Course Name: Essentials of Topology
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Welcome to Lecture 70 on Essentials of Topology.

In this lecture, we will study the formal proof of the Tychonoff theorem. Begin by proving a lemma first, which is based on the concept of Zorn's lemma. The lemma is stated as: Let X be a set and $\mathcal{F}$ be a collection of subsets of X with the finite intersection property. Then, there is a maximal family of subsets of X that contains $\mathcal{F}$ and has the finite intersection property. From the statement itself, it is clear that we are going to use Zorn's lemma in between. So, let $\mathcal{A}$ be the collection of all families of subsets of X, which contains $\mathcal{F}$ and also have FIP. Define a binary relation $\leq$ on $\mathcal{A}$ as: For $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{A}$, $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_1 \subseteq \mathcal{F}_2$. It is easy to see that $\leq$ is a partial order on $\mathcal{A}$. Now, we can justify that $\mathcal{A}$ has a maximal element. The question is, how do we justify it? The answer is, why not let us use Zorn's Lemma? If we are going to use the concept of Zorn's Lemma, let us take any linearly ordered subset and try to justify that every such subset has an upper bound. So, we are going to justify that every linearly ordered subset of $\mathcal{A}$ has an upper bound. Thus, let us take $\mathbb{D}$ as a linearly ordered subset of $\mathcal{A}$. In order to justify that $\mathbb{D}$ has an upper bound, let us take $\mathcal{C} = \bigcup\limits_{D \in \mathbb{D}} D$. The question is, can we justify that this is an upper bound of $\mathbb{D}$. The answer is yes. For which, what do we need to justify? Actually, we have to justify two points. The first one is, $\mathcal{C}$ contains $\mathcal{F}$, and the second one is that $\mathcal{C}$ has the finite intersection property. The question is how to justify that $\mathcal{C}$ contains $\mathcal{F}$. The answer is, it is to be noted that $D \in \mathbb{D}$ and every D is containing $\mathcal{F}$. If it is so, as $\mathcal{C}$ is a union of all such D's, we conclude that $\mathcal{C}$ contains $\mathcal{F}$. Now, if we want to justify that $\mathcal{C}$ has finite intersection property, what to do? Let us take an arbitrary family C_1, C_2, $\ldots$, C_n from $\mathcal{C}$. Now, as $\mathcal{C}$ is a union of such $D, D \in \mathbb{D}$, so we can make a guarantee that for all $1 \leq i \leq n$, there exists some $D_i \in \mathbb{D}$ such that $C_i \in D_i$, where $1 \leq i \leq n$. But $\mathbb{D}$ is linearly ordered, so there exists $D_k \in \mathbb{D}$, which will contain all such D_i, or that $C_1, C_2, ..., C_n \in D_k$. It is to be noted that D_k has the finite intersection property. Therefore, we can conclude that $C_1 \cap c_2 \cap ... \cap C_n \neq \emptyset$. Thus, we can conclude that $\mathcal{C}$ is an element of $\mathcal{A}$, and is an upper bound of $\mathbb{D}$. Thus, each and every linearly ordered subset of $\mathcal{A}$ has an upper bound. Thereby, by using Zorn's lemma, we can conclude that $\mathcal{A}$ has a maximal element.

Having this idea in mind, we are now ready to prove this Tychonoff theorem. So, we have already seen what the statement of this theorem is, that is an arbitrary product of compact topological spaces is compact in the product topology. Let us try to prove it. In order to prove it, why not let us begin with an arbitrary family of compact topological spaces $\{(X_i, \mathcal{T}_i) : i \in I\}$. Now, let $X = \prod_{i\in I} X_i$ and $\mathcal{T}_p$ be the product topology on X. In order to justify that $(X, \mathcal{T}_p)$ is compact, we are going to use the characterization of compactness in terms of finite intersection property. So, the idea is, if we are taking a family $\mathcal{F}$ of closed subsets of X, having finite intersection property, we have to show that $\bigcap\limits_{F\in\mathcal{F}} F \neq \emptyset$. Now, by using the previous lemma, we can conclude that there exists a maximal family, say $\mathcal{M}$ of subsets of X but note that these subsets are not necessarily closed, that contains $\mathcal{F}$ and has FIP. Now, if this is the case and because $\mathcal{M}$ is the maximal one, from the maximality of $\mathcal{M}$, we can conclude that $\bigcap\limits_{M\in\mathcal{M}} M \subseteq \bigcap\limits_{F\in\mathcal{F}} F$, or that $\overline{\bigcap\limits_{M\in\mathcal{M}} M} \subseteq \overline{\bigcap\limits_{F\in\mathcal{F}} F}$, or that $\bigcap\limits_{M\in\mathcal{M}} \overline{M} \subseteq \bigcap\limits_{F\in\mathcal{F}} F$. Now, if we can show that $\bigcap\limits_{M\in\mathcal{M}} \overline{M}$ is nonempty, then obviously $\bigcap\limits_{F\in\mathcal{F}} F$ will be nonempty. So, finally, what is our motive, or what do we want to justify? We want to justify that $\bigcap\limits_{M\in\mathcal{M}} \overline{M}$ is nonempty. For which, we have to show that there exists at least one element in $\bigcap\limits_{M\in\mathcal{M}} \overline{M}$. In order to find that element and before making some final conclusions, we are going to deduce two results, and by using these two, we can find the required element.

