**Course Name: Essentials of Topology**
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Welcome to Lecture 71 on Essentials of Topology.

We have already studied the notion of compactification in Lectures 59 and 60, namely, one-point compactification. In this lecture, we will study the notion of a compactification, called the Stone-Čech compactification. Begin with what we have already studied that if we are having $(\mathbb{R}, \mathcal{T}_e)$, we know that if we are taking $(0, 1]$, this is not a compact subset of $\mathbb{R}$. But $(0, 1] \subset [0, 1]$, that is compact. It means that there exists a compact set containing semi-open interval $(0, 1]$, which is not compact. Specifically, we have seen that $[0, 1]$ is one-point compactification of $(0, 1]$. The question is, is it only the compact set containing this semi-open interval? The answer is no. We can construct a number of compact sets containing this semi-open interval $(0, 1]$. So, the question is, when we are discussing this Stone-Čech compactification, how is this concept going to differ from the concept of one-point compactification? The answer is: actually, whenever we are talking about one-point compactification, this is, in some sense, the minimal compactification. When we are talking about this Stone-Čech compactification, this one is the maximal.

Moving ahead, let us first look at the concept of compactification. So, what does the compactification of a topological space mean? A compactification of a topological space $(X, \mathcal{T})$ is a compact Hausdorff space $(Y, \mathcal{T}')$ with $(X, \mathcal{T})$ as a subspace such that $\bar{X} = Y$. In case, there is more than one compactification, can we say that these compactifications are equivalent? The answer is yes, and that is given here. Two compactifications $(Y_1, \mathcal{T}_1)$ and $(Y_2, \mathcal{T}_2)$ of a topological space $(X, \mathcal{T})$ are said to be equivalent if there is a homeomorphism $h : (Y_1, \mathcal{T}_1) \to (Y_2, \mathcal{T}_2)$ such that $h(x) = x$, for all $x \in X$. Now, the question is: if we are beginning with a topological space $(X, \mathcal{T})$ and this has a compactification, can we fix the nature of this space, or is there any well-known space in topology? The answer is yes, and that is known as completely regular spaces. So, before justifying that if a topological space has a compactification, then it will always be completely regular, let us see what this concept is.

We have already studied the concept of $T_4$-spaces as well as normal spaces. Let us have a look at that. We say that a topological space $(X, \mathcal{T})$ is $T_4$ if for each pair of disjoint closed $A, B \subseteq X$, there exist open sets $G, H \subseteq X$ such that $A \subseteq G$, $B \subseteq H$, and $G \cap H = \emptyset$. We have also seen that a normal space is nothing but a space that is $T_1$ as well as $T_4$. Further, we have seen that instead of open sets for the separation of disjoint closed sets, it can be achieved by using the concept of continuous functions, which is the well-known Urysohn lemma. The lemma is stated as: Let $(X, \mathcal{T})$ be a normal space and $A, B$ be a pair of disjoint closed subsets of $X$. Then there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$, for all $x \in A$; and $f(x) = 1$, for all $x \in B$. Before studying the concept of normal spaces, we have also studied the concept of regular spaces. We say that a topological space $(X, \mathcal{T})$ is regular if it is both $T_1$ as well as $T_3$, where a topological space $(X, \mathcal{T})$ is $T_3$ if for every pair consisting of a point $x \in X$ and a closed set, that is, $F \subseteq X$, which is not containing $x$, there exist open sets $G$ and $H$ such that $x \in G$, $F \subseteq H$, and $G \cap H = \emptyset$. Here we are discussing the separation of a point from a closed set by using open sets. A natural question arises, similar to the case of normal spaces by using the concept of continuous functions, can we separate a closed set from a point by using the concept of a continuous functions. The answer is given in terms of completely regular spaces. Now, let us see the concept of completely regular spaces. Similar to the concept of regular spaces and normal spaces, a topological space $(X, \mathcal{T})$ is completely regular if it is both $T_1$ and $T_{3\frac{1}{2}}$. What is $T_{3\frac{1}{2}}$-space? A topological space $(X, \mathcal{T})$ is $T_{3\frac{1}{2}}$ if for every pair consisting of a point $x \in X$ and a closed set $F \subseteq X$ with $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(F) = \{0\}$ and $f(x) = 1$.

