Course Name: Essentials of Topology
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Welcome to Lecture 9 on Essentials of Topology.

In the previous lecture, we have seen some examples of topologies. In this lecture, we will study some topologies on the set of real numbers, which will continue in the next lecture. Specifically, in this lecture, we will study two topologies, that is the Euclidean topology and the right ray topology. Beginning with the concept of Euclidean topology. Let

$$\mathcal{T} = \{G \subseteq \mathbb{R} : \forall x \in G, \exists a, b \in \mathbb{R} \text{ with } a < b \text{ such that } x \in (a, b) \subseteq G\}.$$

Then, we can show that this $\mathcal{T}$ is a topology on $\mathbb{R}$. This topology is known as Euclidean topology, which is also called standard topology or usual topology.

Let us prove that $\mathcal{T}$ is a topology on $\mathbb{R}$. Beginning with, the empty set will always be a member of $\mathcal{T}$, which is trivial. Also, the set of reals is a member of $\mathcal{T}$, because, for all real numbers x, we can find two real numbers: one is $x - 1$, and another is $x + 1$ such that $x \in (x - 1, x + 1) \subset \mathbb{R}$.

Moving ahead, let us take some $G_1, G_2, ..., G_n \in \mathcal{T}$, that is, we are taking finite members from $\mathcal{T}$. Our motive is to justify that $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. Note that if any of the G_i is empty, then this intersection becomes an empty set, and that will be in $\mathcal{T}$. So, we are assuming that all G_i's are non-empty, $1 \leq i \leq n$. Now, in order to justify that $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$, let us take $x \in G_1 \cap G_2 \cap ... \cap G_n$. Obviously, $x \in G_1$; $x \in G_2$,.., and $x \in G_n$. As $x \in G_1$ and $G_1 \in \mathcal{T}$, we can conclude that we can find two real numbers, a_1 and $b_1, a_1 < b_1$, such that $x \in (a_1, b_1) \subseteq G_1$. Similarly, there exist two real numbers, a_2 and $b_2, a_2 < b_2$, such that $x \in (a_2, b_2) \subseteq G_2$. Also, there exist two real numbers, a_n and $b_n, a_n < b_n$, such that $x \in (a_n, b_n) \subseteq G_n$. Now, let us take a real number a, which is the maximum of these $a_1 a_2, ..., a_n$. Similarly, let us take another real number, b, which is the minimum of these $b_1 b_2, ..., b_n$. So, we can conclude that $x \in (a, b) \subseteq (a_i, b_i) \subseteq G_i$, where $1 \leq i \leq n$. From here, we can conclude that $x \in (a, b) \subseteq G_i$, for all i, where $1 \leq i \leq n$. Therefore,

$x \in (a, b) \subseteq G_1 \cap G_2 \cap ... \cap G_n$, or we can say that $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. So, what have we seen? We have taken some finite number of members from $\mathcal{T}$, and we have shown that their intersection is also in $\mathcal{T}$.

Coming to the next one, let us take a family of subsets $\{G_i : i \in I\}$ of X from $\mathcal{T}$. We have to show that $\cup\{G_i : i \in I\} \in \mathcal{T}$. In order to show it, let us take $x \in \cup\{G_i : i \in I\}$. Then there exists some $k \in I$ such that $x \in G_k$. Note that this $G_k \in \mathcal{T}$. Therefore, there exist two real numbers a and b with a is less than b such that $x \in (a, b) \subseteq G_k$. But note that this G_k is one of the members of $\{G_i : i \in I\}$. Therefore, we can conclude that $x \in (a, b) \subseteq G_k \subseteq \cup\{G_i : i \in I\}$, or $x \in (a, b) \subseteq \cup\{G_i : i \in I\}$. Meaning is, $\mathcal{T}$ is closed under arbitrary union, and therefore, this $\mathcal{T}$ is a topology on the set of real numbers. We will denote this topology by $\mathcal{T}_e$.

Let us discuss some concepts associated with Euclidean topology. Let us take two real numbers a and b, $a < b$, then we can conclude that (a, b) is a $\mathcal{T}_e$-open set. The question is why? Because for all $x \in (a, b)$, we can write $x \in (a, b) \subseteq (a, b)$, and therefore this (a, b) is a member of $\mathcal{T}_e$. Moving ahead, for every real number a, the open intervals of the form (a, ∞) or $(-\infty, a)$ are $\mathcal{T}_e$-open set. The question is, why? If we are taking the first one, (a, ∞), and we are taking any $x \in (a, \infty)$, then $x \in (a, x+1) \subset (a, \infty)$. So, we can conclude that the interval $(a, \infty) \in \mathcal{T}_e$. Similarly, if we are thinking about $(-\infty, a)$, and we are taking any real number x in this interval, then $x \in (x-1, a) \subset (-\infty, a)$. Therefore, $(-\infty, a)$ is also a member of $\mathcal{T}_e$.

Moving ahead, we have seen that every open interval is an open set in the Euclidean topology. The question is, what about the converse? If any set is $\mathcal{T}_e$-open, is it necessary that it will be an open interval? The answer is no. For example, we have seen that $(1, 2)$ and $(3, 4)$ are both members of the Euclidean topology. But if we are taking their union, that is, $(1, 2) \cup (3, 4)$, this is also a member of the Euclidean topology, but this is not an open interval.

