Course Name: Essentials of Topology
Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
Institute Name: Indian Institute of Technology(ISM), Dhanbad
Week: 12
Lecture: 06

Welcome to Lecture 72 on Essentials of Topology.

In this lecture, too, we will continue the study of the concept of Stone-Čech compactification. Begin with, if we are having a topological space $(X, \mathcal{T})$, this topological space has a compactification, then we have shown that this space is always a completely regular space. The question is, if we are beginning with a completely regular topological space $(X, \mathcal{T})$, whether this has a compactification. The answer is yes. We will justify that every completely regular space has a compactification, and that compactification is known as the Stone-Čech compactification.

Before studying such compactification of a completely regular topological space, let us see a concept known as imbedding lemma, which we are not proving here, but we will use this for the study of Stone-Čech compactification. What this lemma is? Let $(X, \mathcal{T})$ be a topological space and let $\{f_i\}_{i\in I}$ be a family of continuous functions $f_i : X \to [0, 1]$. Then the family $\{f_i\}_{i\in I}$ separates points from closed sets if for any $x_0 \in X$ and any closed set $F \subseteq X$ such that $x_0 \notin F$ there is a function $f_j \in \{f_i\}_{i\in I}$ such that $f_j(x_0) > 0$ and $f_j(x) = 0$, for all $x \in A$.

Now, coming to the precise statement of imbedding lemma. This lemma is stated as: Let $(X, \mathcal{T})$ be a T_1-space. If $\{f_i : X \to [0, 1]\}_{i\in I}$ is a family of continuous functions that separate points from closed sets, then the function $f : X \to \Pi_{i\in I}[0, 1] = [0, 1]^I$ such that $f(x) = (f_i(x))_{i\in I}$ is an imbedding. So, what does embedding lemma say? Actually, it guarantees that if we have a family of continuous functions with the condition that this family separates points from closed sets, then we can talk about an imbedding from X to the product of closed intervals. Also, there is a result regarding completely regular spaces, and the proof is from this imbedding lemma itself. It says that a topological space $(X, \mathcal{T})$ is completely regular iff it is homeomorphic to a subspace of $[0, 1]^I$, for some I. We have only stated the result along with the imbedding lemma, but we are not proving it here. For proof, we refer to the book of Munkres.

Now, let us use this lemma and try to see the Stone-Čech compactification of a completely regular topological space. So, we are beginning with this theorem, and this theorem guarantees for the existence of compactification of a completely regular space. The theorem is stated as: Let $(X, \mathcal{T})$ be a completely regular space. Then there exists a compactification $(Y, \mathcal{T}')$ of $(X, \mathcal{T})$ having the property that every bounded continuous function $f : X \to \mathbb{R}$ extends uniquely to a continuous function from Y into $\mathbb{R}$. First, we will discuss the existence of compactification, and after that, we will discuss the uniqueness of this extension.

In order to discuss the compactification $(Y, \mathcal{T}')$ of $(X, \mathcal{T})$, first we will discuss the existence of imbedding. How do we talk about the existence of imbedding? Why not use imbedding lemma. So, we are going to use the Imbedding lemma here, and after using this lemma, we will use the lemma discussed in the previous lecture. For the extension of f, let us begin with the proof of what we have with us, or let us take a family, that is, $\{f_i : i \in I\}$ of all bounded continuous real-valued functions on X. Now, what we can do that for all $i \in I$, let us choose a closed interval. Because we have already used this I for the index set, so, we are using a different notation for an interval. Take the interval as J_i containing $f_i(X)$, where $J_i = [inf f_i(X), sup f_i(X)]$. We can talk about it as we are talking about bounded continuous real value functions. Now, with this interval, let us define a function $h : X \to \prod_{i \in I} J_i$ such that $h(x) = (f_i(x))_{i \in I}$. It is to be noted here that when we are talking about $\prod_{i \in I} J_i$, note that every closed interval is compact; therefore, by using the Tychonoff Theorem, $\prod_{i \in I} J_i$ is also compact. As $(X, \mathcal{T})$ is a completely regular space, we can deduce that because this family $\{f_i : i \in I\}$, this will separate points from closed subsets of X and if this is the case, now we can use imbedding lemma, and by using imbedding lemma, we can conclude that h is an imbedding. Now, let us take this $(Y, \mathcal{T}')$; this is a compactification of topological space $(X, \mathcal{T})$ induced by the imbedding h.

If we are using a lemma that we have discussed in the previous lecture, we can say that there exists an imbedding $H : Y \to \prod_{i\in I} J_i$ such that $H(x) = h(x)$, $\forall x \in X$. Now, what have we assumed? We have a function $f : X \to \mathbb{R}$, it is to be noted that this is real-valued, bounded, and continuous. If this is the case, we can say that $f \in \{f_i\}_{i\in I}$, or $f = f_k$, for some $k \in I$. Now, we can talk about the projection map $\pi_k : \prod_{i\in I} J_i \to J_k$. It is to be noted that this projection map is always continuous. Thus, We can talk about a function $\pi_k \circ H : Y \to J_k$. It is to be noted that this H is continuous and π_k is also continuous; therefore, $\pi_k \circ H$ is a continuous function. Actually, this is precisely the extension which we require. Because for $x \in X$, $(\pi_k \circ H)(x) = (\pi_k \circ H)(x) = \pi_k(h(x)) = \pi_k((f_i(x))) = f_k(x)$. It is to be noted that $f_k = f$. Therefore, $(\pi_k \circ H)(x) = f(x)$. Thus, $\pi_k \circ H$ is a continuous function, which extends the function $f : X \to \mathbb{R}$, and it is to be noted that if such extension exists, what this $(Y, \mathcal{T}')$ is, this is precisely the Stone-Čech compactification of $(X, \mathcal{T})$. The standard notation is given as $\beta(X)$, and let us put the topology $\mathcal{T}'$ on it. As $(X, \mathcal{T})$ is denoted simply by X. So, people use $\beta(X)$ to denote the Stone-Čech compactification of a topological space $(X, \mathcal{T})$. Thus, we have shown the extension of the function f. Now, we have to show that such an extension is unique.

