Course Name: Essentials of Topology
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Welcome to Lecture 8 on Essentials of Topology. In this lecture, we will see some interesting examples from topological spaces. The concepts which will be covered in this lecture are co-finite topology, co-countable topology, particular point topology and excluded point topology, initial segment topology, and final segment topology. We begin with the concept of co-finite topology, which is also known as finite complement topology. This topology is defined in such a manner.

Let X be a non-empty set, and $\mathcal{T} = \{G \subseteq X : G = \emptyset \text{ or } G^c \text{ is finite}\}$. We can see that this $\mathcal{T}$ is a topology on X. Let us see it. This is already given here that the empty set is in $\mathcal{T}$. Also, $X \in \mathcal{T}$ because X^c is an empty set, which is finite.

Coming to the next, let $G_1, G_2, ..., G_n$, that is, a finite number of members from $\mathcal{T}$. Our motive is to justify that their intersection, that is, $G_1 \cap G_2 \cap ... \cap G_n$, is also in $\mathcal{T}$. There will be two cases. The first one is, if any of the G_i is empty set, what will happen? If any of the G_i is empty, $G_1 \cap G_2 \cap ... \cap G_n$ becomes an empty set, and obviously, this is in $\mathcal{T}$. But if all G_i's are non-empty, where $1 \leq i \leq n$, then if we want to justify that this intersection is in $\mathcal{T}$, we have to show that the complement of this intersection is finite. So, let us take the complement of $G_1 \cap G_2 \cap ... \cap G_n$, which will be $G_1^c \cup G_2^c \cup ... \cup G_n^c$. It is to be noted here that these G_i's are in $\mathcal{T}$, and therefore, their complement is finite. So, what we have here, this is nothing but a finite union of finite sets, and we know that a finite union of finite sets is finite, therefore $G_1^c \cup G_2^c \cup ... \cup G_n^c$ is a finite set, and if this is finite, we can say that $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$.

Moving to the next, let us take a collection $\{G_i : i \in I\}, G_i \subseteq X$, where each $G_i \in \mathcal{T}$. Our motive is to justify that $\cup\{G_i : i \in I\} \in \mathcal{T}$. Now, it is clear that if all G_i's are empty sets, then $\cup\{G_i : i \in I\}$ is an empty set, and that is in $\mathcal{T}$. But if at least one of the G_i's is non-empty, let us take $G_k \in \{G_i : i \in I\}$, and G_k is non-empty. Then it means that G_k^c is finite. Now, in order to show that $\cup\{G_i : i \in I\} \in \mathcal{T}$, we have to show that $[\cup\{G_i : i \in I\}]^c$, is finite. Since

$G_k \in \{G_i : i \in I\}$, therefore $G_k \subseteq \cup\{G_i : i \in I\}$, or $[\cup\{G_i : i \in I\}]^c \subseteq G_k^c$. It is to be noted here that if G_k^c is finite, its subset, that is, $[\cup\{G_i : i \in I\}]^c$ is also finite, and therefore $\cup\{G_i : i \in I\} \in \mathcal{T}$. Hence, $\mathcal{T}$ is a topology on X.

Let us see some of the notations and concepts associated with co-finite topology. The first one is: We will denote the co-finite topology using the notation $\mathcal{T}_{cf}$. If X is a finite set, it is interesting to show that $\mathcal{T}_{cf}$ is a discrete topology. How is it possible? The answer is, if you see the definition of co-finite topology, that is, a collection of all subsets of X such that either G is an empty set or G^c is finite. But note that if X itself is a finite set, all the subsets of X will satisfy this property, or $\mathcal{T}_{cf}$ becomes the power set of X, and therefore, $\mathcal{T}_{cf}$ turns out to be a discrete topology. So, whenever we will discuss about co-finite topology, we assume that X is an infinite set. Otherwise, this will always be a discrete topology.

Moving ahead, let us take $X = \mathbb{R}$ and $a, b \in \mathbb{R}$ such that $a < b$. The question is whether the intervals, that is, closed interval $[a, b]$, semi-open interval $(a, b]$, semi-open interval $[a, b)$, and open interval (a, b) are elements of $\mathcal{T}_{cf}$. The answer is no. Why? Because if we are taking any of these and we are trying to find out their complement, their complement is not a finite set. It is clear that none of them is an empty set. Therefore, such intervals cannot be an element of $\mathcal{T}_{cf}$.

