Course Name: Essentials of Topology
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Welcome to Lecture 63 on Essentials of Topology.

In this lecture, we will study some of the results related to Hausdorff spaces. Further, we will discuss the notion of regular spaces along with some examples.

Begin with the concept of convergence of a sequence, which we have already seen. If we recall the example of Sierpinski space, $X = \{a, b\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}\}$. We have seen that if we are taking a sequence (x_n), where $x_n = a$, for all $n \in \mathbb{N}$, this sequence converges to a and b. This means that the limit of a convergent sequence in a topological space is not necessarily unique. So, the problem is regarding the uniqueness. A natural question is: Can we put some restrictions on the topological space so that the limit of any convergent sequence turns out to be unique? The answer is yes, and the idea is T_2-spaces or Hausdorff spaces. Accordingly, let us see this result.

The result is stated as follows: in a Hausdorff topological space, every convergent sequence converges to precisely one point; that is, the limit will always be unique. In order to justify it, let us take a Hausdorff topological space $(X, \mathcal{T})$. Also, let us take a sequence (x_n) in X. Let us assume that, if possible, this sequence is converging to two points, that is, x as well as y, where $x, y \in X$, and these are distinct. Now, what will happen? The first thing is that because this space is Hausdorff, so there exist open sets G and H such that $x \in G$, $y \in H$, and $G \cap H = \emptyset$. Now, what is the meaning of convergence of a sequence? Recall the notion. So, if (x_n) is converging to x, there exists a natural number $n_0 \in \mathbb{N}$ such that $x_n \in G$, $\forall n \geq n_0$. Similarly, if (x_n) is converging to y, we can conclude that there exists, let us take $m_0 \in \mathbb{N}$ such that $x_n \in H$, $\forall n \geq m_0$. Now, let us take $m = max\{n_0, m_0\}$. So, what can we conclude? We can conclude that $\forall n \geq m$, $x_n \in G$, and also, $x_n \in H$. What does it mean? It means that $G \cap H \neq \emptyset$, and it contradicts the fact that the topological space is Hausdorff. So, we reached a contradiction. Hence, sequence (x_n) will converge to a unique point. That is the proof of this result.

Moving ahead, let us see another result on Hausdorff spaces. The result is stated as: let $(X, \mathcal{T})$ be a Hausdorff topological space, then for a compact subset A of X and an element x_0 of X, which is not in A, there exist disjoint open sets G and H, one contains x_0, and one contains A. The proof of this result is similar to the result we proved when we studied the concept of compact topological spaces. At that time, we have shown that the compact subset of a Hausdorff topological space is always closed. So, let us see the proof of this result. What is given to us? We have with us an element $x_0 \in X$, and a compact subset A of X. Also, it is given to us that $(X, \mathcal{T})$ is Hausdorff. Now, if $(X, \mathcal{T})$ is Hausdorff, for each $a \in A$ and x_0, there exist open sets, let us take as G_a and H_a such that $x_0 \in G_a$, $a \in H_a$, and $G_a \cap H_a = \emptyset$. Now, from here, we can conclude that if we are taking a collection $\mathcal{C} = \{H_a : a \in A\}$, this is an open cover of A. If $\mathcal{C}$ is an open cover of A, using the compactness of A, we can deduce that there exist $H_{a_1}, H_{a_2}, ..., H_{a_k}$ in $\mathcal{C}$ such that $A \subseteq H_{a_1} \cup H_{a_2} \cup \cdots \cup H_{a_k}$. Let us take $H = H_{a_1} \cup H_{a_2} \cup \cdots \cup H_{a_k}$. It is clear that this H is open. Also, corresponding to each H_a, we can find some G_a containing x_0. So, why not corresponding to H_{a_1}, let us take G_{a_1}, corresponding to H_{a_2}, let us take $G_{a_2}, \ldots$, and corresponding to H_{a_k}, let us take G_{a_k}. Now, let us take $G = G_{a_1} \cap G_{a_2} \cap \cdots \cap G_{a_k}$. It is to be noted that this G is also an open set, because it is a finite intersection of open sets. Further, we can conclude that $A \subseteq H$. Also, as $x_0 \in G_{a_i}, 1 \leq i \leq n$, so $x_0 \in G$. Thus, we have shown the existence of two open sets G as well as H. The question is only to justify that G and H are disjoint, and that can be done easily. This is possible because of the fact that $G_a \cap H_a = \emptyset$, $\forall a \in A$. Just think about it. How to do it? This type of result we have already proved when we have studied the notion of compactness.

