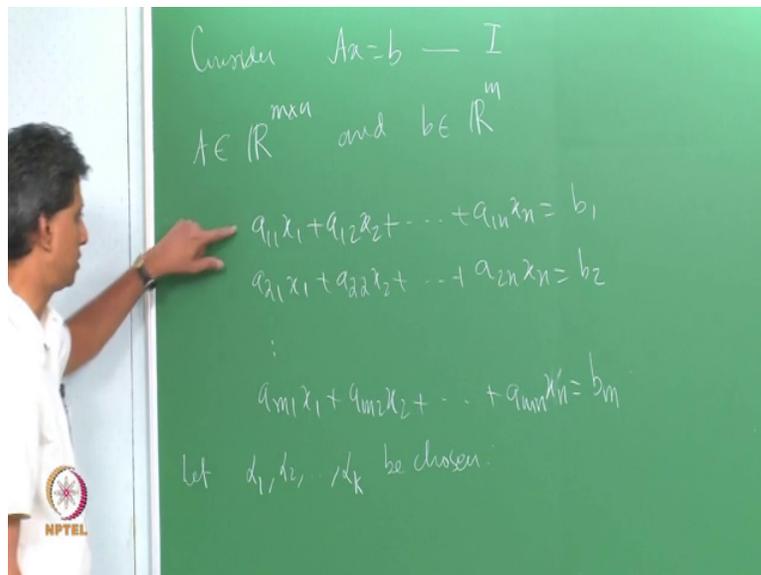


Linear Algebra
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Module 1
Lecture 3a
Equivalent Systems of Linear Equations 1

In the last lecture we were discussing the notion of elementary operations there were three elementary operations that were defined the formulas for these elementary operations were also given today we will continue with our discussion on these elementary row operations and how they help in reducing a system of equations $Ax = b$ into a system of equations of the form $Cx = d$ where the reduction here means that the matrix C has a simpler structure than A . Now what exactly is this structure that will be made precise a little later, so we will discuss this problem of reducing a system to a simpler system reducing a given system to a simpler system.

What we would also like to do is see there is an intermediate notion of two systems being equivalent, so we will discuss this notion before we get into proving before getting into the problem of elementary row operations and what it does to a system of linear equations, okay. So what I want to discuss now is given two systems of linear equations when are they called equivalent, so we will discuss the notion of equivalence of two linear systems, okay.

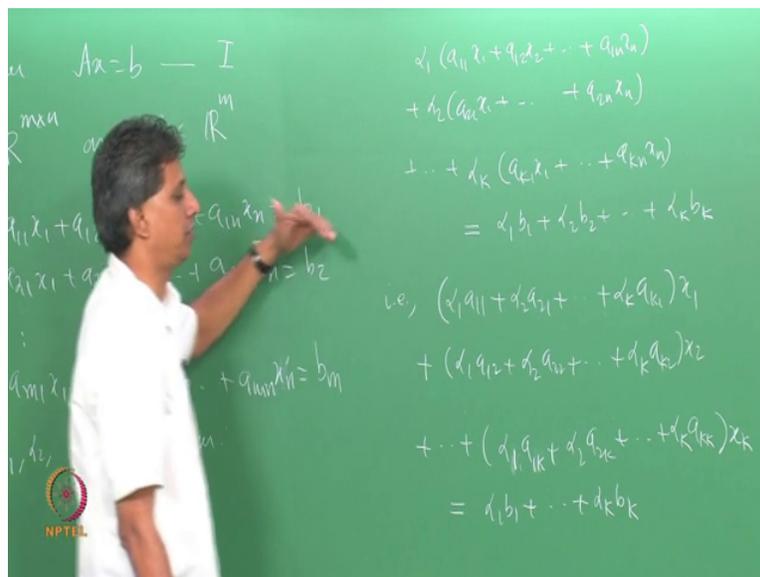
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So let us consider the following so before we had the system $Ax = b$ which we will call as 1 where A is a rectangular matrix m rows and n columns and the right hand requirement vector is has m coordinates it is a vector in R^n . Let us write this equation on full so we have something like the following $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$ the second equation $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$ and let me just write down the last equation $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$, okay so this is the expanded version of this equation $Ax = b$, okay.

Now suppose I choose certain scalars let us say I have α_1, α_2 etcetera α_k I chose k scalars less than m we chosen suppose that I multiply the first equation by α_1 , second equation by α_2 etcetera the k th equation by α_k then I get the following equation, okay I multiply the first equation the α_1 , second equation by α_2 etcetera the k th equation by α_k and then form a new equation this equation is called a linear combination of the first k equations of this system, okay.