We will justify the uniqueness by using a result, stated as: Consider the topological spaces $(X, \mathcal{T})$ and $(Z, \mathcal{T}')$, where $(Z, \mathcal{T}')$ is Hausdorff. Also, let $A \subseteq X$ and $f : A \to Z$ be a continuous function. Then there is at most one extension of f to a continuous function $g : \bar{A} \to Z$. It is to be noted that uniqueness in our case will followed by taking $Z = \mathbb{R}$, and $A = X$. Now, let us prove it. If possible, assume that there are two different extensions $g, g' : \overline{A} \to Z$ of f. These are continuous functions, and these extend f. Now, as g, g' are different, there exists $x \in \overline{A}$ so that $g(x) \neq g'(x)$. But, it is to be noted that $g(x)$ and $g'(x)$ are elements of Z. What this Z is? This space is Hausdorff. Because $(Z, \mathcal{T}')$ is Hausdorff, there exist two open subsets, say, G and H of Z such that $g(x) \in G$, $g'(x) \in H$, and $G \cap H = \emptyset$. Now, let us take an open subset O containing x such that $g(O) \subseteq G$, and $g'(O) \subseteq H$. Can we take it? The answer is yes because g is continuous. Now, as O is an open set containing x and $x \in \overline{A}$. Therefore, $O \cap A \neq \emptyset$. If this is nonempty, let $y \in O \cap A$. Now, if $y \in O$, we can conclude that $g(y) \in G$ and $g'(y) \in H$. At the same time, it is to be noted that $y \in A$. Because $y \in A$, therefore we can say that $g(y) = f(y)$ as well as this $g'(y) = f(y)$, as g and g' are extensions of $f : A \to Z$. From here, we can conclude that $g(y) = g'(y)$. If $g(y) = g'(y)$, note that $g(y) \in G$, $g'(y) \in H$. So, $G \cap H \neq \emptyset$, contradicting the fact that G and H are disjoint, by using the Hausdorffness of the topological space $(Z, \mathcal{T}')$. Therefore, what we have assumed that there are two different continuous extensions, $g, g' : \overline{A} \to Z$, is not possible; that is, there will be at most one extension of f to a continuous function, that is, $g : \overline{A} \to Z$.

Now, let us see one of the properties of Stone-Čech compactification. What have we discussed? So, begin with a completely regular space $(X, \mathcal{T})$, and let us assume that $(Y, \mathcal{T}')$ is a compactification. As we have discussed in the previous theorem, we call it the Stone-Čech compactification. If this is the case, we can talk about the extension of the function with these assumptions. Now, what we can deduce is that if we have any continuous function $f : X \to C$, where $(C, \mathcal{T}'')$ is a compact Hausdorff space. The function f extends uniquely to a continuous function g from Y to compact Hausdorff space C. In order to prove it, let us begin with $(C, \mathcal{T}'')$, which is a compact Hausdorff space. We have already discussed that every compact Hausdorff space is normal, and every normal space is completely regular. So, $(C, \mathcal{T}'')$ is a completely regular topological space. If this is a completely regular topological space, we recall that this topological space can be imbedded in $[0, 1]^I$, for some index set I. So, why not let us take C as a subset $[0, 1]^I$? Now, note that $f : X \to C$ is a function, C is completely regular, which is a subset of $[0, 1]^I$. Now, if this is the case, what we can say is that each component, let us take f_i of this function f, is a bounded continuous real-valued function on X. So, by our assumption, from the previous theorem, the function f_i can be extended to a continuous function $g_i : Y \to \mathbb{R}$. It is to be noted that such extensions will always be unique. Now, having the function $g_i : Y \to \mathbb{R}$, corresponding to f_i, we can define a function $g : Y \to \mathbb{R}^I$ such that $g(y) = (g_i(y))_{i \in I}$. It is to be noted that $\mathbb{R}^I$ is endowed with the product topology. Therefore, we can conclude that the function g is a continuous function. Now, $g(Y) = g(\overline{X}) \subseteq \overline{g(X)} = \overline{f(X)} = \overline{C} = C$. Thus, $g : Y \to C$ is continuous, and its uniqueness follows from the uniqueness of g_i. Therefore, g is the required extension.

Moving ahead, if we are taking two such compactifications $(Y_1, \mathcal{T}_1)$, and $(Y_2, \mathcal{T}_2)$ of a completely regular topological space $(X, \mathcal{T})$. Then the question is: can we show that such compactifications are equivalent? The answer is yes. Just think about it.

These are the references.

That's all from this lecture. Thank you.