Moving ahead, till now, what have we seen? We have seen that if we are taking a topological space $(X, \mathcal{T})$, and we are taking two of its elements, we can separate them by using open sets. We have studied three such criteria, but what we are going to do now is we are thinking of the separation of a point and a closed set (not containing the point) by using the concept of open sets. Such spaces are known as regular spaces. But this is not precisely the regularity, this concept is known as the concept of T_3-spaces, and T_3-spaces together with T_1-spaces are known as regular spaces. Formally, T_3-spaces are defined as: A topological space $(X, \mathcal{T})$ is T_3 if for every pair consisting of a

point $x \in X$ and a closed set $F \subset X$ with x is not an element of this closed set, there exist open sets G as well as H such that $x \in G$, $F \subseteq H$, and $G \cap H = \emptyset$. Further, $(X, \mathcal{T})$ is regular if it is both T_1 and T_3.

It is to be noted that here we studied two concepts: one is the concept of T_3-spaces, and the second is the concept of regular spaces. The question is how are they related to T_2-spaces or Hausdorff spaces. The answer is: there is no relationship between T_3 and T_2-spaces; that is, a T_2-space may not be a T_3-space. Even what we can see that a T_3-space may not be a T_2-space. But we will see that if the space is regular, it will always be Hausdorff, that is T_2. Also, we can show that a T_2-space may not be regular. So, we will justify all of these one by one. At least this first one can be directly deduced from the definition itself because if we are taking a regular topological space $(X, \mathcal{T})$, meaning is, it is T_1 and T_3. Now, if we are taking two elements $x, y \in X$ such that $x \neq y$, we can say that $\{y\}$ is a closed set. Why? Because the space is a T_1-space. Also, it is to be noted that if $x \neq y$, x is not an element of $\{y\}$. Therefore, by using the definition of a T_3-space, we can conclude that there exist open sets G and H such that $x \in G$, $\{y\} \subseteq H$ with $G \cap H = \emptyset$, or that, $x \in G, y \in H$, and $G \cap H = \emptyset$. Therefore, we can conclude that $(X, \mathcal{T})$ is a T_2-space. Thus, a regular space is always a T_2-space.

Let us take some examples of T_3 and regular spaces and try to justify what we have stated here. The first such example is an indiscrete topological space $(X, \mathcal{T})$, where $|X| \geq 2$. We have seen that this is not a Hausdorff space. This space is T_3 because X is only the non-empty closed set and there are no points outside it. Thus, a T_3-space may not be a T_2-space. Let us take another example. The well-known Sierpinski space $X = \{a, b\}$ with $\mathcal{T} = \{\emptyset, X, \{a\}\}$. It is to be noted that this is not a T_2-space. Even we can easily justify that this is not T_3, too. Next, let us take discrete topological spaces. We can justify that such spaces are T_2 and T_3 both. Further, let us take $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. This topological space is a T_3-space. It is simple to justify that this space is not a T_2-space because we cannot separate a from b.

Moving ahead, let us show that the set of real numbers, along with Euclidean topology, is T_3. Even, we can also justify that this is regular. Why? Because we know that this space is always a T_1-space. Let us see the justification for it. In order to justify it, let us take a closed subset F of $\mathbb{R}$. Also, let us take

a real number x so that $x \notin F$. Now, if $x \notin F$, obviously x cannot be an element of F', because if F is closed, F' will always be a subset of F. If x is not an element of F', there exists an open set G containing x such that $G \cap F = \emptyset$. It is to be noted that $x \in G$, therefore there exists an open interval $(x - \epsilon, x + \epsilon)$ such that $(x - \epsilon, x + \epsilon) \cap F = \emptyset$. Now, let us construct two open sets; the first open set is H, which is precisely $(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2})$. Also, let us take another one, which is $H' = \cup\{(y - \frac{\epsilon}{2}), (y + \frac{\epsilon}{2}) : y \in F\}$. From these two constructions, it is clear that $x \in H$ and $F \subseteq H'$. The question is: whether H and H' are disjoint. The answer is yes, and we can justify it. How to justify it? Let us go through contradiction. For example, if possible, let us assume that $H \cap H' \neq \emptyset$. So, we can find a real number z, which is a member of $H \cap H'$, or $z \in H$ as well as $z \in H'$. Now, if $z \in H$, $|x - z| < \epsilon/2$. Also, if $z \in H'$, there exists $y_0 \in F$ such that $|z - y_0| < \epsilon/2$. Now, if we are computing $|x - y_0|$, then $|x - y_0| \leq |x - z| + |z - y_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. From here, we can conclude that $y_0 \in (x - \epsilon, x + \epsilon)$. But it is to be noted that $y_0 \in F$, and $(x - \epsilon, x + \epsilon) \cap F = \emptyset$. So, we reached to a contradiction. This contradiction is because of the fact that $H \cap H' \neq \emptyset$. So, our assumption is wrong. Hence, $H \cap H' = \emptyset$, and the space, which is the real line with the Euclidean topology, is T_3. As stated earlier that, this space is T_1, too. Therefore, this space is regular also. It is to be noted that the set of real numbers, along with Euclidean topology, is Hausdorff. So, what have we seen? We have seen a number of examples. Some of the examples of the spaces which are both Hausdorff as well as regular, even they are also T_3. Also, we have seen that there exist some T_3-spaces which are not Hausdorff. Still, a question we have not answered, that is the construction of a counter-example that a T_2-space is not necessarily T_3, or a Hausdorff space may not be a regular space. Let us see an example.