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So I get the new equation which is so α_1 into $a_{11}x_1 + a_{12}x_2$ etcetera $a_{1n}x_n$ plus α_2 into $a_{21}x_1$ etcetera $a_{2n}x_n$ etcetera plus α_k into $a_{k1}x_1$ etcetera $a_{kn}x_n$ I formed this left hand side is linear combination of the left hand side of the first k equations the right hand side also I take a similar linear combination $\alpha_1 b_1 + \alpha_2 b_2$ etcetera plus $\alpha_k b_k$ I can rewrite this as that is this can be written as α_1 into, okay I will write this as $\alpha_1 a_{11} +$

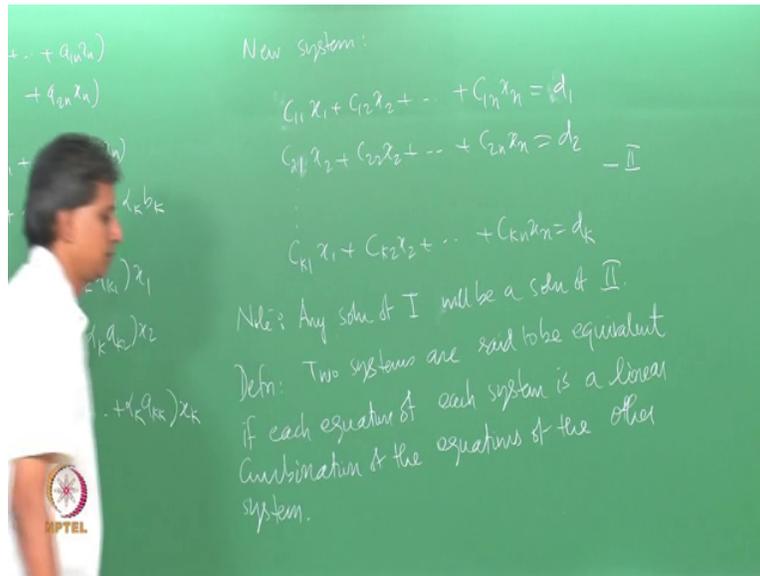
$\alpha_2 a_{21}$ etcetera plus $\alpha_k a_{k1}$ into x_1 plus so all the time I am trying to do is collect the coefficients of x_1 , collect the coefficients of x_2 etcetera and then see what that equation is.

So the next equation is $\alpha_1 a_{12}$ plus $\alpha_2 a_{22}$ etcetera plus $\alpha_k a_{k2}$ into x_2 plus etcetera $\alpha_k a_{k1}$ $\alpha_1 a_{11}$ plus $\alpha_2 a_{21}$ etcetera $\alpha_1 a_{12}$ plus $\alpha_2 a_{22}$ etcetera the k th equation is $\alpha_k a_{k1}$ plus $\alpha_2 a_{2k}$ plus etcetera plus α_1 , sorry this is $\alpha_1 a_{1k}$ plus $\alpha_2 a_{2k}$ etcetera plus $\alpha_k a_{kk}$ into x_k equals $\alpha_1 d_1$ etcetera $\alpha_k b_k$, okay. So all that I have done is all that I have done is to consider a new equation apart from these m equations this is a new equation some constant times x_1 , constant times x_2 etcetera constant times x_2 right hand side is of this form the same constants right hand side is of this form, okay.

Now the point that I am trying to make is the following you take any solution x_1 etcetera x_n that satisfies these m equations you take any solution x_1 etcetera x_n that satisfies these equations than that solution will satisfy this equation also any solution in fact the way we have taken any solution satisfying the first k equations will be a solution for this single equation will be a solution for this single equation that means what so this is called the thing that we have written down here is a linear combination of the equations of the first k equations of the system one this single equation is a linear combination of the first k equations of this system.

So any solution of system one will also be a solution of an equation obtained by taking linear combinations of the equations of system 1, okay.

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So consider let us consider this new system consider the new system lets call it by something else maybe $c_{11}x_1$ plus $c_{12}x_2$ plus etcetera plus $c_{1n}x_n$ equals d_1 , $c_{21}x_1$ plus $c_{22}x_2$ plus etcetera $c_{2n}x_n$ equals d_2 I am just going to consider k equations for the moment $c_{k1}x_1$ plus $c_{k2}x_2$ plus etcetera plus $c_{kn}x_n$ equals d_k .