Let us take one more example. If we are taking $X = \mathbb{R}$, again, whether the set of natural numbers, set of integers, set of rational numbers, or set of irrational numbers are elements of co-finite topology, the answer is no. Why? Because their complement cannot be finite. So, what type of sets will be members of $\mathcal{T}_{cf}$? One of the examples is here. If we are taking $X = \mathbb{R}$, let us take the sets of the form $\mathbb{R} - \{x_1, x_2, ..., x_n\}$. They are elements of co-finite topology because if we are finding out the complement of this set, this complement becomes $\{x_1, x_2, ..., x_n\}$, which is finite, and therefore, these types of sets will be members of co-finite topology on the set of real numbers.

In the following picture, if we are taking the real line, which is punctured at $-4, -1, 2,$ and 4, that is, this is precisely $\mathbb{R} - \{-4, -1, 2, 4\}$. Then it will belong to co-finite topology on the set of real numbers, because if we are finding out its complement, the complement will be a finite set.

The next one is the concept of co-countable topology or countable complement topology. How is this topology defined? Let us take X as a non-empty set and $\mathcal{T} = \{G \subseteq X : G = \emptyset$ or G^c is countable $\}$. Then, similar to the concept of co-finite topology, one can justify that $\mathcal{T}$ is a topology on X. This topology differs from co-finite topology in the sense of only the complement of subsets. In co-finite topology, we have seen that G^c is a finite set, where instead of taking the finite set, we are taking G^c as a countable set in this particular topology. As the proof is similar to that one, I am not giving here the proof in detail.

Let us see again some notations and concepts associated with co-countable topology. So, we shall denote this topology by $\mathcal{T}_{cc}$. Now, if X is a countable set, then $\mathcal{T}_{cc}$ is always a discrete topology. The question is, why? The answer is, if we are looking $\mathcal{T}_{cc}$, we have already seen that this is a collection of all subsets G of X such that G is an empty set or G^c is countable. But note that if X itself is countable, then all subsets of X will satisfy this property, that is, $\mathcal{T}_{cc}$ is precisely all those G such that $G \subseteq X$, which is nothing but the power set of X. Therefore, this co-countable topology turns out to be discrete, and accordingly, if we want to discuss co-countable topology, we will always take X as an uncountable set. Otherwise, there will be no difference between co-countable topology and discrete topology.

Let us take some of the examples. For example, if $X = \mathbb{R}$ and $a, b \in \mathbb{R}$ such that $a < b$. The question is whether closed interval $[a, b]$, semi-open interval $(a, b]$, semi-open interval $[a, b)$ and open interval (a, b) are elements of co-countable topology. The answer is no. The question is, why? Because neither of these intervals is an empty set. If we are looking for their complements, the question is whether that is countable. The answer is no. Accordingly, none of these intervals can be a member of co-countable topology.

Moving ahead, let us take $X = \mathbb{R}$ and check whether the set of natural numbers, set of integers, set of rational numbers, and set of real numbers are elements of this co-countable topology. Obviously, no in the case of $\mathbb{N}$, as its complement is not countable. No, in the case of the set of integers, because its complement is not countable. Even no in the case of the set of rationals, because its complement is the set of irrationals. But yes, in the case of the set of irrationals. Because, whenever we are looking for the complement of the set of irrationals, that becomes the set of rationals, and we know that this set is countable. So, finally, the conclusion is: Among these four sets, only this set of irrationals is a member of the co-countable topology.

Moving ahead, let us discuss one more topology, known as particular point topology. This topology is defined as: If we are having a non-empty set X and a particular point $x_0 \in X$ and $\mathcal{T} = \{G \subseteq X : G = \emptyset \text{ or } x_0 \in G\}$. Then $\mathcal{T}$ is a topology on X, and this topology is known as the particular point topology, because the definition itself depends on this particular point, that is x_0.

