Course Name: Essentials of Topology
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Welcome to Lecture 64 on Essentials of Topology.

In the previous lecture, we studied the concept of regular spaces and saw a number of examples. In this lecture, we will study some of the results on regular spaces. Further, we will also study the concept of normal spaces.

Begin with, let us see, a result on regular spaces. Precisely, this is a characterization of a T_3 space. The result is stated as: A topological space $(X, \mathcal{T})$ is T_3 if and only if for each $x \in X$ and for each open set G containing x, there exists an open set H containing this x such that $\overline{H} \subseteq G$. In order to justify it, let us assume that $(X, \mathcal{T})$ is T_3. Also, let G be an open set containing x. Note that the given space is T_3. So, if we want to use the power of a T_3-space, we require a closed set. Why not let us take $F = X - G$. Then F is a closed set, and x cannot be an element of F. So, we can use the definition of a T_3-space. By using the definition of a T_3-space, we can conclude that there exist open sets H and H' such that $x \in H$, $F \subseteq H'$, and $H \cap H' = \emptyset$. Now, if $H \cap H' = \emptyset$, then $H \subseteq H'^c$. Also, it is to be noted that H' is open, so its complement is closed. Therefore, $\overline{H} \subseteq H'^c$. Further, as $F \subseteq H'$, $H'^c \subseteq F^c$, or $H'^c \subseteq G$, i.e., $\overline{H} \subseteq G$.

In order to prove the converse part, let us assume that for each $x \in X$ and for each open set G such that $x \in G$, there exists an open set H containing x with the property that $\overline{H} \subseteq G$. Now, we have to justify that the topological space is a T_3-space. In order to justify that $(X, \mathcal{T})$ is a T_3-space, begin with an element $x \in X$ and a closed subset $F \subseteq X$, which does not contain x. Now, let $G = F^c$, then G is an open set. Further, $x \in G$. As per assumption, there exists an open set, say H, such that H contains x and $\overline{H} \subseteq G$. Now, if $\overline{H} \subseteq G$, $\overline{H}^c \supseteq G^c = F$. Also, it is to be noted that $x \in H$. Thus, we have an open set H, which contains x. Also, let us take $\overline{H}^c = H'$. So, $F \subseteq H'$. What is H'? This is also an open subset of X because $\overline{H}$ is a closed set. Further, $H \cap H' = \emptyset$. Therefore, $(X, \mathcal{T})$ is a T_3-space.

Moving ahead, we have already seen that a Hausdorff topological space may not be a regular space. But, we can show that a compact Hausdorff topological space is regular. In order to justify it, let us take a compact Hausdorff topological space $(X, \mathcal{T})$. Now, if we are taking a closed subset $F \subseteq X$, let us take $x \in X$ such that $x \notin F$. As $(X, \mathcal{T})$ is compact, F will also be a compact subset of X. In the previous lecture, we have already studied that if the topological space is Hausdorff and we have a compact subset $F \subseteq X$, which is not containing x, there exist open sets G and H such that $x \in G$, $F \subseteq H$, and $G \cap H = \emptyset$. Thus, $(X, \mathcal{T})$ is a regular space.

Moving ahead, let us see some of the results. These results are not proven here as they are similar to what we have seen in the case of Hausdorff topological spaces. The first result is that a subspace of a regular space is regular. The second one is that regularity is a topological property, and finally, the product of regular spaces is also regular. So, just think about proof of these results.

Till now, what have we studied? We have seen that if we are having a topological space $(X, \mathcal{T})$, and if we are having an element as well as a closed set which is not containing the element, we can construct two disjoint open sets. One is containing the point, and one is containing the closed set. Now, we are moving one step ahead, and what we want to see is whether we can separate two disjoint closed sets. Meaning is to say that if we are having a topological space $(X, \mathcal{T})$, let us take two disjoint closed sets A and B. The question is: can we construct two disjoint open sets, one containing A and one containing B? If this is possible, we say that $(X, \mathcal{T})$ is a T_4-space. Also, if we are adding one more concept, that is the concept of T_1-spaces, such spaces are known as normal spaces. Formally, we say that a topological space $(X, \mathcal{T})$ is T_4 if for each pair of disjoint closed subsets A and B of X, there exist open subsets G and H of X such that $A \subseteq G$, $B \subseteq H$, and $G \cap H = \emptyset$. Finally, we say that $(X, \mathcal{T})$ is normal if it is both T_1 as well as T_4.

