Course Name: Essentials of Topology
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Welcome to Lecture 7 on Essentials of Topology. In the previous lectures, we have recalled the concepts associated with sets, functions, and metric spaces, which we have to use in the next lectures. Even we require some more concepts, which we will recall as per their requirements. In this lecture, we will discuss the concept of topological spaces. The topics, which we will discuss are topology, open sets, as well as some examples of topology. Begin with the concept of open sets in a metric space.

We have seen in the last lecture that in a metric space (X, d),

- $\emptyset$ and X are open sets;
- finite intersection of open sets is open; and
- arbitrary union of open sets is open.

In the next, what we are going to do is, forget the structure d available with X. We will begin with a nonempty set. We will take a collection of some subsets of X, and after that, we will put some restrictions like these three and get the idea of topology.

So, let us begin with the formal definition of topology. Let X be a nonempty set and $\mathcal{T} \subseteq P(X)$. Then $\mathcal{T}$ is called a topology on X if

- $\emptyset, X \in \mathcal{T}$;
- $\mathcal{T}$ is closed under finite intersection, i.e.,

$$\text{if } G_1, G_2, ..., G_n \in \mathcal{T}, \text{ then } G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T};$$

- $\mathcal{T}$ is closed under arbitrary union, i.e.,

$$\text{if } G_i \in \mathcal{T} \text{ for } i \in I, \text{ then } \cup\{G_i : i \in I\} \in \mathcal{T}.$$

If $\mathcal{T}$ is a topology on a set X, then this pair $(X, \mathcal{T})$ is called a topological space. Furthermore, each G in $\mathcal{T}$ is called a $\mathcal{T}$-open set or simply an open set.

Let us take some of the examples of topology. The first one is, why not take the topology associated with metric spaces? As we have seen, the definition of topology itself is inspired by the concept of open sets in metric spaces. So, let (X, d) be a metric space and $\mathcal{T} = \{G \subseteq X : \forall x \in G, \exists r > 0 \,\text{such that}\, B(x, r) \subseteq G\}$. If we are looking at this definition, this can be rewritten as $\mathcal{T} = \{G \subseteq X : G \text{ is an open set in metric space } (X, d)\}$. As we have seen, the empty set and X are open sets in metric space (X, d), which means that the empty set and X are in $\mathcal{T}$. We also know that the finite intersection of open sets in a metric space (X, d) is open. So from here, we can conclude that if $G_1, G_2, ..., G_n \in \mathcal{T}$, then $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. We also have seen that an arbitrary union of open sets in (X, d) is open. So, we can conclude that if $G_i \in \mathcal{T}$ for $i \in I$, then $\cup\{G_i : i \in I\} \in \mathcal{T}$, or we can say that $\mathcal{T}$ is a topology on X. We call this topology the metric topology.

Let us discuss some more concepts on metric topology. We have seen that if we begin with a metric space (X, d), we can put a topology on X, called metric topology. We will use a fixed notation for it. We will denote a metric topology induced by metric d on X by $\mathcal{T}_d$. Second, we already know that an open ball is an open set. Therefore, we can conclude that the open ball centered at x with radius r always belongs to this $\mathcal{T}$, $x \in X$, and $r > 0$. We will discuss more concepts about metric topology, metric spaces, and topological spaces.

Moving ahead, let us take other examples, which are free from a metric. So, let us take $X = \{a, b\}$. We are defining two topologies on X.

- $\mathcal{T}_1 = \{\emptyset, X, \{a\}\}$, and
- $\mathcal{T}_2 = \{\emptyset, X, \{b\}\}$.