Let us see how this $\mathcal{T}$ is a topology on X. Begin with the first one. Note that the empty set is a member of $\mathcal{T}$ because this is given to us. X is also in $\mathcal{T}$ because $x_0 \in X$. Moving ahead, let us take a finite number of subsets $G_1, G_2, ..., G_n$ of X, which are in $\mathcal{T}$. Our motive is to justify that $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. The question is, how is it possible? The answer is, if any of G_i is empty, there is no need to discuss about this intersection because, in that case, this intersection becomes an empty set, which is always a member of $\mathcal{T}$. But if all of G_i's are non-empty, then from the definition of $\mathcal{T}$, $x_0 \in G_i$, $1 \leq i \leq n$, and if $x_0 \in G_i$, for all $1 \leq i \leq n$. Thus $x_0 \in G_1 \cap G_2 \cap ... \cap G_n$, and therefore $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$.

Coming to the last point, let us discuss how an arbitrary union of members of $\mathcal{T}$ is a member of $\mathcal{T}$. So, let us take a family of subsets of X, that is $\{G_i : i \in I\}$, where $G_i \in \mathcal{T}$. Our motive is to justify that $\cup\{G_i : i \in I\} \in \mathcal{T}$. Note that if all G_i's are empty sets, then $\cup\{G_i : i \in I\}$ becomes an empty set, that belongs to $\mathcal{T}$. Now, if G_k is non-empty, for some $k \in I$, where G_k is one of the members of $\{G_i : i \in I\}$. Then, we can conclude that x_0 belongs to G_k and $G_k \subseteq \cup\{G_i : i \in I\}$. Thus $\cup\{G_i : i \in I\}$ contains x_0; whereby, $\cup\{G_i : i \in I\} \in \mathcal{T}$. Hence, $\mathcal{T}$ is a topology on X.

There is one more concept known as excluded point topology. This topology depends on the exclusion of a fixed point. So, let us take X as a non-empty set and a fixed-point x_0 in X. Define $\mathcal{T} = \{G \subseteq X : G = X \text{ or } x_0 \notin G\}$. Then, we can see that this $\mathcal{T}$ is a topology on X, known as the excluded point topology. Let us see a glimpse of proof that how this is a topology. The answer is that the empty set is a member of $\mathcal{T}$ because it doesn't contain x_0, and $X \in \mathcal{T}$;

this is already given to us.

Moving ahead, if we are taking some finite number of elements $G_1, G_2, ..., G_n$ from $\mathcal{T}$, we need to justify that their intersection is in $\mathcal{T}$. Let us see it. First, if all of G_i's equals X, $1 \leq i \leq n$. Then $G_1 \cap G_2 \cap ... \cap G_n = X$, which is already in $\mathcal{T}$. But if there exists some G_i among $G_1, G_2, ..., G_n$ such that G_i is not equal to X, then x_0 cannot be an element of this particular G_i. Thus x_0 cannot be an element of $G_1 \cap G_2 \cap ... \cap G_n$. Therefore, $G_1 \cap G_2 \cap ... \cap G_n$ will be a member of $\mathcal{T}$. Now, let us take a collection of subsets $G_i \subseteq X$ indexed by a set I, where every $G_i \in \mathcal{T}$. Our motive is to justify that $\cup\{G_i : i \in I\} \in \mathcal{T}$. Now, there will be two cases. The first one is, if $G_i = X$, for some $i \in I$. Then it is simple to say that $\cup\{G_i : i \in I\} = X$, which belongs to $\mathcal{T}$. But, if G_i is not equal to X, for all $i \in I$, then it is clear that x_0 cannot be an element of G_i, as $G_i \in \mathcal{T}$ for all $i \in I$. Thus, if x_0 is not an element of G_i, for all $i \in I$, we can say that $\cup\{G_i : i \in I\}$ does not contain x_0, and therefore, $\cup\{G_i : i \in I\} \in \mathcal{T}$. Hence, $\mathcal{T}$ is a topology on X.