As stated earlier, a completely regular space means that this is $T_1$ and $T_{3\frac{1}{2}}$. Now, we can conclude that every normal space $(X, \mathcal{T})$ is a completely regular space. The question is, how is it possible? The answer is given by Urysohn lemma and the fact that for a closed subset $F \subseteq X$ and $x \notin F$, $\{x\}$ is also closed. Why? Because $(X, \mathcal{T})$ is a $T_1$-space. Also, we can justify that completely regular spaces are regular. The question is, how is it possible? The answer is, if we are taking a completely regular space, this is $T_1$ and $T_{3\frac{1}{2}}$. Now, let us see the definition of $T_{3\frac{1}{2}}$-spaces; what we say that if we are taking a closed set $F$ and an element $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(F) = \{0\}$ and $f(x) = 1$. Now, if we are taking a set

$G = f^{-1}([0, \frac{1}{2}))$, then $F \subseteq G$. Also, let us take a set $H = f^{-1}((\frac{1}{2}, 1])$. Then $x \in H$. From the construction, it is clear that $G \cap H = \emptyset$. Also, $G$ and $H$ are open subsets of $X$. Why? Because $f$ is continuous, and $[0, \frac{1}{2})$ as well as $(\frac{1}{2}, 1]$, both are open subsets of $[0, 1]$. Thus, we can say that every completely regular space is a regular space.

Now, let us see a simple result of a completely regular space that we have to use. The result is: every subspace of a completely regular space is completely regular. In order to justify this result, let us take a completely regular space $(X, \mathcal{T})$ and $Y \subseteq X$. Our interest is to show that $(Y, \mathcal{T}_Y)$ is also completely regular. It is to be noted that if $(X, \mathcal{T})$ is $T_1$, then $(Y, \mathcal{T}_Y)$ will always be a $T_1$-space. So, what exactly do we have to justify that $(Y, \mathcal{T}_Y)$ is a $T_{3\frac{1}{2}}$-space. How do you justify it? Let us take a $\mathcal{T}_Y$-closed subset $F \subseteq Y$ and $y \notin F$. As $F$ is $\mathcal{T}_Y$-closed, so $Cl_Y(F) = F$. Note that $Cl_Y(F) = Y \cap Cl_X(F)$, so $F = Y \cap Cl_X(F)$. Now, $Cl_X(F) \subseteq X$ is closed and $y \notin Cl_X(F)$. As $(X, \mathcal{T})$ is a $T_{3\frac{1}{2}}$-space, there exists a continuous function $f : X \to [0, 1]$ such that $f(Cl_X(F)) = \{0\}$ and $f(y) = 1$. Now, let us take the restriction of this function $f$ on $Y$. Obviously, $f|_Y : Y \to [0, 1]$ is a continuous function. Also, $f|_Y(F) = \{0\}$ and $f|_Y(y) = 1$. Thus, $(Y, \mathcal{T}_Y)$ is a $T_{3\frac{1}{2}}$-space. Therefore, $(Y, \mathcal{T}_Y)$ is completely regular.

Having this idea of completely regular spaces in mind, let us move to the concept of compactification again. What we can show is that if any topological space is having a compactification, that space will always be completely regular. In order to justify it, let us take a topological space $(X, \mathcal{T})$, and let us take this space $(Y, \mathcal{T}')$; this is a compactification of topological space $(X, \mathcal{T})$. Now, if it is a compactification, we can say that $(Y, \mathcal{T}')$ is a compact Hausdorff space. We have already studied that every compact Hausdorff space is normal. So, $(Y, \mathcal{T}')$ is normal. Also, what we have studied is that every normal space is completely regular. Thus, $(Y, \mathcal{T}')$ is completely regular. At the same time, $(X, \mathcal{T})$ is a subspace of $(Y, \mathcal{T}')$. But we have studied the fact that a subspace of a completely regular space is completely regular. Therefore, $(X, \mathcal{T})$ is also completely regular. That's the justification of the statement that if $(X, \mathcal{T})$ has a compactification, then $(X, \mathcal{T})$ will always be a completely regular space. The question is, what about the converse of this statement? Whether every completely regular space has a compactification. The answer is yes, and we can see that. Before proving the statement, let us prove a lemma, and that lemma uses a concept, that is, the concept of imbedding.

We have already studied the concept of homeomorphism. From there, we can see the concept of imbedding. Actually, an imbedding, or topological imbedding of a topological space $(X, \mathcal{T})$ into in the topological space $(Y, \mathcal{T}')$, is a function $f : X \to Y$, which sends $X$ homeomorphically onto the subspace $f(X)$ (with relative topology) of $(Y, \mathcal{T}')$. With this concept of imbedding, let us see this lemma. The lemma states that for topological spaces $(X, \mathcal{T})$ and $(Z, \mathcal{T}'')$, where $(Z, \mathcal{T}'')$ is a compact Hausdorff space, let $h : (X, \mathcal{T}) \to (Z, \mathcal{T}'')$ be an imbedding. Then there exists a compactification $(Y, \mathcal{T}')$ of $(X, \mathcal{T})$ having the property that there is an imbedding $H : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$ that equals $h$ on $X$, and the compactification $(Y, \mathcal{T}')$ is uniquely determined up to equivalence. Before proving this lemma, let us use this terminology that $(Y, \mathcal{T}')$ is called compactification induced by imbedding $H$. Now, let us see the proof of this lemma.