So, what exactly we are going to do is that first, we are making a claim that if we are taking a finite number of members $M_1, M_2, ..., M_n \in \mathcal{M}$, their intersection, that is, $M_1 \cap M_2 \cap ... \cap M_n \in \mathcal{M}$. So, how do you justify it? Let us take $M_1 \cap M_2 \cap ... \cap M_n = M$. Now, if $M \notin \mathcal{M}$, let $\mathcal{M}' = \mathcal{M} \cup \{M\}$. It is to be noted that $\mathcal{M} \subset \mathcal{M}'$. So, what features does this $\mathcal{M}'$ have? The first one is, $\mathcal{M}'$ contains $\mathcal{F}$, and the second one is, $\mathcal{M}'$ has FIP. How do you justify it? Why not let us take $M_1', M_2', ..., M_p' \in \mathcal{M}'$. Also, let none of these M_i's are equal to $\mathcal{M}$. Then, as $M_1', M_2', ..., M_p' \in \mathcal{M}$. $M_1' \cap M_2' \cap ... \cap M_p' \neq \emptyset$. Now, if any of M_i's is equal to M. Then, note that $M = M_1 \cap M_2 \cap ... \cap M_n$ and $M_1, M_2, ..., M_n$ are members of $\mathcal{M}$. As $\mathcal{M}$ has FIP, so in this case, too, their intersection will be nonempty, that is, $M'_1 \cap M'_2 \cap ... \cap M \cap .. \cap M'_p \neq \emptyset$. If this is nonempty, we can conclude the $\mathcal{M}'$ is containing $\mathcal{F}$ and having FIP. But, it is to be noted that $\mathcal{M}$ is maximal. Now, if $\mathcal{M}$ is maximal, by the maximality of $\mathcal{M}$, we conclude that $\mathcal{M} = \mathcal{M}'$, which is a contradiction because $\mathcal{M} \subset \mathcal{M}'$. Why does this problem arise? Because we have assumed that M is not a member of $\mathcal{M}$. Therefore, M is a member of $\mathcal{M}$, or $M = M_1 \cap M_2 \cap ... \cap M_n$ is an element of $\mathcal{M}$, where $M_1, M_2, ..., M_n$ are members of $\mathcal{M}$.

Now, let us make the second claim. What is it? This is, let $N \subseteq X$ such that it intersects nontrivially every member of $\mathcal{M}$. Then $N \in \mathcal{M}$. The question is, how do we justify it? Similar to the previous one, what will happen if $N \notin \mathcal{M}$? So, let us take $\mathcal{M}'' = \mathcal{M} \cup \{N\}$. Then, it is clear that $\mathcal{M} \subset \mathcal{M}''$. Now, what are the features of $\mathcal{M}''$? The first thing is, $\mathcal{M}''$ contains $\mathcal{F}$. The question is, whether $\mathcal{M}''$ has FIP. Let us try to justify it. For which, let us take $M''_1, M''_2, ..., M''_k \in \mathcal{M}''$. Now, there will be two cases. The first one is, if $M''_1, M''_2, ..., M''_k \in \mathcal{M}$, that is, none of these are equal to this N. Then, we can deduce that $M''_1 \cap M''_2 \cap ... \cap M''_k \neq \emptyset$. This is because $\mathcal{M}$ has FIP. Also, if any of $M''_1, M''_2, ..., M''_k$ is equal to N, what will happen? Even if we are taking $N \cap (M''_1 \cap M''_2 \cap, ..., \cap M''_k)$, and if we are replacing one of the elements by N, then $N \cap (M''_1 \cap M''_2 \cap, ..., \cap M''_k)$ will always be nonempty. Why? Because of our assumption that N intersects nontrivially with every member of $\mathcal{M}$. Therefore, $\mathcal{M}''$ has FIP. We have already seen that it contains $\mathcal{F}$. Also, it is to be noted that $\mathcal{M} \subset \mathcal{M}''$. But $\mathcal{M}$ is maximal. So, we again reach a contradiction because no superset of $\mathcal{M}$ can have FIP and contains $\mathcal{F}$. Therefore, we conclude that $N \in \mathcal{M}$.