Let us see one more example. This is an example of topology on the set of natural numbers. For $n \in \mathbb{N}$, let us take the sets of the form $H_n = \{1, 2, ..., n\}$, and $\mathcal{T} = \{\emptyset, \mathbb{N}\} \cup \{H_n : n \in \mathbb{N}\}$. We can see that this $\mathcal{T}$ is a topology on $\mathbb{N}$, known as the initial segment topology. Before seeing the proof that $\mathcal{T}$ is a topology, let us visualize the elements of $\mathcal{T}$, which consists of an empty set, the set of natural numbers, and the rest of the sets lie between the empty set and the set of natural numbers, beginning with $\{1\}$, two-point set $\{1, 2\}$, ... From here, we can conclude that the elements of $\mathcal{T}$ are nested. Now, with this idea, let us see how this $\mathcal{T}$ is a topology on $\mathbb{N}$. So, the first thing is that the empty set and set of natural numbers are both in $\mathcal{T}$, as this is already given to us. Now, if there is a finite number of members in $\mathcal{T}$, that is, $G_1, G_2, ..., G_n$ are in $\mathcal{T}$, and if you want to find out the intersection of these, that is, $G_1 \cap G_2 \cap ... \cap G_n$. Then this intersection will be nothing but the smallest subset among all G_i's, because the subsets in $\mathcal{T}$ are nested, and if that is the smallest subset among all G_i's, that will be one among $G_1, G_2, ..., G_n$, and hence in $\mathcal{T}$. Similarly, if we are taking a family of subsets $G_i \subseteq X$ in $\mathcal{T}$ indexed by I, and we are looking at the structure of $\cup\{G_i : i \in I\}$. Then, this will be nothing but the largest subset in the collection, or it may be $\mathbb{N}$ itself. Therefore, this union is in $\mathcal{T}$. Hence $\mathcal{T}$ is a topology on the set of natural numbers.

Similar to the concept of initial segment topology, there is another concept known as final segment topology. Here, on the set of natural numbers, $\mathcal{T}$ is defined as $\mathcal{T} = \{\emptyset\} \cup \{H_n : n \in \mathbb{N}\}$. Here $H_n = \{n, n+1, n+2, ...\}, n \in \mathbb{N}$. Again, before going to a formal proof that this $\mathcal{T}$ is a topology on $\mathbb{N}$, let us see the nature of elements of $\mathcal{T}$. Note that, $\mathcal{T}$ is consisting of an empty set. It is also consisting H_n, $n \in \mathbb{N}$, and what are the relationship between these H_n's. Actually, $H_n \subseteq H_{n-1} \subseteq ...H_2 \subseteq H_1 = \mathbb{N}$. So, similar to the previous case, the elements of $\mathcal{T}$ are nested in this case too.

Now, see this empty set and $\mathbb{N}$ both are members of $\mathcal{T}$, because $\mathbb{N}$ is nothing but H_1 and $H_1 \in \mathcal{T}$. If we are taking some finite members from $\mathcal{T}$, that is $G_1, G_2, ..., G_n$ are in $\mathcal{T}$, then their intersection, that is, $G_1 \cap G_2 \cap ... \cap G_n$, will be the smallest set among G_i's, and therefore that will be in $\mathcal{T}$. Finally, if we are taking an arbitrary collection of G_i from $\mathcal{T}$, where $i \in I$. Then $\cup\{G_i : i \in I\}$ will be the largest subset in this sub-collection, and even because all the subsets available in $\mathcal{T}$ are nested, that may be $\mathbb{N}$ itself. Therefore, $\cup\{G_i : i \in I\} \in \mathcal{T}$, and hence $\mathcal{T}$ given here is a topology on the set of natural numbers.

Let us take the last example. This example shows that the collection $\mathcal{T} = \{\emptyset, \mathbb{N}\} \cup \{G \subset \mathbb{N} : G \text{ is finite }\}$ is not a topology on the set of natural numbers. The question is why; the answer is, if we are taking the sets $\{3\}, \{4\}, ..., \{n\}, ...$ from $\mathcal{T}$. Then, note that each of these sets is a finite set, and their union is an infinite set $\{3, 4, 5, ...\}$, which is not in $\mathcal{T}$, because as per the definition, $\mathcal{T}$ consists of an empty set, the set of natural numbers, or all subsets of $\mathbb{N}$, which are finite in nature. Meaning is, we can find a collection of subsets from $\mathcal{T}$ whose union is not in $\mathcal{T}$. Therefore, $\mathcal{T}$ is not a topology on $\mathbb{N}$.

These are the references.

That's all from today's lecture. Thank you.