Course Name: Essentials of Topology
Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
Institute Name: Indian Institute of Technology(ISM), Dhanbad
Week: 11
Lecture: 06

Welcome to Lecture 66 on Essentials of Topology.

Continuing with the concept of separation axioms, in this lecture, we will discuss the formal proof of the well-known Urysohn lemma. In the last lecture, we have already seen the statement of this beautiful result. The statement is: Let $(X, \mathcal{T})$ be a normal space and A, B be a pair of disjoint closed subsets of X. Then there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$, for all $x \in A$; and $f(x) = 1$, for all $x \in B$. In order to prove it, what we need to justify is the existence of a continuous function that satisfies two conditions. In order to justify the existence of function, we will use a result on normal spaces discussed in the previous lecture, along with the theory of dyadic rationals.

Let us begin with a normal space $(X, \mathcal{T})$ and a pair of disjoint closed subsets A, B of X. As A and B are disjoint, $A \subseteq B^c$. It is to be noted that this A is closed, and B^c is an open subset of X. So, why not let us take $B^c = G_1$? Also, let us use the result on normal spaces, which we have discussed in the previous lecture, that is, there exists an open set G_0 such that $A \subseteq G_0 \subseteq \overline{G_0} \subseteq G_1$. Again, it is to be noted that $\overline{G_0}$ is a closed set, and G_1 is an open subset of X. So, again, if we are applying the same result, we can deduce that there exists another open set $G_{1/2}$ such that $\overline{G_0} \subseteq G_{1/2} \subseteq \overline{G_{1/2}} \subseteq G_1$. What exactly are we doing? Note that we are choosing the suffix of open sets from the interval itself; that is, some real numbers between 0 and 1. Continuing the same process, there exists open sets $G_{1/4}$ and $G_{3/4}$ such that $\overline{G_0} \subseteq G_{1/4} \subseteq \overline{G_{1/4}} \subseteq G_{1/2} \subseteq \overline{G_{1/2}} \subseteq G_{3/4} \subseteq \overline{G_{3/4}} \subseteq G_1$. We thus define open sets G_r for each $r \in \mathcal{D}$ such that $\overline{G_s} \subseteq G_t$ if $s < t, s, t \in \mathcal{D}$.

Having this idea in mind, let us try to construct a function. We have seen a glimpse of this in the previous lecture, too. So, we are defining a function

$f : X \to [0, 1]$ such that

$$f(x) = \begin{cases} inf\{r \in D : x \in G_r\}, & if x \in G_1, \\ 1, & otherwise, \end{cases}$$

Now, for $x \in A$, $f(x) = inf\{r \in \mathcal{D} : x \in G_r\}$, because if we are taking $x \in A$, note that how the sequence is given, that is, $A \subseteq G_0 \subseteq \overline{G_0} \subseteq G_1$. So, x will always be an element of G_1. Thus, if we are taking the infimum of all such $r \in D$, that is obviously 0. So, for all $x \in A$, $f(x) = 0$. Also, for $x \in B$, note that x cannot be an element of B^c. What is B^c? Note that $B^c = G_1$. It means that x is not an element of G_1, and if $x \notin G_1$, $f(x) = 1$. Thus, what we can say that for all $x \in B$, $f(x) = 1$.

The following points need to be noted here.

- If $f(x) < r$. Then $inf\{t \in D : x \in G_t\} < r$. Thus, $x \in G_r$.
- If $x \notin G_r$. Then $f(x) \geq r$.
- If $x \in G_r$. Then $f(x) \leq r$.
- If $f(x) > r$. Then $x \notin G_r$, or $x \in X - G_r$.

So, these are some of the observations from the definition of function. We will use these observations to justify that the function we have constructed is continuous.

