Course Name: Essentials of Topology
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Welcome to Lecture 68 on Essentials of Topology.

In this lecture, we will discuss formal proof of the Tietze extension theorem, for which we have already made a foundation in the previous lecture. Begin with the statement of the Tietze extension theorem, which we have already seen in the previous lecture. The statement was: Let $(X, \mathcal{T})$ be a normal space, $A \subseteq X$ be closed, and $f : A \to [a, b]$ be a continuous function. Then there exists a continuous function $g : X \to [a, b]$ such that $g(x) = f(x)$, for all $x \in A$. To prove this theorem, we have already seen a lemma in the previous lecture that we will use here. But before starting from the lemma, what exactly will we do? Instead of proving this theorem for a general $[a, b]$, we will take the closed interval as $[-1, 1]$. There is no harm in doing so, as we know that closed intervals are homeomorphic.

Now, let us recall the lemma which we have seen in the previous lecture. We have seen that if $(X, \mathcal{T})$ is a normal space and $A \subseteq X$, which is closed too, and if we are taking a function, that is $f : A \to [-r, r]$, also if this function is continuous, then there exists a continuous function, let us take that continuous function as $g : X \to \mathbb{R}$ such that $|g(x)| \leq (1/3)r$, for all $x \in X$ and $|f(x)-g(x)| \leq (2/3)r$, for all $x \in A$. Just to make a difference that this x is an element of A and x is an element of X, we are writing $|f(a) - g(a)| \leq (2/3)r$, for all $a \in A$. Now, let us use this lemma to prove the Tietze extension theorem. We are going to take a function $f : A \to [-1, 1]$. Now, we have to construct a function g by using the lemma. It is trivial or what we can do that why not let us take $r = 1$ in the lemma. So, there exists a continuous function, say $g_1 : X \to \mathbb{R}$ such that $|g_1(x)| \leq 1/3$, for all $x \in X$, and $|f(a) - g_1(a)| \leq 2/3$, for all $a \in A$. Now, as $|f(a) - g_1(a)| \leq 2/3$, define a new function $f - g_1 : A \to [-2/3, 2/3]$. As f and g_1, both are continuous, therefore, $f - g_1$ is continuous, too. If $f - g_1$ is a continuous function, we can again use the lemma for this function, too.

Now, if we are using the lemma for $f - g_1$, it is to be noted that $r = 2/3$. If $r = 2/3$ and we are using the above lemma, we can conclude that there exists a continuous function, say $g_2 : X \to \mathbb{R}$ such that $|g_2(x)| \leq (1/3)r = (1/3)(2/3)$, for all $x \in X$, and $|f(a) - g_1(a) - g_2(a)| \leq (2/3)r = (2/3)(2/3)$, for all $a \in A$. Repeat the process that we have carried out. What we can see is $f - g_1 - g_2 : A \to [-(2/3)^2, (2/3)^2]$ is again a continuous function. So, again, we can use the lemma, and by using that lemma there exists continuous functions g_1, g_2, ..., g_n such that $|f(a) - g_1(a) - g_2(a) - ... - g_n(a)| \leq (2/3)^n$, where $f - g_1 - g_2 - ... - g_n : A \to [-(2/3)^n, (2/3)^n]$ is a continuous function. Inductively, we can say that there exists a continuous function $g_{n+1} : X \to \mathbb{R}$ such that $|g_{n+1}(x)| \leq (1/3)(2/3)^n$, and $|f(a) - g_1(a) - g_2(a) - ... - g_{n+1}(a)| \leq (2/3)^{n+1}$. That is, we can construct a function g_n; this g_n is a continuous function for all $n \in \mathbb{N}$.

Now, our motive is to construct a function $g : X \to [-1, 1]$. The question is, how do we construct this g? The answer is, let us take a function $g : X \to \mathbb{R}$ such that $g(x) = \sum_{n=1}^{\infty} g_n(x)$. The question is, how can we justify that g is continuous, $g(x) = f(x)$, for all $x \in A$; and $g : X \to [-1, 1]$. Let us discuss one by one. In order to justify the continuity of g, we are going to use the Weierstrass M-test, which we have also recalled in the previous lecture. What exactly have we recalled there? Given a sequence of functions $f_n : X \to \mathbb{R}$, let $s_n(x) = \Sigma_{i=1}^{n} f_i(x)$. If $|f_i(x)| \leq M_i$, for all $x \in X$ and for all i, and if the series ΣM_i converges, then the sequence (s_n) converges uniformly to a function s.