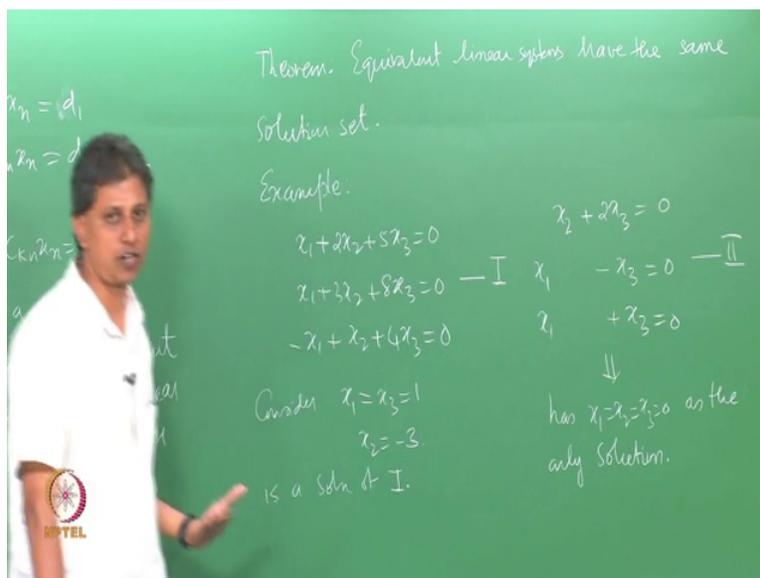
So I am considering a new system which has the property that each equation of this new system is obtained by system 1 by considering linear combinations that is take the first equation that is obtained by a certain linear combination of the m equations of system 1, similarly equation 2 is certain linear combination of the m equations of system 1 etcetera. So each equation here is obtained as a linear combination of a certain linear combination of the m equations of system 1 then what follows is that any solution of system 1 will be a solution of system 2 I will call this 2 so this is a point any solution of system 1 will be a solution of system 2 but the converse is in general false the converse is in general false.

However, suppose that these two systems are related by the fact that each equation of system 2 is a linear combination of the equations of the system 1 as well as the fact that each equation of system 1 is a linear combination of equations of system 2 if that happens then from whatever we have discussed till now it follows that this two equations have the same solutions, okay when we go from system 1 to system 2 what we are trying to do is to obtain each is to observe that each equation system 1 has been obtained as a linear combination of the equation of system 1 and so

the any solution of the system 1 must be a solution of system 2 all the time I am trying to do is to go the other way around and I want to conclude that these two systems are the same set of solution this can be done if I am able to verify so this is true if it holds that every solution of system every equation of system 1 is a linear combination of the equations of the system 2, okay there is a name for two systems that are related by this condition that is called equivalent system so let me just give definition, two systems are said to be equivalent if each equation of each system is a linear combination of the equations of the other system.

So I have two system they are related by this condition such systems will be called equivalent systems.

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So the argument that I have given now can be formalized in the following statement, theorem equivalent linear systems have the same solution set, now this is a fundamental lemma that will be useful to us. Let me just give a numerical example to kind of illustrate what is going on. Let us consider the following equations, so first equation is x_1 plus $2x_2$ plus $5x_3$ equals 0 the second equation is x_1 plus $3x_2$ plus $8x_3$ equal to 0 the third equation is minus x_1 plus x_2 plus $4x_3$ equals 0 so this is a third equation.

Now this is system 1 let us say let me also write system 2 so system 2 for me is so let me write it on this side x_2 plus $2x_3$ equals 0 so this is a one equation x_1 minus x_3 equals 0 and then x_1 plus x_3 equals 0 I will call this system 2 and make the following observation which is so let me make

the following observation which is that let us look at this equation 2 is can be solved quite easily, okay let us do that quickly.

From the second equation I have x_1 equals let us say x_3 equals x_1 I substitute into this then the third equation gives me $2x_1$ equal to 0 that is x_1 equal to 0 go back to the second equation I get x_3 equal to 0 from this equation it follows that x_2 is 0. So system 2 has 0 x_1 equal to x_2 equal to x_3 equal to 0 as the only solution this is the only solution for system 2 as 0 as the only solution, okay. On the other hand let us look at system 1, okay let us look at system 1 I will try and see whether each equation here is obtained as a linear combination of the equations of system 1, okay see my objective is to solve equation 1 it might turn out that these two systems are equivalent I do not know we need to verify that, okay but let us look at the first equation x_2 plus $2x_3$ equals 0 I will leave for you to verify that the first equation and the second equation are linear are certain linear combinations of the three equations of system 1, okay I will leave this as an exercise for you to verify you can choose easy constants and then multiply the equation by the constant just by another constant add subtract do these operations you will be able to show that the first two equations are linear combinations of certain linear combinations of the three equations of system 1.