The example which will help us to justify it, is the set of real numbers along with a topology $\mathcal{T}_K$ on it, that we have already studied. We have seen that $\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}\} \cup \{(a, b) - K : a, b \in \mathbb{R}\}$, where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ is a basis for this K-topology $\mathcal{T}_K$. We can justify that this space is T_2 but not T_3. The question is, how do we justify it? In order to justify that this space is T_2, we recall the fact that the Euclidean topology is always coarser than this K-topology, and it is also be noted that the real line with Euclidean topology is a T_2 space. So, what will happen? for all $x, y \in \mathbb{R}, x \neq y$, there exist open sets G and H such that $x \in G, y \in H$ with $G \cap H = \emptyset$. Therefore, $(\mathbb{R}, \mathcal{T}_K)$ is a T_2-space. Now, in order to justify that this space is not a T_3-space, let us

try to find a closed set, take a point that is not in that closed set, and show that we cannot separate these two by disjoint open sets. The question is: how do we find that closed set? The answer is here itself. We can show that K is a $\mathcal{T}_K$-closed. In order to justify that K is closed, we have to justify that it contains all of its limit points. What exactly we can justify for it that if we are taking any element outside of K, that cannot be the limit point of K, that is, if we are taking any $x \in \mathbb{R} - K$, $x \notin K'$.

- Let $x \in \mathbb{R} - (K \cup \{0\})$. Then there exists $a, b \in \mathbb{R}$, $a < b$ such that this $((a, b) - \{x\}) \cap K = \emptyset$.

- $0 \notin K'$ because there exists an open set $G = (-1, 1) - K$ such that $(G - \{0\}) \cap K = \emptyset$.

Thus, no point outside of K can be a limit point of K. Therefore, K is $\mathcal{T}_K$-closed, and note that 0 is not an element of K. In order to justify that $(\mathbb{R}, \mathcal{T}_K)$ is not a T_3 space, we can show that by using disjoint open sets, we cannot separate 0 from K.

If possible, let us assume that there exist open sets G and H such that $0 \in G$, $K \subseteq H$, with $G \cap H = \emptyset$. Now, what will happen? As G is an open set, there exists $B \in \mathcal{B}$ such that $0 \in B \subseteq G$. Now,

- if B is of the form (a, b), then for $B = (-\epsilon, \epsilon)$, using the Archimedean property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$; that is, $\frac{1}{n} \in B$, or that $\frac{1}{n} \in G$. Also, note that $K \subseteq H$. So, $\frac{1}{n} \in H$. Therefore, $G \cap H \neq \emptyset$, hence we reach to a contradiction.

- If B is of the form $(a, b) - K$, let us take some large value of n such that $\frac{1}{n} \in (a, b)$; we can do it by using the Archimedean property. Further, it is to be noted that H is also open. Because H is containing K, so we can say that $\frac{1}{n} \in H$. Now, if $\frac{1}{n} \in H$, there exists $B' \in \mathcal{B}$ such that $\frac{1}{n} \in B' \subseteq H$. The question is, what will be the look of this B'. With the trivial reason, this B' will always be of the form of this (c, d) because if we are taking it as something like $(c, d) - K$, then $\frac{1}{n} \notin B'$. That's why we are taking B' of the form (c, d), so, $\frac{1}{n} \in (c, d) \subseteq H$. Now, let us choose a real number z such that $\max\{c, \frac{1}{n+1}\} < z < \frac{1}{n}$. Then $z \in H$. Also, $z \in G$. Thus, $G \cap H \neq \emptyset$. Therefore, the set of real numbers with K-topology is not a T_3-space.

These are the references.

That's all from this lecture. Thank you.