From the definition, it is clear that if a topological space is normal, then that space will always be regular. The question is: how is it possible? The answer is, if we are taking a normal topological space $(X, \mathcal{T})$, a closed subset $F \subseteq X$, and $x \in X$, which is not in F. Then, it is to be noted that $\{x\}$ is a closed set. Why? Because $(X, \mathcal{T})$ is T_1-space. So, we have two closed sets with us. Now, let us use the definition of a T_4-space. So, what can we conclude? We can conclude that there exist open subsets G and H of X such that $\{x\} \subseteq G$, $F \subseteq H$, and $G \cap H = \emptyset$. From here, we can conclude that $x \in G$, $F \subseteq H$, and $G \cap H = \emptyset$. Thus, $(X, \mathcal{T})$ is a regular space. Now, the question is, whether the regular spaces are normal, too. The answer is no. That is, if a space is regular, it doesn't mean that space will be normal. Even, before discussing normality as well as regularity, it is important to see the relationship between T_3 as well as T_4-spaces. Similar to the relationship between T_2 and T_3-spaces, we can justify that if a space is T_3, it may not be T_4. Even a space that is T_4 is not necessarily T_3. So, let us see some examples of normal spaces, and in between, we will try to justify these statements.

Begin with the first example. We have already seen the concept of right-ray topology $\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ on $\mathbb{R}$. The question is, what about disjoint closed subsets of $\mathbb{R}$? Note that we cannot find such closed sets. Thus, the set of real numbers, along with right-ray topology, is trivial, a T_4-space. It is to be noted that this space is not a T_3-space. The question is, why? The answer is, if we are taking a closed set $(-\infty, a]$ and a real number b, where $a < b$, obviously, $b \notin (-\infty, a]$. Then, can we find two disjoint open subsets of $\mathbb{R}$, one containing this closed set, and another containing this b? The answer is no. Because the open set containing this closed set will also contain b. Therefore, this space is not a T_3-space. The question is whether this space is normal. Just think about it.

Moving ahead, let us take a metrizable space $(X, \mathcal{T}_d)$, where d is metric on X. We can show that $(X, \mathcal{T}_d)$ is a normal space. We have already discussed that this space is a T_1-space. Thus, we have to only justify that $(X, \mathcal{T}_d)$ is a T_4-space. In order to justify that this space is T_4, let us take two disjoint closed subsets $A, B \subseteq X$. Then $A \cap B' = \emptyset$ and $A' \cap B = \emptyset$. Because A and B, both being disjoint closed subsets of X, they are separated, too. Therefore, A will not contain the limit points of B, and also B will not contain the limit points of A. If this is the case, what we can say is that for all $a \in A$, we can construct an open ball $B(a, r_a), r_a > 0$ such that $B(a, r_a) \cap B = \emptyset$. Similarly, for all $b \in B$, there exists some $r_b > 0$ such that $A \cap B(b, r_b) = \emptyset$. Now, from here, let us try to construct open subsets of X containing A as well as B. How we can do it, the answer is, let us take $G = \bigcup_{a \in A} B(a, r_a/2)$.

Then G is an open subset of X and $A \subseteq G$. Further, let $H = \bigcup_{b \in B} B(b, r_b/2)$. Then $B \subseteq H$. Finally, we need to justify that $G \cap H = \emptyset$. In order to justify it, let us see what will happen if there exists some element $z \in G \cap H$? It means that $z \in G$, and $z \in H$. Now, if $z \in G$ and $z \in H$, there exist $a \in A$, and also some $b \in B$ such that $z \in B(a, r_a/2)$ and $z \in B(b, r_b/2)$. If these are the cases, we can conclude that $d(a, z) < r_a/2$ and $d(z, b) < r_b/2$. Thus, $d(a, b) \leq d(a, z) + d(z, b) < \frac{r_a + r_b}{2}$. Now, if $r_a = max\{r_a, r_b\}$, then $d(a, b) < r_a$, and if $d(a, b) < r_a$, it means that $b \in B(a, r_a)$. But at the same time, it is to be noted that $b \in B$. That is, $B \cap B(a, r_a) \neq \emptyset$, which is a contradiction. Similarly, in other case, if $r_b = max\{r_a, r_b\}$, we can reach to a contradiction. Thus, in any case, $G \cap H = \emptyset$ and therefore, $(X, \mathcal{T}_d)$ is a T_4- space and hence, normal. From here, it is clear that if we are talking about this real line with Euclidean topology, this space is also a normal space.