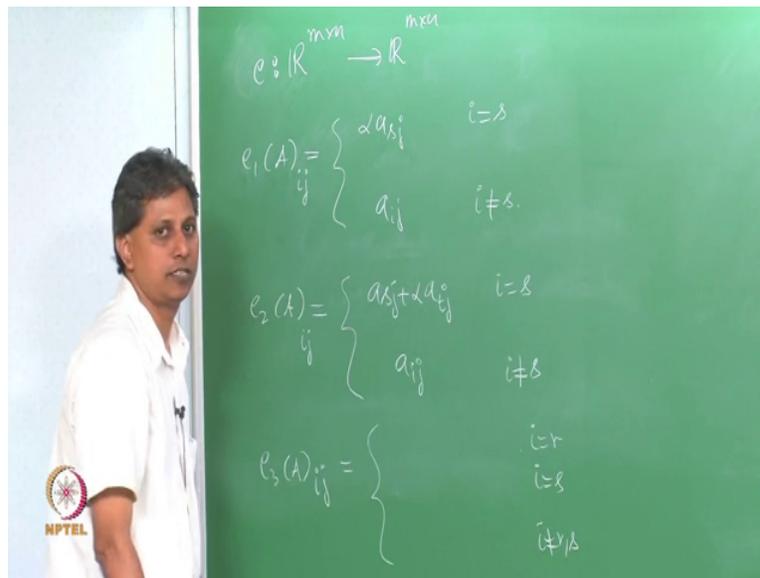
So any solution of system 1 will satisfy the first two equations but the third equation I am claiming that is not a linear combination of the three equations of system 1 the third equation here in system 2 is not a linear combination of the three equations of system 1. Now this is difficult to prove but what we can do is to make the following observation, okay this is in general difficult to prove but what we can do is to observe that so, okay what I will try and do is to give a solution of equation system 1 which does not satisfy the third equation it will then follow that this is not a linear combination of the three equation of system 1, okay.

So let us consider the case I just give one set of values let us say x_1 equal to x_3 equal to 1 and x_2 equals minus 3, okay x_1 equal to x_3 equal to 1 x_2 is minus 3, then let us look at the first equation x_1 plus $2x_2$ is 6 minus 6 plus 6 that is 0 the second equation is x_1 plus $3x_3$ that is 1 minus 9 plus 8 that is 0 the third equation is minus x_1 minus 1 minus 3 minus 4 plus 4 is 0, so this solves 1 this is a solution of system 1 you can now observe that this is obviously not a solution of system 2 the third equation is not satisfied because x_1 equal to x_3 equal to 1 and so this is not satisfied.

And so now it follows, okay it now follows that see if the third equation here where a linear combination of the three equations of system 1 then any solution of system 1 would have to be a solution of the third equation now that is not that does not happen in this example and so system 2 is not equivalent to system 1, okay let us also observe that the general solution set of this system is of this form I will call s as the solution set that is of the form $\alpha \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ where α is an arbitrary real number, okay it can be proved by the usual method of eliminating the variables that this is the most general this is the solution set any solution will be of this type let us observe that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is contained in this for the choice $\alpha = 0$, okay which means that any equation here is a linear combination of the equations of two but not the other way around, okay so these two systems are not equivalent, okay.

So this is just to illustrate given example of two systems which are not equivalent, so these two systems are not equivalent, when they are equivalent the solution sets will be the same, okay. We were discussing the concept elementary row operations in the last class and then there is a pause we are looking at something else systems two systems being equivalent, okay the thing is that these notions are related considering linear combinations of equations of a system is mounted to doing a certain elementary row operation or a sequence of elementary row operations, okay we will make this precise we will make this notion we will make this idea precise but before that let us go back and recall the definition of the elementary row operations, okay.

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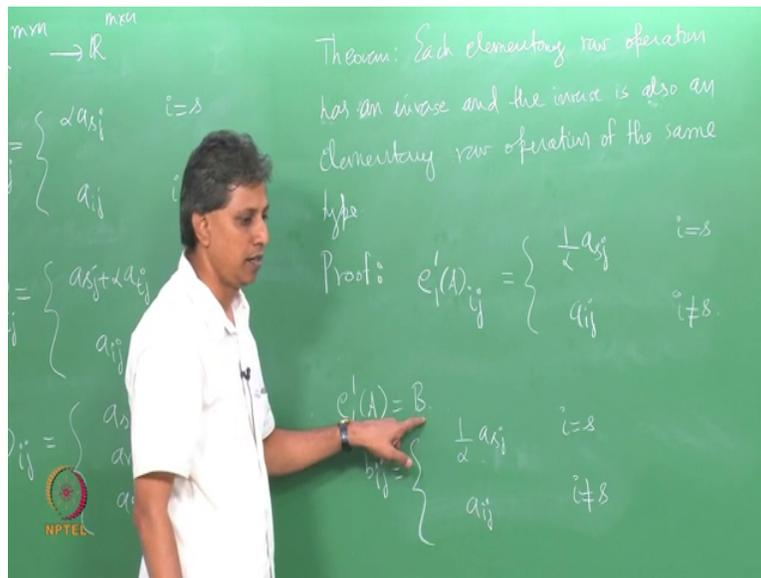