Moving ahead, we can find some other intervals which are not members of the Euclidean topology. The examples are: $[a, b], [a, b), (a, b], a, b \in \mathbb{R}, a < b$. The question is: what is the problem? The answer is, the elements which is at the end, where the interval is closed, there will always be a problem. We cannot construct an open interval containing that point, which is a subset of these intervals. Let us see it mathematically. For example, let us take the first one, i.e., $[a, b]$. We are justifying that this one is not an element of Euclidean topology. Why? Let us prove it by contradiction. It is clear from here that $a \in [a, b]$. If possible, let us take c and d as two real numbers with c is less than d alongwith the condition that $a \in (c, d) \subseteq [a, b]$. Then what will happen? This is clear from here that $c < a < d$. From this inequality, we can conclude one more thing that: $c < (c+a)/2 < a < d$. From here, we have two conclusions. One is, $(c + a)/2$ is a member of this interval (c, d), but at the same time, the question is whether $(c+a)/2$, a member of this closed interval $[a, b]$. The answer is no because this real number is always less than a. So, what we have assumed is wrong, and therefore, we cannot construct any such open interval, and that's why this closed interval $[a, b]$ is not a member of $\mathcal{T}_e$. Similarly, we can discuss for the semi-open intervals.

Moving ahead, the question is whether finite sets, the set of natural numbers, the set of integers, and the set of rational numbers are members of the Euclidean topology. The answer is no in all the cases. Why? Because if we are taking A as a finite set and any member $x \in A$. The question is: Can we construct an open interval by finding two real numbers, a and b, such that a is less than b and $x \in (a, b) \subseteq A$? If A is finite, the answer is no, as A is a finite set, and the interval (a, b) is an infinite set. The same will happen when we are going to take the set of natural numbers. Here, if we are trying to construct such open intervals by taking two real numbers, a and b, note that (a, b) will consist of rational and irrational numbers. So, this (a, b) cannot be a subset of $\mathbb{N}$. If we are thinking about the set of rationals and constructing such intervals, note that such intervals will also contain irrational numbers. So, this is not possible. If we are looking for the set of irrationals, the same problem is here. We cannot construct any open interval because that open interval will also contain rationals. So, the final answer is whatever the sets mentioned here, none of these can be a member of $\mathcal{T}_e$, that is, the Euclidean topology.

Moving to the next, what we have seen in the case of Euclidean topology is that we have taken some subsets of $\mathbb{R}$ satisfying some property, and that property was given in terms of some open interval. The question is: if we are taking some $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(a, b) : a, b \in \mathbb{R}, a < b\}$, is it a topology? The answer is no. Why? It's simple to justify that because the union of two open intervals may not be an open interval. We have also seen that in the case of Euclidean topology, two intervals, that is, intervals of the form (a, ∞) and intervals of the form $(-\infty, a)$, are also members of the Euclidean topology. The question is, instead of taking this (a, b), if we are taking the intervals of the form (a, ∞) or intervals of the form $(-\infty, a)$, can we construct a topology on $\mathbb{R}$? The answer is here. If we are taking $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$, then this is a topology on $\mathbb{R}$, known as right ray topology. Let us see the justification behind this concept.

It is already given that empty set and $\mathbb{R}$ are members of $\mathcal{T}$. If we are taking some finite number of sets $G_1, G_2, ..., G_n$ from this $\mathcal{T}$, is $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$? The answer is yes. How? If any G_i is an empty set, what will be this $G_1 \cap G_2 \cap ... \cap G_n$? This intersection will also be an empty set. Therefore, $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. So, what are we taking? We are taking that these G_i's are non-empty for all $1 \leq i \leq n$. Now, $G_1 \cap G_2 \cap ... \cap G_n$ will look like $(a_1, \infty) \cap (a_2, \infty) \cap ... \cap (a_n, \infty)$. This can be written as (a, ∞), where a is nothing but a maximum of $a_1, a_2, ..., a_n$. Therefore, $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. Moving to the next one. If we are taking a family of subsets $\{G_i : i \in I\}$ of $\mathbb{R}$ indexed by I, where G_i is from $\mathcal{T}$, whether $\cup\{G_i : i \in I\}$, is in $\mathcal{T}$? The answer is yes. There will be different cases. For example, if any of G_i becomes $\mathbb{R}$, what about the union? The union will be nothing but $\mathbb{R}$ itself. What about if all G_i's are the empty set? What about the union? The union will be nothing but the empty set. But if we are taking G_i's, which are not equal to $\mathbb{R}$ and non-empty, then how will $\cup\{G_i : i \in I\}$ look like? This will be something like the $\cup\{(a_i, \infty) : a_i \in \mathbb{R}, i \in I\}$, which can be written as (a, ∞). What is this a? a is be nothing but the infimum of $\{a_i : i \in I\}$. Therefore, $\cup\{G_i : i \in I\}$, is in $\mathcal{T}$. Hence, $\mathcal{T}$ is a topology on the set of real numbers.

These are the references.

That's all from today's lecture. Thank you.