Course Name: Essentials of Topology
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Welcome to Lecture 65 on Essentials of Topology.

In this lecture, we will discuss the well-known Urysohn lemma and will establish a foundation for its proof. In the previous lecture, we studied the concept of normal spaces and went through a number of examples. What the concept of normal spaces is, we have seen that a topological space is normal if it is both T_1 and T_4-space. Coming to the concept of T_4-spaces. We say that the space $(X, \mathcal{T})$ is T_4 if for two disjoint closed subsets A and B of X, there exist open sets, say G and H such that $A \subseteq G$, $B \subseteq H$, and $G \cap H = \emptyset$. The question is, can we construct such open sets by using the concept of continuous functions? The answer is yes, and that is stated here. The statement is: Let $(X, \mathcal{T})$ be a topological space, A, B be a pair of disjoint closed subsets of X and $f : X \to [0, 1]$ be a continuous function such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then there exist open sets $G, H \subseteq X$ such that $A \subseteq G$, $B \subseteq H$ and $G \cap H = \emptyset$. The question is, how is it possible? Let us see it.

As $f(A) = \{0\}$, so $A \subseteq f^{-1}(\{0\})$. Similarly, as $f(B) = \{1\}$, we can conclude that $B \subseteq f^{-1}(\{1\})$. It is to be noted that $[0, 1] \subset \mathbb{R}$, and the topology on $\mathbb{R}$ is the standard topology. Thus, we can talk about the relative topology on this $[0, 1]$. Now, let us take two open subsets of $[0, 1]$ as $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$. Then, as $\{0\} \subseteq [0, \frac{1}{2})$, $f^{-1}(\{0\}) \subseteq f^{-1}([0, \frac{1}{2}))$. Similarly, as $\{1\} \subseteq (\frac{1}{2}, 1]$, $f^{-1}(\{1\}) \subseteq f^{-1}((\frac{1}{2}, 1])$. If $G = f^{-1}([0, \frac{1}{2}))$ and $H = f^{-1}((\frac{1}{2}, 1])$. Then, by the continuity of f, G and H are open subsets of X. Also, $A \subseteq G$, $B \subseteq H$ and it is clear that $G \cap H = \emptyset$. Thus, we found a pair of disjoint open sets, one containing the closed set A, and another containing the closed set B. The question is: what about its converse? That is, if we are taking a T_4-space, or in general, let us take a normal space, can we construct a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$, and $f(B) = \{1\}$, this concept is known as the well-known Urysohn lemma.

The lemma is stated as: Let $(X, \mathcal{T})$ be a normal space and A, B be a pair

of disjoint closed subsets of X. Then there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$, for all $x \in A$; and $f(x) = 1$, for all $x \in B$. Before discussing the proof of this lemma, let us take an example of such continuous functions in the case of a metrizable space $(X, \mathcal{T}_d)$, where d is metric on X. We have shown that $(X, \mathcal{T}_d)$ is a normal space. Now, let us take two disjoint closed subsets, A and B of X. Note that $\overline{A} = A$ and $\overline{B} = B$. Our motive is to construct a continuous function $f : X \to [0,1]$ such that $f(x) = 0$, for all $x \in A$; and $f(x) = 1$, for all $x \in B$. In order to construct such a function, we will use two results. One we have already seen, that is, for a metric space (X, d), if take $A \subseteq X$ and define a function $f : X \to \mathbb{R}$ such that $f(x) = d(x, A)$, $\forall x \in X$. Then, this function is always a continuous function. Also, from the concept of metric spaces, we know that $d(x, A) = 0$ if and only if $x \in \overline{A}$. As in our case $\overline{A} = A$ and $\overline{B} = B$, whereby, we can always say that if $x \in A$, $d(x, A) = 0$, and similarly, if $x \in B$, $d(x, B) = 0$.

Now, let us define a function $h : X \to [0,1]$ such that $h(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$. It is to be noted that $0 \le h(x) \le 1$. The question is, why? Because d is a metric. So, $\frac{d(x,A)}{d(x,A)+d(x,B)}$ will always be greater than or equal to 0. Further, $d(x, A) + d(x, B) \ge d(x, A)$, therefore $\frac{d(x,A)}{d(x,A)+d(x,B)} \le 1$. Next, we want to show that h is continuous. What exactly do we require? Note that h will always be a continuous function, provided $d(x, A) + d(x, B) \neq 0$. The question is: how do we justify it? The answer is here.

- If $x \in A$, then $x \notin B$, as A and B are disjoint. Therefore, $d(x, B) > 0$, or that $d(x, A) + d(x, B) > 0$.

- If $x \in B$, then $x \notin A$, as A and B are disjoint. Therefore, $d(x, A) > 0$, or that $d(x, A) + d(x, B) > 0$.

- If $x \notin A$ and $x \notin B$. Then $d(x, A) > 0$ and $d(x, B) > 0$, or that $d(x, A) + d(x, B) > 0$.