With these two concepts, let us try to find an element of $\bigcap_{M \in \mathcal{M}} \overline{M}$. For which, we are going to take the help of projection maps $\pi_i : (X, \mathcal{T}_p) \to (X_i, \mathcal{T}_i)$, $i \in I$. We have already seen that the family which we have chosen, that is, $\mathcal{M}$, was a maximal family. This is a collection of subsets of X, and at the same time, it contains $\mathcal{F}$, and it also has FIP. Now, by using projection maps, let us choose a family of subsets $\{\pi_i(M) : M \in \mathcal{M}\}$ of X_i. It is to be noted that this family has FIP. But the question is, whether each $\pi_i(M)$ is closed. The answer is not necessarily. So, what to do? Why not let us take a family of subsets of X_i, given by $\{\overline{\pi_i(M)} : M \in \mathcal{M}\}$? Now, what about this family? We can say that this is obviously a family of closed subsets of X_i, and this family has FIP. Now, it is to be noted that this $(X_i, \mathcal{T}_i)$ is compact. So, let us use the compactness of $(X_i, \mathcal{T}_i)$, whereby $\bigcap_{M\in\mathcal{M}} \overline{\pi_i(M)} \neq \emptyset$. Thus, for each $i \in I$, we can find $x_i \in \bigcap_{M\in\mathcal{M}} \overline{\pi_i(M)}$. Now, let $x = (x_i)_{i\in I}$. It is clear that $x \in X = \prod_{i\in I} X_i$. The question is, whether x is an element of $\bigcap_{M\in\mathcal{M}} \overline{M}$. The answer is yes, and we can justify it. For justification, let us take an open subset G of X and $x \in G$. Then, there exists a basic open set, that is, $\bigcap_{i\in J} \pi_i^{-1}(G_i)$ such that $x \in \bigcap_{i\in J} \pi_i^{-1}(G_i) \subseteq G$.

It is to be noted here that $G_i \in \mathcal{T}_i$ and $x_i \in G_i$. Also, $J \subseteq I$, is finite. The question is, why are we choosing the basic open set in this particular form? The answer is because G_i is a member of $\mathcal{T}_i$, a collection of such elements forms a sub-basis for product topology, and if we are taking their finite intersection, that will be a member of the basis for product topology. Now, what can we conclude from here? It is to be noted that $x_i \in G_i$, and what we have already studied that x_i is an element of $\bigcap_{M\in\mathcal{M}} \overline{\pi_i(M)}$. Thus, $x_i \in \overline{\pi_i(M)}$. Therefore, by using the definition of closure, $G_i \cap \pi_i(M) \neq \emptyset$, and this is true for all $M \in \mathcal{M}$. From here, we can conclude that $\pi_i^{-1}(G_i) \cap M \neq \emptyset$, and this is true for all $M \in \mathcal{M}$. Thus, we can make a guarantee by the result which we have discussed. That is, $\pi_i^{-1}(G_i) \in \mathcal{M}$, for all $i \in J$. It is to be noted that this $\mathcal{M}$ has FIP. Therefore, the finite intersection of members of the form $\pi_i^{-1}(G_i)$ will always be nonempty. Thus, $(\bigcap_{i\in J} \pi_i^{-1}(G_i)) \cap M \neq \emptyset$, for all $M \in \mathcal{M}$, or that $G \cap M \neq \emptyset$, for all $M \in \mathcal{M}$. What G is? G is an arbitrary open subset of X. Thus, we can conclude that $x \in \overline{M}$. But note that $x \in \overline{M}$, for every $M \in \mathcal{M}$. Therefore, $x \in \bigcap_{M\in\mathcal{M}} \overline{M}$. Thus, we have shown that $\bigcap_{M\in\mathcal{M}} \overline{M} \neq \emptyset$. Also, if you recall what we have deduced that $\emptyset \neq \bigcap_{M\in\mathcal{M}} \overline{M} \subseteq \bigcap_{F\in\mathcal{F}} F$, therefore the conclusion, that is, $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$. Hence, the product space $(X, \mathcal{T}_p)$, that is, X with the product topology, this is compact. This is proof of the Tychonoff theorem.

These are the references.

That's all from this lecture. Thank you.