It can be seen that $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X. As an empty set and X, both are already members of $\mathcal{T}_1$ as well as $\mathcal{T}_2$. If we find their finite intersection, that will be an empty set, X, or singleton set $\{a\}$, in the first case. Also, that will be the empty set, X or singleton set $\{b\}$, in the second case and similarly for the union. That's why $\mathcal{T}_1$ and $\mathcal{T}_2$, both are topology on X. Now, if these are topology, the open sets with respect to the first topology are empty set,

X, and singleton set $\{a\}$. These are $\mathcal{T}_1$-open sets, or we can say that these are open sets with respect to topology $\mathcal{T}_1$. But note that the singleton set $\{b\}$ is also a subset of X; it is not a member of $\mathcal{T}_1$. So, singleton set $\{b\}$ is not $\mathcal{T}_1$-open. Similarly, in the second topology, the open sets are empty set, X and $\{b\}$. These are $\mathcal{T}_2$-open sets. Note that the singleton set $\{a\}$ is a subset of X, but the singleton set $\{a\}$ is not a member of $\mathcal{T}_2$. Therefore, $\{a\}$ is not $\mathcal{T}_2$-open.

Moving ahead, let us take some more examples. Begin with a nonempty set X, and let us assume $\mathcal{T} = \{\emptyset, X\}$. It can be seen that $\mathcal{T}$ is a topology on X, as this contains an empty set as well as X. Finite intersection and arbitrary union of members of $\mathcal{T}$ are obviously in $\mathcal{T}$. Therefore, this $\mathcal{T}$ is a topology on X. It has a special name, called indiscrete topology or trivial topology. Even, we cannot think about the topology, not containing the empty set as well as X. So, we can have one more conclusion that the indiscrete topology is the smallest topology.

Now, begin with a nonempty set X and $\mathcal{T} = \{G : G \subseteq X\}$. Precisely, this $\mathcal{T}$ is nothing but the power set of X. Then $\mathcal{T}$ is a topology on X. Note that $\emptyset, X \subseteq X$; therefore, the empty set and X are members of $\mathcal{T}$. If we are taking $G_1, G_2, ..., G_n \in \mathcal{T}$, it means that $G_1, G_2, ..., G_n \subseteq X$, and their intersection is also a subset of X. Therefore, $G_1 \cap G_2 \cap ... \cap G_n \in \mathcal{T}$. Finally, if we take a family of members G_i of $\mathcal{T}$ indexed by set I. Then note that $G_i \subseteq X$, for all $i \in I$. Therefore, $\cup\{G_i : i \in I\} \subseteq X$, or we can conclude that $\cup\{G_i : i \in I\} \in \mathcal{T}$. Therefore, $\mathcal{T}$ defined here is a topology on X. This topology is known as discrete topology. Note that we have taken $\mathcal{T}$ as a power set of X. Thus, we can conclude that $\mathcal{T}$ is the largest topology on X.

Moving ahead, let us take one more example. Let us take a set $X = \{a, b, c\}$. Note that the number of elements in X is 3. If we are talking about the number of elements in the power set of X, that will be 2^3, that is 8. The question is, how many topologies on X can be constructed? Two topologies are natural, that is, the indiscrete topology or trivial topology, and the second is discrete. It is to be noted that trivial topology is the smallest one, and discrete topology is the largest one. Can we construct some more topologies in between these two? The answer is yes.

For example, let us take one such topology, $\mathcal{T}_1$ as $\{\emptyset, X, \{a\}\}$. One can check that this is a topology. Let us take $\mathcal{T}_2$ as $\{\emptyset, X, \{a, b\}\}$. This is also a topology on X. Let us take $\mathcal{T}_3$ as $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. This is also a topology on X. So, in this way, we can construct a number of topologies on X. The number of such topologies on X is 29. Here, we have constructed only a few. One can construct the remaining topologies on X. Here, it is to be noted that if we are taking $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{b, c\}\}$, or we are taking $\mathcal{T} = \{\emptyset, X, \{a\}, \{b\}\}$. Note that neither the first nor the second is a topology on X. The question is, why? Here in the first case, $\{a, b\}$ is a member of $\mathcal{T}$, and $\{b, c\}$ is also a member of $\mathcal{T}$. But if we are finding $\{a, b\} \cap \{b, c\}$, which is a singleton set $\{b\}$, that is not in $\mathcal{T}$. Therefore, $\mathcal{T}$ is not a topology. Similarly, in the second case, the singleton set $\{a\}$ and the singleton set $\{b\}$ are in $\mathcal{T}$. But if we are looking for their union, which is a set $\{a, b\}$, that is not in $\mathcal{T}$. Therefore, this $\mathcal{T}$ is also not a topology.