Now, if we want to use this test in our case, what we have seen is that the function that we have constructed, that is, $g_i : X \to \mathbb{R}$ satisfying the property that $|g_i(x)| \leq (1/3)(2/3)^{i-1}$. Why not let us take $(1/3)(2/3)^{i-1} = M_i$. We know that this series converges. So, we can use this Weierstrass M-test, and by using this test, we can conclude that the sequence (s_n), where $s_n = \sum_{i=1}^{n} g_i(x)$, converges uniformly. Also, the question is, what about the limit? The answer is, this is nothing but g. Further, what can we say about (s_n)? Note that this is a sequence of functions. Further, it is to be noted that each g_i is a continuous function. So, (s_n) is a sequence of continuous functions. If so, what is the limit? The limit is nothing but g. So, g is also a continuous function. We can also justify the continuity of g in a different way. What we can say is that if we are taking a natural number k, $k > n$. Then $|s_k(x) - s_n(x)| =$

$|\sum_{i=n+1}^{k} g_i(x)| \leq (1/3) \sum_{i=n+1}^{k} (2/3)^{i-1} < (1/3) \sum_{i=n+1}^{\infty} (2/3)^{i-1} = (2/3)^n$. Now, for fixed n, let us take that $k \to \infty$. Then, $s_n \to g$ uniformly. Further, as we have already discussed that the sequence (s_n) is a sequence of continuous functions; by using the uniform limit theorem, it makes a guarantee that g will always be a continuous function.

Moving ahead, let us show that $g(a) = f(a)$, for all $a \in A$. We have already seen that $|f(a) - g_1(a) - g_2(a) - ... - g_n(a)| \leq (2/3)^n$, or $|f(a) - \sum_{i=1}^{n} g_i(a)| \leq (2/3)^n$, i.e., $|f(a) - s_n| \leq (2/3)^n$. Now, as $n \to \infty$, $g(a) = \lim_{n\to\infty} s_n(a) = f(a)$. Thus, for all $a \in A$, $g(a) = f(a)$. One more thing we need to justify is that the codomain of g is $[-1, 1]$. This can be seen easily. Let us try to find the modulus value of $g(x)$. Now, $|g(x)| = |\sum_{i=1}^{\infty} g_n(x)| = |\lim_{n\to\infty} \sum_{i=1}^{n} g_n(x)| = \lim_{n\to\infty} |\sum_{i=1}^{n} g_n(x)| \leq \lim_{n\to\infty} \sum_{i=1}^{n} |g_n(x)| \leq \lim_{n\to\infty} \sum_{i=1}^{n}(1/3)(2/3)^{n-1} = 1$. Thus, $|g(x)| \leq 1$, or we can claim that $g : X \to [-1, 1]$ is a function. Hence, the proof of this beautiful theorem, known as the Tietze extension theorem.

There is another version of this theorem that can be derived from the previous one. The statement of this version is: Let $(X, \mathcal{T})$ be a normal space, $A \subseteq X$ be closed, and $f : A \to \mathbb{R}$ be a continuous function. Then there exists a continuous function $g : X \to \mathbb{R}$ such that $g(x) = f(x)$, for all $x \in A$. In order to prove it, we are going to take the function $f : A \to (-1, 1)$, and we will show the existence of a continuous function $g : A \to (-1, 1)$. Similar to the previous version of this theorem, as $\mathbb{R}$ is homeomorphic to $(-1, 1)$, the rest of the proof is obvious. So, let us try to prove the theorem.

Begin with the continuous function $f : A \to (-1, 1) \subseteq [-1, 1]$. If $f : A \to [-1, 1]$ is a continuous function, by using the previous version of this theorem, there exists a continuous function $h : X \to [-1, 1]$ such that $f(a) = h(a)$, for all $a \in A$. But what is our motive? Our motive is to construct the continuous function $h : X \to (-1, 1)$. So, what is the problem here? The problem is created by the end points, that is -1 and 1. How do we handle this problem? The answer is, why not let us take $B = h^{-1}(\{-1, 1\})$. It is to be noted that B is a closed subset of X. The question is, why? Because $\{-1, 1\}$ is a closed set in $[-1, 1]$ and h is continuous. Further, what we can say that A and B are disjoint sets. Thus, we have two disjoint closed sets with us and if they are disjoint and closed, we can use the Urysohn lemma. Accordingly, there exists a continuous function $h' : X \to [0, 1]$ such that $h'(b) = 0$, for all $b \in B$ and $h'(a) = 1$, for all $a \in A$. With these two functions, that is, h' as well as h, let us define a new function that will serve our purpose. So, we are defining a function $g : X \to \mathbb{R}$ such that $g(x) = h(x).h'(x)$. Now, if we are taking $a \in A$, then $g(a) = h(a).h'(a) = f(a).1 = f(a)$. So, we have shown that $g(x) = f(x)$, for all $x \in A$. But the problem is, whether the codomain is $(-1, 1)$. Let us see it. If $x \in B$, as A and B are disjoint, x cannot be an element of A. Also, what we have seen is that if $b \in B$, $h'(b) = 0$. From here, we can conclude that $g(x) = h(x).h'(x) = 0$. Also, if x is not an element of B, then $h(x) \neq 1$. Even $h(x) \neq -1$, that is, $-1 < h(x) < 1$. Further, we have already seen the nature of h'. Note that $h' : X \to [0, 1]$. So, what we can conclude from here is that if x is not an element of B, in this case, $g(x)$ will always lie between -1 and 1. Hence, $g : X \to (-1, 1)$ such that $g(x) = f(x)$, for all $x \in A$. That's the proof of another version of the Tietze extension theorem.

These are the references.

That's all from this lecture. Thank you.