Moving ahead, and let us take one more example, i.e., $(\mathbb{R}, \mathcal{T}_l)$? It is to be noted that this is normal. We can even show that this is regular, too. How is this normal? Note that for normality, we have to show that this is T_1 and T_4. This space is T_1 because the topology $\mathcal{T}_l$ is finer than the Euclidean topology. We have already shown that $\mathbb{R}$ with the Euclidean topology is a T_1-space. Now, we have to justify that $(\mathbb{R}, \mathcal{T}_l)$ is a T_4-space. In order to justify it, let us take two disjoint closed subsets $A, B \subseteq \mathbb{R}$. With the same logic we used in the case of metrizable spaces, we can say that $A \cap B' = \emptyset$, and $A' \cap B = \emptyset$. Thus, we can find some basic open sets, that is, for all $a \in A$, there exists $\epsilon_a > 0$ such that $[a, a + \epsilon_a) \cap B = \emptyset$, and for all $b \in B$ there exists $\epsilon_b > 0$ such that $[b, b + \epsilon_b) \cap A = \emptyset$. Now, let $G = \bigcup_{a \in A} [a, a + \epsilon_a)$ and $H = \bigcup_{b \in B} [b, b + \epsilon_b)$. Then G and H open, $A \subseteq G$, and $B \subseteq H$. Finally, the question arises whether G and H are disjoint. The answer is yes, and we can show it. How do you justify it? Let us see. If possible, $G \cap H \neq \emptyset$. Then there exist $a \in A$ and $b \in B$ so that $[a, a + \epsilon_a) \cap [b, b + \epsilon_b) \neq \emptyset$. Now, let $b = max\{a, b\}$. If this is the case, so what will happen? This $b \in [a, a + \epsilon_a)$, and $b \in [b, b + \epsilon_b)$, or $b \in B$. But $[a, a + \epsilon_a) \cap B = \emptyset$. Thus, we reach to a contradiction. Similarly, if we are taking $a = max\{a, b\}$, we can reach to a contradiction. Thus, $G \cap H = \emptyset$, or that $\mathbb{R}$ with the lower limit topology, is a normal space. Also, this will always be a regular space. The question is, what about the product of these two spaces, i.e., $\mathbb{R}^2$, when the topology on $\mathbb{R}$ is the lower limit topology. We have already seen that the product of regular spaces is a regular space. Therefore, this space will always be regular. But interestingly, this is not normal. Just think about it. This is not straightforward to justify $\mathbb{R}^2$ is not normal, when $\mathbb{R}$ is equipped with the lower limit topology $\mathcal{T}_l$. From here, one more thing is clear that the product of normal spaces need not be normal.

Moving ahead, similar to the concept of regular spaces. What we can justify that every compact Hausdorff space is normal. In order to justify it let us take a compact Hausdorff topological space $(X, \mathcal{T})$. We have already shown that this space is regular. Now, if we want to justify that this space is normal, too, let us take two disjoint closed subsets $A, B \subseteq X$. It is to be noted that for all $a \in A$, a cannot be an element of B, as A and B are disjoint. So, by regularity of $(X, \mathcal{T})$, there exist open sets, say, G_a and H_a such that $a \in G_a$, $B \subseteq H_a$, and $G_a \cap H_a = \emptyset$. Now, let $\mathcal{C} = \{G_a : a \in A\}$. Note that this is an open cover of A. What A is? Note that $A \subseteq X$ is closed. What X have we taken? This is compact. So, we can conclude that this A is compact, too. Thus, by using the compactness, there exists a subcover of $\mathcal{C}$. That is, $G_{a_1}, G_{a_2}, ..., G_{a_k} \in \mathcal{C}$ such that $A \subseteq G_{a_1} \cup G_{a_2} \cup ... \cup G_{a_k}$. Also, note that corresponding to every G_a, there is a H_a, and $G_a \cap H_a = \emptyset$. So, why not let us take open sets H_{a_1}, H_{a_2}, ..., H_{a_k} corresponding to G_{a_1}, G_{a_2}, ..., G_{a_k}, respectively. Also, let $G = G_{a_1} \cup G_{a_2} \cup ... \cup G_{a_k}$ and $H = H_{a_1} \cap H_{a_2} \cap ... \cap H_{a_k}$. Then $A \subseteq G$, $B \subseteq H$. Again, the big question is, what about $G \cap H$? Note that this will always be an empty set. Such types of justifications have also been discussed earlier. Therefore, $(X, \mathcal{T})$ is a T_4-space. Further, being a Hausdorff space, $(X, \mathcal{T})$ is T_1, too. Therefore, we can conclude that $(X, \mathcal{T})$ is a normal space.

These are the references.

That's all from this lecture. Thank you.