So what were the operations? We call these as e_1 , e_2 , e_3 so each elementary row operation takes a matrix of order m cross n to another matrix of order m cross n the definition is the first elementary row operation is multiplying a row by a non-zero scalar. So let us say that I multiply the s th row by a non-zero scalar α then that row changes to αa_{sj} all the other rows remain the same when i is not equal to s this is the first operation. The second operation is to replace the s th row by s th row plus α times the r th row, so when i is equal to s I am replacing the s th row so this corresponds to the first index so I am actually writing down the ij th entry of e_1 , ij th entry of e_2 etcetera.

So the first index corresponds to the row and second one corresponds to the column. So when i is equal to s it is replacing the s th row by s th row plus so that is the a_{sj} plus α times the t th row that is what I remember having used earlier and then the other rows are kept as their if i is not equal to s then it is just a_{ij} this is the second elementary row operation that we discussed last time. The third one is just interchange of rows let us say we interchange s th row and the r th row for instance then the ij th entry of this matrix is if i is equal to r i is equal to s i is not equal to r comma s so I am simply interchanging the rows r and s .

So when i is equal to r it is a_{sj} and i is equal to s it is a_{rj} all the other entries remain the same this is a_{ij} these are three elementary row operations I made the statement last time that the these functions each of these functions has an inverse and inverse is also of the same type.

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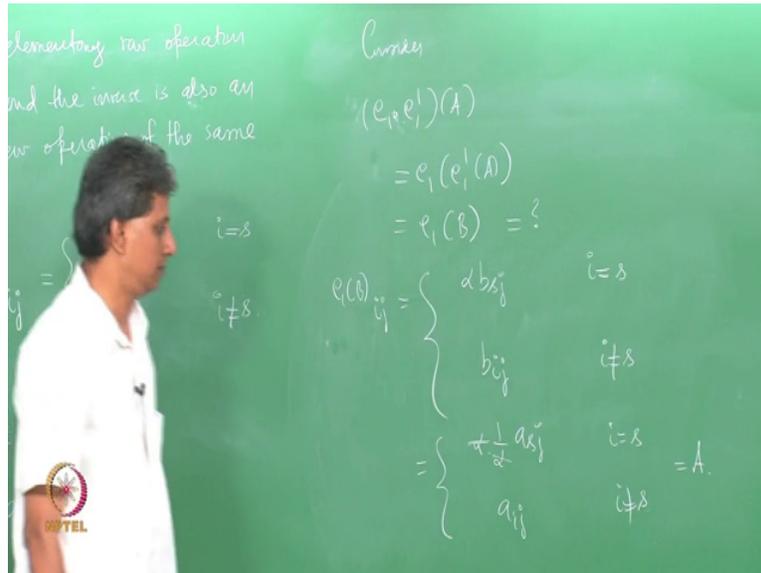


So let us see that quickly so that is going to be my next result, each elementary row operation has an inverse let me say has an inverse and the inverse is also an elementary row operation what is more important elementary row operation what is more important is that it is an elementary row operation of the same type it is also an elementary row operation of the same type, okay let us prove this let us dispose off the easy cases let us look at the first elementary row operation where we multiply a row by a non-zero scalar, okay. So all that I will do is explicitly write down the inverse functions. So let me use prime denote to denote the inverse functions so I am saying that the inverse of the first elementary row operation e_1 will be denoted by e_1 prime of A I will simply give the formula for that then we can probably verify one or two of them.

So e_1 prime is I must see the inverse operation must give me the matrix A that I started with. So given a matrix A I do the elementary row operation e_1 on that and then I need to do another elementary row operation which will nullify the effect, so we should give me back the matrix A , so obvious since α is not 0 it will be just this definition so look at one by αa_{sj} , right? So I am replacing I am multiply the s th row by the constant times 1 by α this operation is allowed this is multiplying a row by a constant, right? That is elementary row operation of the first type of course the other entries are left as they are, so this is my e_1 prime let us quickly verify that e_1 into e_1 prime gives me the identity function, okay.

