Course Name: Essentials of Topology
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Welcome to Lecture 61 on Essentials of Topology.

In this lecture, we will initiate the study of the concept of separation axioms. Begin with what separation axioms mean. The idea behind the concept is the separation of distinct points, or points and closed sets, by using open sets. It may happen that these open sets may intersect, or they may be disjoint. That will depend on the axioms we study. Even in the literature, there are more than ten separation axioms. Some of them are known as T_0, T_1, T_2, T_3, and T_4. We have also studied some of the separation axioms in the previous lectures.

During the study of the concept of compactness, what did we use? We used two notions: the concept of T_1-spaces and the concept of T_2-spaces. Even there is another such concept, known as T_0-spaces. So, let us begin with such concept. We say that a topological space $(X, \mathcal{T})$, is T_0 if for each pair of distinct points of X, there is an open set that contains one point but not the other. Formally, what exactly we have to do is for all $x, y \in X, x \neq y$, there exists an open set, let us take this as G such that $x \in G$, but $y \notin G$, or there exists an open set H such that $y \in H$, but $x \notin H$. So, either of the cases is enough to justify that the space is a T_0-space.

Let us take some of the examples. The well-known example is indiscrete topology on a set X, where $|X| \geq 2$. If we are taking $x, y \in X, x \neq y$, there does not exist an open set which contains one element but not the other. Thus, such spaces are not T_0. Now, let us take discrete topological spaces; that is, the topology on X is discrete. Such spaces are always T_0. Why? The answer is simple because if we are taking $x, y \in X, x \neq y$, then there exists an open set $\{x\}$, which contains x but does not contain y. Let us take one more example, which is the well-known Sierpinski space. What the space is? It is $X = \{a, b\}$ with the topology $\mathcal{T} = \{\emptyset, X, \{a\}\}$. This space is T_0 space. Why? Because if we are taking two elements of X, let us take a and b. Now, there exists an open set, that is, $\{a\}$, and it is to be noted that this contains a, but b is not an element of it.

Moving ahead, let us take some more examples. Metrizable spaces are always T_0. The question is, why? The answer is simple because if we are taking a metric space (X, d), and the metrizable space, that is, $(X, \mathcal{T}_d)$. Then, for $x, y \in X$ such that $x \neq y$, what about this $d(x, y)$? This will always be greater than zero. If $d(x, y)$ is greater than zero, why not let us take $d(x, y) = r$. Now, if we are looking for $B(x, r/2)$, it will always contain x. But at the same time y cannot be a member of this open ball. Therefore, $(X, \mathcal{T}_d)$ is a T_0-space. Even from here it is also clear that the real line with Euclidean topology is also a T_0-space.

Moving ahead, and let us take another example. What we are taking is X with co-finite topology. Obviously, the set X here what we are considering, that is an infinite set. Even if that will be finite, there is no problem because the topology will be discrete, and we have shown that discrete topological spaces are T_0. In the case when X is an infinite set, this is also T_0. Why? The answer is simple because if we are taking any two elements x and y in X, $x \neq y$; we can think about $X - \{x\}$. It is to be noted that this is an open set, and the feature of this open set is that it contains y, but note that x cannot be an element of this set. Therefore, this space is a T_0-space.

Now, let us move to the concept of T_1-spaces, whose definition as well as results we used when we studied the concept of compactness. We say that a topological space $(X, \mathcal{T})$ is T_1 if for each pair of distinct elements $x, y \in X$, there is an open set containing x but not y, and there exists another open set containing y but not x; i.e., $(X, \mathcal{T})$ is T_1 if for all $x, y \in X, x \neq y$, there exist open sets G and H, such that $x \in G, y \notin G$ and $y \in H, x \notin H$.

From this definition, it is clear that every T_1-space is T_0. The question is: what about the converse? It seems that the converse is not necessarily true. Let us look at some of the examples. The well-known example is, why not let us take Sierpinski space, i.e., $X = \{a, b\}$ with the topology $\{\emptyset, X, \{a\}\}$. We have already studied that this space is T_0. This space is not T_1. The question is, why? The answer is: The open set $\{a\}$ is containing a, but this is not containing b. But the question arises: what about an open set containing b? It is to be noted that X is the only open set which is containing b, but at the same time, X also contains a. Therefore, this space is not T_1.

There are some more examples that we have already studied in the case of T_0-spaces. For example, if we are taking indiscrete topological spaces $(X, \mathcal{T}), |X| \geq 2$, it is to be noted that these spaces are not T_1. The justification is similar to what we have seen in the case of T_0-spaces. Even we can justify that discrete topological spaces are T_1. If we are talking about metrizable spaces, metrizable spaces will always be T_1. Why? The justification is similar to that of T_0-spaces. Still, let us see it. If we are taking a metric space (X, d), and the metrizable space $(X, \mathcal{T}_d)$. We are taking two elements $x, y \in X$, and these two are distinct. If x and y are distinct, we can conclude that $d(x, y) > 0$. So, why not let us take $d(x, y) = r$, $r > 0$. Now, we can construct open balls $B(x, r/3)$ and $B(y, r/3)$ such that $x \in B(x, r/3), y \notin B(x, r/3)$, and $y \in B(y, r/3), x \notin B(y, r/3)$. Therefore, $(X, \mathcal{T}_d)$ is a T_1-space. From here, we can also conclude that if we are talking about real line or the set of real numbers with Euclidean topology, this is also T_1. Even, if we are taking a set X with co-finite topology, this space is also a T_1-space.