Now, we have to justify that the function $f : X \to [0, 1]$. Note that $[0, 1] \subseteq \mathbb{R}$, and this $\mathbb{R}$ is endowed with Euclidean topology. So, in the subspace topology, if we want to see the elements of sub-basis for the relative topology on $[0, 1]$, that will be of the form, $[0, a)$, or $(b, 1]$, where $0 < a, b < 1$. Now, if we can justify that the inverse images of these sets are open subsets of X, then obviously, f is continuous. So, what precisely are we going to justify? We are going to justify that for all such a and b in between 0 and 1; their inverse images will always be $\mathcal{T}$-open. So, let us take the semi-open interval $[0, a)$, and instead of taking it, why not let us take the intervals of the form $[0, s)$, where $s \in \mathcal{D}$. Now, we van justify that $f^{-1}([0, s)) = \cup\{G_r : r < s, r \in \mathcal{D}\}$. In order to justify it, let $x \in f^{-1}([0, s))$. Then $f(x) < s$. As $\mathcal{D}$ is dense in $[0, 1]$, there exists $r \in \mathcal{D}$ such that $f(x) < r < s$. Now, if $f(x) < r$, then $x \in G_r$, or that $x \in \cup\{G_r : r < s, r \in \mathcal{D}\}$, that is, $f^{-1}([0, s)) \subseteq \cup\{G_r : r < s, r \in \mathcal{D}\}$.

Further, let $x \in \cup\{G_r : r < s, r \in \mathcal{D}\}$. Then there exists $t \in \mathcal{D}$ such that $x \in G_t, t < s$. Now, if $x \in G_t$, $f(x) \leq t$, or that $f(x) < s$. If $f(x) < s$, then $x \in f^{-1}([0, s))$. Thus $\cup\{G_r : r < s, r \in \mathcal{D}\} \subseteq f^{-1}([0, s))$. Therefore, $f^{-1}([0, s))$ is a union of $\mathcal{T}$-open sets. As we know that the arbitrary union of open sets is open. Thus, we can conclude that $f^{-1}([0, s))$ is open subset of X.

What remains to justify that inverse image of members of the form $(b, 1]$, are also $\mathcal{T}$-open. We will justify it in two steps. So, first, we are going to justify that $f^{-1}((s, 1]) = \cup\{X - G_r : r > s, r \in \mathcal{D}\}$. In order to justify it, let $x \in f^{-1}((s, 1])$. Then $f(x) > s$. By using the denseness of $\mathcal{D}$, there exists $r \in \mathcal{D}$ such that $f(x) > r$, and $r > s$. Now, as $f(x) > r$, $x \notin G_r$, or $x \in X - G_r$. It is to be noted that $r > s$. Thus, $x \in \cup\{X - G_r : r > s, r \in \mathcal{D}\}$, whereby $f^{-1}((s, 1]) \subseteq \cup\{X - G_r : r > s, r \in \mathcal{D}\}$. For the converse part, let $x \in \cup\{X - G_r : r > s, r \in \mathcal{D}\}$. Then $x \in X - G_t$ or $x \notin G_t$, where $t > s$ and $t \in \mathcal{D}$. Note that if $x \notin G_t$, then $f(x) \geq t$. At the same time, it is to be noted that $t > s$. So, $f(x) \geq t > s$, that is, $f(x) > s$; and if $f(x) > s$, we can conclude that $x \in f^{-1}((s, 1])$. Thus, $\cup\{X - G_r : r > s, r \in \mathcal{D}\} \subseteq f^{-1}((s, 1])$, or that $f^{-1}((s, 1]) = \cup\{X - G_r : r > s, r \in \mathcal{D}\}$. But the problem is here. By using $f^{-1}((s, 1]) = \cup\{X - G_r : r > s, r \in \mathcal{D}\}$, can we justify that $f^{-1}((s, 1])$ is open? It is a difficult job for us. So, we are going to use this result, and we justify that $f^{-1}((s, 1]) = \cup\{X - \overline{G_r} : r > s, r \in \mathcal{D}\}$.