So let us look at let us call it e_1 prime of A as B then, okay then I will write down the entries of B so b_{ij} b_{ij} by definition is so e_1 prime is the new matrix B, okay b_{ij} by definition is just this 1 by alpha a_{sj} if i is equal to s it is a_{ij} if i is not equal to s so I am just calling this as a new matrix B.

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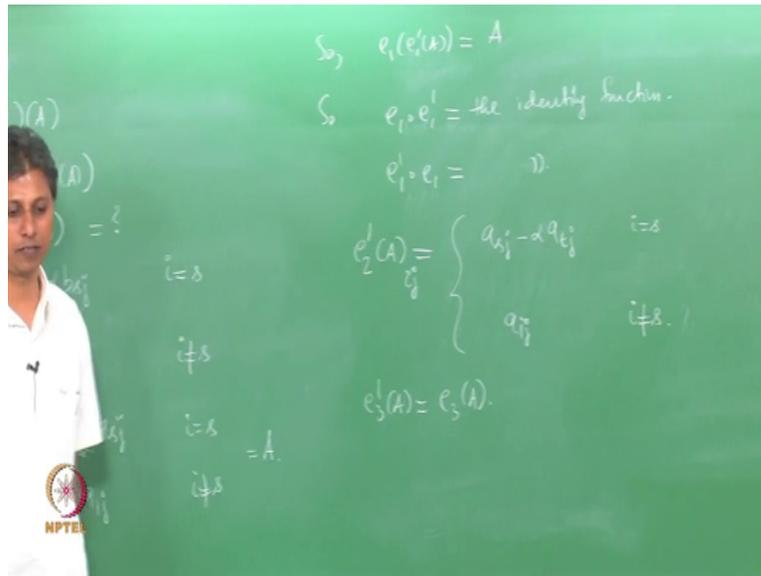


Now for this matrix B I will do the operation e_1 that is now I am asking you to consider e_1 of see this is a circle e_1 e_1 prime of A I want to show that this is equal to A it would then follow that this composition is the identity mapping.

So this is e_1 of so it is a composition function that is e_1 of I am calling this as the matrix B e_1 prime of A as B and so now I will do this elementary row operation on B, now how is this defined? So just look at e_1 of B the ij th entry so you need to go back to this definition multiply the s th row by alpha leave all the other rows as they are. So e_1 be ij multiply the s th row that is B_{sj} by alpha this is when i is equal to s all the other entries are left as they are that is B_{ij} so I am now doing an elementary row operation 1 on B so I am preferring to use b_{ij} so when i is not equal to s .

Now go back to this and then see what these entries are, so this is equal to alpha times B_{sj} what is B_{sj} when i is equal to s that gives me B_{sj} that corresponds to this alpha into 1 by alpha into A_{sj} so this is when i is equal to s when i is not equal to s it is just a_{ij} this is equal to a_{ij} when i is not equal to s now you can see that this is precisely the matrix A.

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So what we have shown is that e_1 of e_1 prime of A we have shown that it is equal to A and so it follows that $e_1 \circ e_1^{-1}$ equals the identity function we can show in a similar manner that $e_1^{-1} \circ e_1$ is also the identity function I will leave that part that is very similar to this, okay.

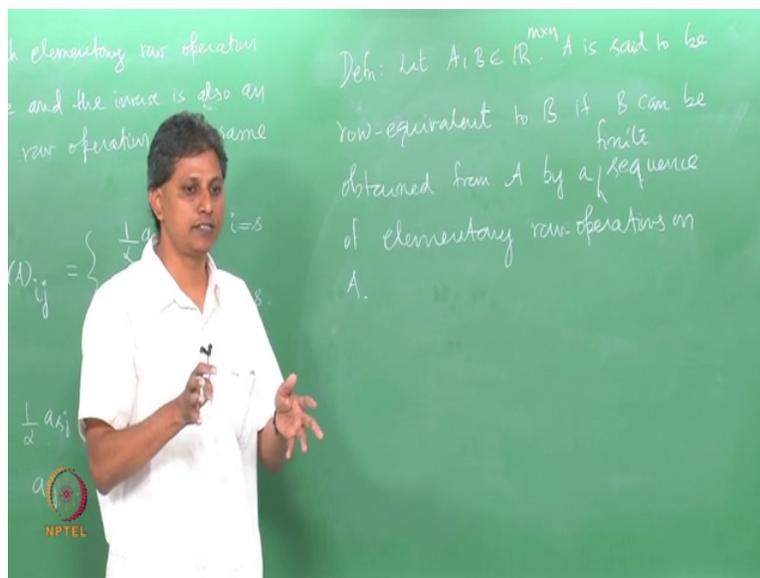
So you must observe here that second part is an elementary row it is invertible not only that it is an elementary row operation of the same type, type here means multiplying the s th row by a non-zero scalar, okay we can similarly verify that the other two operations are also invertible and that the inverses are of the same type I will leave this part as an exercise without getting into the details all that I will do is to write down the inverse formula for the inverse e_2 prime of A , now e_2 is this operation replacing the s row by s th row plus (α) constant non-zero constant times t_j . Now e_2 prime the inverse of that can be shown to satisfy this, so when i is equal to s it is again the s th row that is a_{ij} this time it will minus α times a_{tj} the other row elements are kept as they are i is not equal to s .