Begin with what we have in our hand, that $h : (X, \mathcal{T}) \to (Z, \mathcal{T}'')$ is an imbedding. Note that $h(X) \subseteq Z$. Now, let $h(X) = X_0$ and $\overline{X_0} = Y_0$. Thus, $Y_0 \subseteq Z$. Even, we can talk about the relative topology on $Y_0$, and we can say that $(Y_0, \mathcal{T}''_{Y_0})$ is a subspace of $(Z, \mathcal{T}'')$. What is $(Z, \mathcal{T}'')$? It is to be noted that this is a compact Hausdorff space. Obviously, we can say that $(Y_0, \mathcal{T}''_{Y_0})$ is Hausdorff. Further, because $Y_0 = \overline{X_0}$, this is a closed subset of a Hausdorff space. So, we can also say that $(Y_0, \mathcal{T}''_{Y_0})$ is compact, too. Thus, $(Y_0, \mathcal{T}''_{Y_0})$ is compact Hausdorff, and what about the $\overline{X_0}$, that is $Y_0$. From here, what we can say is that $(Y_0, \mathcal{T}''_{Y_0})$ is a compactification of $X_0$ with the relative topology on it. Now, our motive is to construct this set $Y$. The question is how to construct it? The idea is, in order to construct $(Y, \mathcal{T}')$, what we are going to justify is that the pair $(X, Y)$ is homeomorphic to $(X_0, Y_0)$. In order to justify it, let us take a set $A$. How is this going to help us? We will see it, but what are the features of this $A$. The first one is, $A \cap X = \emptyset$, and the second one is, this $A$ is in the bijective correspondence with this difference, that is $Y_0 - X_0$. It means that there exists a bijective function, say $k : A \to Y_0 - X_0$. Now, let us take $Y = X \cup A$, and define a bijective function $H : Y \to Y_0$ such that

$$H(y) = \begin{cases} h(y), & y \in X, \\ k(y), & y \in A, \end{cases}$$

Also, let us define a topology $\mathcal{T}'$ on $Y$ as: We say that $G$ is a member of $\mathcal{T}'$ if and only if $H(G)$ is an open subset of $Y_0$. Now, from here, one thing is clear: this $H$ is a homeomorphism. As this map is bijective, we can see how this is defined in terms of $h$ as well as $k$. Also, the second thing is that $(X, \mathcal{T})$, this is a subspace of $(Y, \mathcal{T}')$. Why? Because what we can say is that this restriction of $H$ over $X$, that is nothing but $H|_X = h$. Thus, what we have in our hands is that $H : Y \to Y_0$ is a homeomorphism. Now, if we expand it, we can conclude that $H : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$ is an imbedding. That's the proof of the first part.

In order to prove the second one, let us take two spaces $(Y_1, \mathcal{T}_1)$ and $(Y_2, \mathcal{T}_2)$. What we are saying is that these two are compactifications of $(X, \mathcal{T})$. Now, let us define two maps: $H_1 : Y_1 \to Z$ and $H_2 : Y_2 \to Z$, these are imbedding, and these are extensions of $h$, too. Now, for $i = 1, 2$, $H_i(X) = h(X) = X_0$. The question is, can we justify from here that for $i = 1, 2$, if we are looking at the image of $Y$ under $H_i$, that is precisely $H_i(Y) = \overline{X_0}$. The answer is yes, and we can justify it. The question is how? It follows from the fact that $H_i$ is continuous. It is to be noted that $X_0 \subseteq H_i(Y)$. The question is whether $H_i(Y)$ is closed. The answer is yes. Why? Because it is compact. Why is it compact? Because this is a continuous image of a compact set, and compact subspaces of a Hausdorff space are closed. Therefore, $H_i(Y)$ is a closed set. Now, using the concept of closure, we can deduce that $H_i(Y) = \overline{X_0}$, for $i = 1, 2$. Now, $H_2^{-1} \circ H_1 : (Y_1, \mathcal{T}_1) \to (Y_2, \mathcal{T}_2)$ is a function such that for $x \in X$, $H_2^{-1} \circ H_1(x) = x$, that is, this is identity on $X$, and that's all to justify that this compactification $(Y, \mathcal{T}')$ is uniquely determined up to equivalence. Hence, we can conclude that the compactification is uniquely determined up to equivalence. That's the proof of this lemma. In the next lecture, we will see that every completely regular space has a compactification.

These are the references.

That's all from this lecture. Thank you.