In order to prove $f^{-1}((s, 1]) = \bigcup\{X - \overline{G_r} : r > s, r \in D\}$, let $x \in f^{-1}((s, 1])$. It is to be noted that $f^{-1}((s, 1]) = \cup\{X - G_r : r > s, r \in \mathcal{D}\}$. Thus, we can say that $x \in X - G_t$, for some $t \in \mathcal{D}$ and $t > s$. Now, by using the denseness of $\mathcal{D}$, there exists $r \in \mathcal{D}$ such that $t > r > s$. Just recall what we have already justified that if $s < t$, $\overline{G_s} \subseteq G_t$. If we are using this fact, we can say that $\overline{G_r} \subseteq G_t$, or $X - \overline{G_r} \supseteq X - G_t$, as $t > r$. It is to be noted that $x \in X - G_t$. Therefore, $x \in X - \overline{G_r}$. It is to be noted that $r > s$. Thus, $x \in \cup\{X - \overline{G_r} : r > s, r \in \mathcal{D}\}$. Therefore, $f^{-1}((s, 1]) \subseteq \cup\{X - \overline{G_r} : r > s, r \in \mathcal{D}\}$. For the converse part, let $x \in \cup\{X - \overline{G_r} : r > s, r \in \mathcal{D}\}$. Then $x \in X - \overline{G_t}$, for some $t \in \mathcal{D}$, where $t > s$, or we can say that $x \notin \overline{G_t}$. If $x \notin \overline{G_t}$, then $x \notin G_t$, and if $x \notin G_t$, then $f(x) \geq t$. It is to be noted that $t > s$. So, $f(x) > s$, or $x \in f^{-1}((s, 1])$. Thus, $\cup\{X - \overline{G_r} : r > s, r \in \mathcal{D}\} \subseteq f^{-1}((s, 1])\}$. Hence, $f^{-1}((s, 1]) = \bigcup\{X - \overline{G_r} : r > s, r \in D\}$. Now, it is to be noted that $\overline{G_r}$ is a closed subset of X. Therefore, $X - \overline{G_r}$ is an open subset of X, or $f^{-1}((s, 1])$ is an open subset of X. Thus, $f^{-1}([0, s)])$ and $f^{-1}((s, 1])$ are $\mathcal{T}$-open sets. If these are $\mathcal{T}$-open sets, we can conclude that f is a continuous function. We have already justified that $f(x) = 0$, for all $x \in A$. Also, we had justified that $f(x) = 1$, for all $x \in B$. That's the proof of this well-known Urysohn lemma.

It is to be noted that in the statement of Urysohn lemma, we have taken the topological space $(X, \mathcal{T})$ as a normal space, i.e, $(X, \mathcal{T})$ is both T_1 and T_4. While in the proof of lemma, we have used the concepts associated with T_4-spaces. So, we can say that Urysohn lemma also holds for T_4-spaces, because in the proof, there is no use of the concepts associated with T_1-spaces. Further, one can construct a continuous function $\phi : X \to [a, b]$. How is it possible? The answer is, because what we have seen is that if we have a normal space $(X, \mathcal{T})$, and let us take two disjoint closed subsets $A, B \subseteq X$, we have already seen that we can construct a function, that is, a continuous function $f : X \to [0, 1]$. Now, let us take a function $h : [0, 1] \to [a, b]$ such that for all $t \in [0, 1]$, $h(t) = a + (b - a)t$. Note that we have already seen that this h is a homeomorphism. Therefore, it is continuous too. Thus, if we are taking the composition of these two functions, that is, $\phi = h \circ f$, note that $\phi : X \to [a, b]$ is a function; this is continuous. Now, for all $x \in A$, $\phi(x) = (h \circ f)(x) = h(f(x)) = h(0) = a$. Similarly, for all $x \in B$, $\phi(x) = b$. That is, one can talk about the existence of a continuous function $\phi : X \to [a, b]$, and it maps each element of the set A to a and each element of B to b.

These are the references.

That's all from this lecture. Thank you.