Thus, $d(x, A) + d(x, B) \neq 0$. Hence, the function h is continuous.

Finally, for $x \in A$, $h(x) = 0$, and for $x \in B$, $h(x) = \frac{d(x,A)}{d(x,A)+0} = 1$. Thus, by using the normality, we can construct a function h satisfying $h(x) = 0$, for all $x \in A$, and $h(x) = 1$, for all $x \in B$. When we want to prove this result for a general topological space, we require some more concepts.

Let us see the first result, which we will use to prove Urysohn's lemma. This result is stated as: Let $(X, \mathcal{T})$ be a normal space. Then, for each closed subset $F \subseteq X$ and for each open set G containing F, there exists an open set H such that $F \subseteq H \subseteq \overline{H} \subseteq G$. In order to prove it, let us begin with $F \subseteq G$. Then $F \cap G^c = \emptyset$. Now, use the normality of the space $(X, \mathcal{T})$. What we can say is that there exist two open sets, say H and H' such that $F \subseteq H$, $G^c \subseteq H'$, and $H \cap H' = \emptyset$. Now, as $H \cap H' = \emptyset$, we can conclude that $H \subseteq H'^c$, or that $\overline{H} \subseteq H'^c$, as H'^c is a closed set. Now, we can conclude that $F \subseteq H$, H is always a subset of its closure, and the closure of H is a subset of H'^c. But, it is to be noted that $H'^c \subseteq G$. Thus, $F \subseteq H \subseteq \overline{H} \subseteq G$. That was our requirement to justify this result. We will use this result to prove Urysohn lemma.

Now, the next concept we will use is the concept of dyadic rationals. Note that our motive is to construct a continuous function from $X \to [0, 1]$. So, what are we going to discuss? We are going to discuss about the rationals in $[0, 1]$. Let us take the rational numbers of the form $\frac{m}{2^n}$, where $0 \leq m \leq 2^n$. The collection of such such rational numbers, we will denote by $\mathcal{D}$, and these rationals are known as dyadic rationals. Now, let us see that how these rationals will look like.

- For $n = 1$, note that $0 \leq m \leq 2$. In this case $\frac{m}{2^n}$ will be $0, 1/2, 1$.

- For $n = 2$, note that $0 \leq m \leq 2^2 = 4$. In this case $\frac{m}{2^n}$ will be $0, 1/4, 1/2, 3/4, 1$.

Also, it is to be noted that $\mathcal{D}$ is a countable dense subset of $[0, 1]$.

Before providing the formal proof of Urysohn lemma, let us see a glimpse of how we will use these two results in the proof. Note that what is given to us in Urysohn lemma that we are having with us a normal space $(X, \mathcal{T})$, and also, we are having two disjoint closed subsets A and B of X. Now, if A and B are disjoint, then $A \subseteq B^c$. Let us take $B^c = G_1$. So, what we can say that the closed set A is contained in an open set. Now, by using the result that we have already discussed, there exists an open set G_0 such that $A \subseteq G_0 \subseteq \overline{G_0} \subseteq G_1$. Again, it is to be noted that G_1 is open, and $\overline{G_0}$ is a closed set. If this is the case, again use the same result. Therefore, there exists another open set, let us take that open set as $G_{1/2}$ such that $A \subseteq G_0 \subseteq \overline{G_0} \subseteq G_{1/2} \subseteq \overline{G_{1/2}} \subseteq G_1$. From here, what are we getting? We are getting open sets such that $G_0 \subseteq G_{1/2} \subseteq G_1$.

Now, see the suffix of these open sets. Note that we are getting 0; we are getting $1/2$, and 1. If we are continuing this process, we can get $1/4, 3/4$, and other dyadic rationals in between 0 and 1. Let us only use the above open sets here.

Now, let us see how we can construct a function. We are defining the function $f : X \to [0, 1]$ such that $f(x)=\min\{r : x \in G_r\}$, when $x \in G_1$; and $f(x) = 1$, when $x \notin G_1$. Note that if $x \in A$, and if we want to find out the value of $f(x)$, it will be the minimum of the rationals, that is, $0, 1/2, 1$. This is because $A \subseteq G_0 \subseteq G_{1/2} \subseteq G_1$. If we are taking $x \in A$, x is a member of $G_0, G_{1/2}, G_1$, and the minimum of $0, 1/2, 1$ is 0. Thus, for all $x \in A$, $f(x) = 0$. Also, if $x \in B$, note that $x \notin B^c$, or that $x \notin G_1$. Now, if $x \notin G_1$, $f(x) = 1$. The question is whether this function is continuous, which is not easy to justify here. We will justify this when we discuss the formal proof of Urysohn lemma. Still, from here, it is clear that how we will use both the concepts, one is a result regarding normal spaces, and the second is the concept of dyadic rationals.

These are the references.

That's all from this lecture. Thank you.