Now this is an elementary we have to show that this is an inverse of e_2 but let us say we have shown that then it follows that the inverse is also an elementary row operation of the same type that is it replaces the s th row by s th row plus a constant times the t th row, okay so this is also an elementary row operation of the same type as e_2 finally e_3 prime is straight forward is again this

is the ij th entry e_{33} prime is straight forward you interchange two rows you do the same interchange once again you will get the matrix A .

So e_{33} prime of A this time I will write it is just e_{33} of A this is such a thing is called a self-invertible map σ is its own inverse, okay. So that is again if e_{33} is interchanging row r and row s then e_{33} prime the inverse is interchanging again row r and row s so it is again an elementary row operation of the same type as e_{33} , okay so that proves this theorem. Now this gives us a this leads to another definition called row equivalent matrices.

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So let me write the definition here the definition of row equivalent matrices, $\mathbb{R}^{m \times n}$ for me will be the set of all matrices having m rows and n columns so this will be my notation this notation will be considered again when we discuss the notion of vector spaces, okay. Suppose I have two matrices then we say that A is said to be row equivalent to B remember one equivalence that we considered earlier is that of systems being equivalent, okay we are now considering equivalence of two matrices of the same order, however we will show that these two notions are related to each other, okay A is said to be row equivalent to B if B can be obtained from the matrix A what is the condition by a sequence of by a finite sequence always of elementary row operations by a finite sequence of elementary row operations on the matrix A B is said to be row equivalent A is said to be row equivalent to B if B can be obtained from A by a finite sequence of elementary row operations on the matrix A .

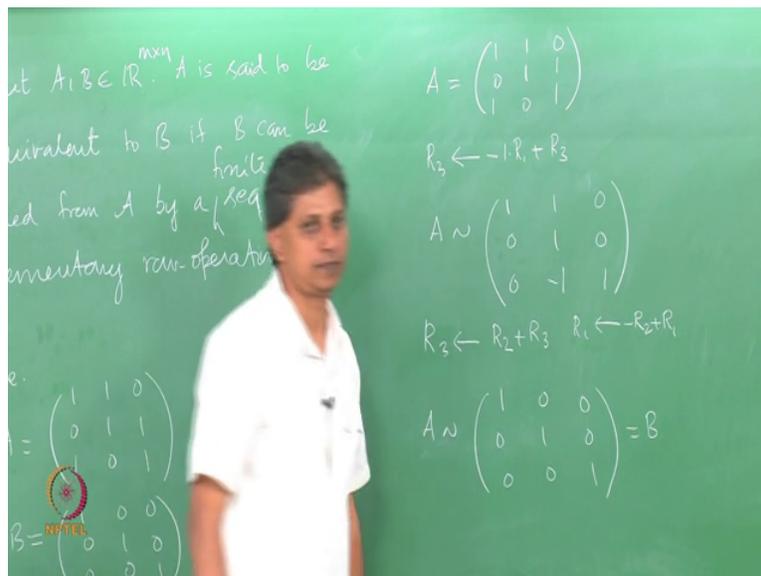
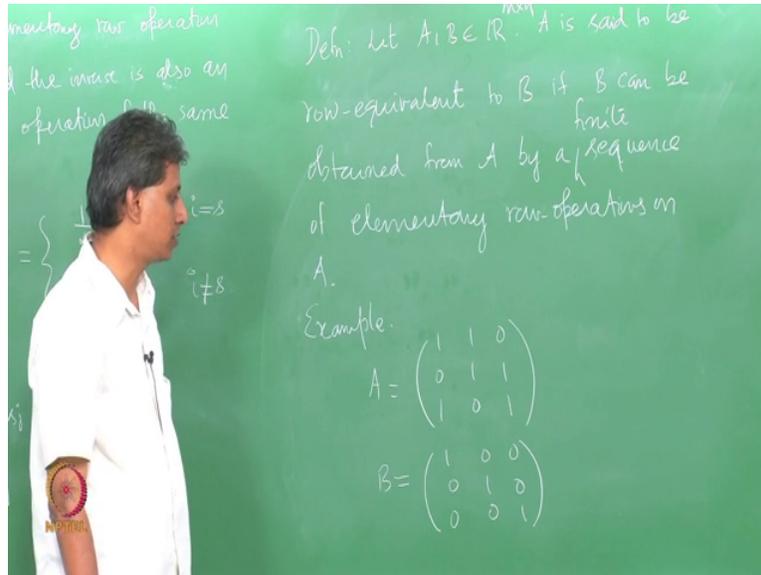
So starting with A I arrive at the matrix B then I say that A is row equivalent to B , okay. Then one would like to ask the question as to whether one can go the other way around if A is row equivalent to B can it happen that B is row equivalent to A , okay. Obviously A is row equivalent to itself do not do any operation you leave it as it is then A is row equivalent to A a finite sequence of operations has been performed, so A is row equivalent to itself the question is if A is row equivalent to B then can we go the other way around that is whether B is row equivalent to A we will answer this question a little later in the next couple of lectures by observing that the process of going from A to B can be reversed in fact this can be done by means of the elementary row operations that we have discussed just now we will look at what is called as the notion of elementary matrices.

