex normal matrix on if you have a complex normal matrix then it can be diagonalized by a unitary matrix.

There is also another way of stating this which is that if T is a normal operator on a complex finite dimensional inner product space V then there is a basis orthonormal basis B of V which has the property that each vector in the basis is an eigenvector for T we have seen this before, we have seen the statement before so let me not emphasize ok, how does the proof go? It makes use of the previous two results the Shur Triangularization theorem and the fact that if the matrix of a linear transformation corresponding to an orthonormal basis is triangular then the operator is normal if and only if the matrix is diagonal ok.

So let me say by Shur's theorem, the Shur's theorem is applicable for any operator by Shur's theorem the matrix of I will simply say there exists a basis such that the matrix of t relative to B let us call it as A is upper triangular. So that is Shur's theorem. Now A is upper triangular appeal to one of the results the earlier result that we proved, since T is normal it follows that A is diagonal which is what we wanted to show that is there exists an orthonormal basis B of the vector finite dimensional vector space inner product space V with a property that the matrix of T relative to this basis is a diagonal matrix ok.

Now you must compare the statement with the statement for a self adjoint operator. See I will leave it as an exercise for you to write down the matrix versions, the matrix version is something that I told you immediately after writing down this theorem so that is anyway there. Now you must compare this with the spectral theorem for a self adjoint operator where the underline inner product space was not assume to be complex ok.

To summarize this is what we have. If you have a self adjoint operator on a finite dimensional inner product space it doesn't matter whether it is a real inner product space or a complex inner product space it always has the property that there is an orthonormal basis for the space V which has the which also satisfy the additional condition that each vector of the orthonormal basis is an eigenvector for the operator T.

For the case of complex no for the case of normal operators I have already made the statement that if you have a normal operator on a real space then it is not necessarily diagonalizable that is if T satisfies T into T transpose equal to T transpose into T then doesn't necessarily follow that T is diagonalizable the example that was given was that of the rotation operator which we also see which we have also seen in the beginning of today's lecture. But if it is a complex inner product space and T is a normal operator then there is an orthonormal basis with a property that each vector is an eigenvector ok let me stop.