Moving ahead, let us discuss more concepts about metric spaces and topological spaces. We are beginning with some of the examples. Let us take X as the set of integers with metric d as the discrete metric on $\mathbb{Z}$. This metric d is defined as:

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y. \end{cases}$$

If we are looking for an open ball centered at an integer x, and $r = 1/2$. Then we have already seen that this is $\{y \in \mathbb{Z} : d(x, y) < 1/2\}$, or this is simply $\{y \in \mathbb{Z} : d(x, y) = 0\}$. From the definition, this will be a singleton set $\{x\}$. Also, we have seen that open balls are always members of the topology. Thus, corresponding to this metric, if $\mathcal{T}_d$ is the topology, then this singleton set $\{x\}$ will always be a member of this $\mathcal{T}_d$, for all $x \in \mathbb{Z}$. We can conclude from here that this $\mathcal{T}_d$ is a discrete topology, which is possible because if singleton sets are members of topology, their union will also be a member of topology. So all two-point sets, three-point sets, and, in general, all subsets of X, or all subsets of $\mathbb{Z}$, will also be members of topology. Meaning is, if we are beginning with the discrete metric on $\mathbb{Z}$, the topology that we are getting on $\mathbb{Z}$, is a discrete topology.

Let us take another example on the same set, i.e, X is again the set of integers, but the metric which we are taking is absolute metric, or that, $d(x, y) = |x - y|$. Now, again, if we are looking for the open ball centered at x and radius $1/2$. This is a collection of all integers such that $d(x, y) < 1/2$, or this is a collection of all integers so that $|x-y| < 1/2$, or this is a collection of all integers y such that $y \in (x-1/2, x+1/2)$, that is nothing but singleton set $\{x\}$. With the same justification as in the previous example, what we have discussed in this example, the singleton set $\{x\}$ will belong to $\mathcal{T}_d$. If the singleton set is in $\mathcal{T}_d$, we can conclude that $\mathcal{T}_d$ is a discrete topology.

From these two examples, we can conclude that distinct metrics may induce the same topology. We have seen that corresponding to every metric on X, one can find a topology. But what about the converse? Whether corresponding to every topological space, there exists a metric. The answer is negative. Let us take an example, which is a simple one. Assume $X = \{a, b\}$ with indiscrete topology. We can justify that even indiscrete topology cannot be induced by any metric on X. If possible, let d be a metric on X inducing indiscrete topology $\mathcal{T}$. Then, we can show some contradiction. How is it possible? We have assumed that d is a metric, so let us take $d(a, b) = r, r > 0$. Now, we are constructing an open ball centered at a with radius $r/2$. This will be a collection given as $\{y \in X : d(a, y) < r/2\}$. But note that here is only two elements, a and b, with distance r. Thus, this set is precisely $\{y \in X : d(a, y) = 0\}$, or it will be nothing but a singleton set $\{a\}$. But as we already know that every open ball is an open set and that will always be a member of topology. Here, note that the singleton set $\{a\}$ is not a member of this $\mathcal{T}$. Therefore, no such metric is possible.

So, what we can conclude about metric spaces and topological spaces, the answer is given by this diagram. Let us take the first one as a collection of metric spaces and the second one as a collection of topological spaces. One thing is clear: corresponding to every metric, there exists a topology that we call metric topology. Also, corresponding to two metrics, there may exist only one topology, meaning is, two distinct metrics may induce the same topology. Finally, what we have seen is that, if we are taking a set X with an indiscrete topology $\mathcal{T}$, then corresponding to this topology, there doesn't exist any metric on X. Thus, we can conclude from here that if we are defining a function from a collection of metric spaces to a collection of topological spaces, then this function is neither one-one nor onto.

These are the references which we have used and will continuously use throughout the lectures. The book on Introduction to Topology, Pure and Applied by

Adams and Franzosa. The book on Topology by Munkers and the book on General Topology by Willard.

That's all from this lecture. Thank you.