So each elementary row operation gives rise to an elementary matrix using the elementary matrices we will show that if A is row equivalent to B then B is row equivalent to A we will also show that if A row is row equivalent to B and B is row equivalent to C then A is row equivalent to C , okay. Now this is intuitively obvious but it can again be a formal proof can again be given for this that is if on the matrix A you perform a finite sequence of elementary row operations to get the matrix B and then further down you perform a finite sequence of elementary row operations on B to get the matrix C then obviously you do the same the sequence one after the other sequence of elementary row operations one after the other on the matrix A you will end up with the matrix C .

So this transitive relationship is obvious from the definition, alright? So this operation this relation equivalence on the set of all m cross n matrices is really an equivalence relation, right? Reflexive, anti-symmetric transitive relation is called an equivalence relation so this actually an equivalence relation that justifies the name equivalence the only this is there an adjective here because you are doing elementary row operations. So row equivalence is actually an equivalence relation, okay. Now maybe I can give a simple example of obtaining a matrix B from a matrix A so let us look at one simple numerical example and then in the next lecture I will consider the problem of how the notion of equivalence systems of linear equations related to row equivalence of two specific matrices, okay this link this relationship is an important relationship where we will be able to consolidate this seemingly relate unrelated notions of row equivalent matrices and equivalence systems of linear equations, okay.

So let me conclude today's lecture by considering a simple example of constructing a matrix B that is row equivalent to a matrix A for instance.

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So let us take A to be let us say 110, 011, 101 and let me take B to be the matrix 100, 010, 001, okay the entries are all very simple they are either 0 or 1 note that B is the identity matrix of order 3, okay now let us do so all that I need to show is the claim is that A is row equivalent to B that is I will show that upon a sequence of finite sequence of elementary row operations on A I will get the matrix B, okay let us see how this is done.

So let me write A once again A is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ now I will do the following elementary row operation on the matrix A that is I will replace row 3 by minus 1 times row 1 plus row 3 and leave the second row as it is. Then A is equivalent to so I will use this symbol then A is equivalent row equivalent to the matrix the first row entry is the same the second row is also the same the third row is minus this row plus this row so that gives me $\begin{pmatrix} 0 & -1 & 1 \end{pmatrix}$ then I do the following operation row 3 is row 2 one time row 2 plus row 3, okay and I should also do the following operation on row 1, row 1 will be minus 1 times row 2 plus row 1, so I am keeping row 2 fixed when I do this then I get the following matrix which is row equivalent to the matrix A so the second row is kept as it is so I will write that as it is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ the third row becomes row 2 plus so I have to just add these two rows that gives me $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ and row 1 is replaced by minus row 2 plus row 1 so this is $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ this is precisely the matrix B and so A is row equivalent to B, okay.

Now in this example can verify by doing the inverse operations say I have done elementary row operations I know that each elementary row operation is invertible and that inverse is also of the same type, so starting from B I can do the inverse elementary row operations and end up with the matrix A, okay. So it can be verified in this numerical example that A is row equivalent to B and that B is row equivalent to A, okay. So let me stop today's lecture with this example the relationship between equivalent systems of linear equations and row equivalent matrices I have two systems $ax = b$ and $cx = d$ I will actually show that if $ax = b$ and $cx = d$ are equivalent then the matrix A comma D, sorry A comma B and the matrix C comma D are actually row equivalent, okay.

So this will help us to formalize Gaussian elimination that will be done in the next few lectures, so let me stop here.