With these, let us see some of the results, the first result which we have also used. The result is: a topological space $(X, \mathcal{T})$ is T_1 if and only if every single point subset of X is closed. In order to justify it, let us first assume that the space $(X, \mathcal{T})$ is T_1. Our motive is to justify that every single point subset of X is closed; that is, if we are taking any $x \in X$, let us try to justify that $\{x\}$ is closed. If we want to justify that $\{x\}$ is closed, this is equivalent to justifying that its complement, $X - \{x\}$ is open, which is equivalent to justifying that $X - \{x\}$ is a neighborhood of each of its points. In order to show that $X - \{x\}$ is a neighborhood of each of its points, let us take any $y \in X - \{x\}$. It is clear that $y \neq x$. If $y \neq x$, because $(X, \mathcal{T})$ is T_1, we can say that there exists an open set; let us take that open set as G_y such that $y \in G_y$, but $x \notin G_y$. It is to be noted that if $x \notin G_y$, what we can say is that this G_y will always be a subset of $X - \{x\}$. As $y \in G_y$, we can write $y \in G_y \subseteq X - \{x\}$. From here, it is clear that $X - \{x\}$ is a neighborhood of y, and therefore $X - \{x\}$ is an open set.

Let us see the converse part of this result. We are assuming that for all $x \in X$, $\{x\}$ is closed. Our motive is to justify that $(X, \mathcal{T})$ is a T_1-space. In order to justify it, let us take two elements, that is $x, y \in X$, and these two are distinct. Now, we can talk about these two singleton sets, $\{x\}$ and $\{y\}$. It is to be noted that both will be closed, and if both are closed, we can conclude that $X - \{x\}$ and $X - \{y\}$, both are open. Also, it is clear that $x \in X - \{y\}$, but $y \notin X - \{y\}$. At the same time, $y \in X - \{x\}$, but $x \notin X - \{x\}$. Therefore, $(X, \mathcal{T})$, is a T_1-space.

Moving ahead, and let us see another result. This result also we have used. The result is stated as: if we are having a topological space $(X, \mathcal{T})$, and we are having a subset A of X, then x is a limit point of A if and only if each open set containing x contains infinitely many points of A. Now, let us assume that each open set containing x contains infinitely many points of A, and try to justify that x is a limit point of A. This is simple to justify because if we are taking any open set G containing x. Then $(G - \{x\}) \cap A \neq \emptyset$, therefore $x \in A'$. For the converse, let us assume that $x \in A'$. We have to justify that each open set containing x contains infinitely many points of A. In order to prove it, let us assume that there exists an open subset of $G \subseteq X$, containing x such that $G \cap A$ is finite. Now, let $F = (G \cap A) - \{x\}$. Then F is a finite set. The question is: is F a closed subset of X. The answer is yes. Why? Moving to the previous result, what we have shown there is that every singleton subset of X is closed, and if every singleton subset of X is closed, their finite union is closed, too. Therefore, F is closed. Now, let us take $H = G - F = G \cap F^c$. Note that F^c is open, G is already open. So, what is this H? H is an open set. Also, it is to be noted that $x \in H$, that is, H, is an open set containing x, but $(H - \{x\}) \cap A = \emptyset$. Therefore, x cannot be a limit point of A. But what we have assumed is that $x \in A'$; thus, we have reached a contradiction. Therefore, every open set containing x contains infinitely many points of A.

Moving ahead, let us check the T_1-ness of the continuous image of a T_1-space. Interestingly, the continuous image of a T_1-space is not necessarily T_1. The question is, why? The answer is simple. Let $X = \{a, b\}$ with two topologies; the first one is the discrete topology $\mathcal{T}$, and the second one, we are taking $\mathcal{T}'$, where $\mathcal{T}' = \{\emptyset, X, \{a\}\}$. Now, if we are defining a function $f : (X, \mathcal{T}) \to (X, \mathcal{T}')$ such that $f(x) = x$. Then f is a continuous function. Also, it is to be noted that $(X, \mathcal{T})$ is a T_1-space, while $(X, \mathcal{T}')$ is not a T_1-space. Hence, a continuous image of a T_1-space is not necessarily T_1.

The question is, what type of images of a T_1-space will be T_1. The answer is: the closed image of a T_1-space is a T_1-space. In order to justify it, let us take a closed surjective function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$. We are taking f as surjective so that $f(X) = Y$. Also, let us assume that $(X, \mathcal{T})$ is a T_1-space. We have to justify that $(Y, \mathcal{T}')$ is also T_1. In order to justify it, let us take two distinct elements $y_1, y_2 \in Y$. Because of surjectiveness, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$, and $f(x_2) = y_2$. Also, because $(X, \mathcal{T})$ is a T_1-space, we can conclude that the singleton sets $\{x_1\}$ and $\{x_2\}$, are $\mathcal{T}$-closed sets. As the function f is closed, we can say that $\{f(x_1)\}$ and $\{f(x_2)\}$ are $\mathcal{T}'$-closed. If these are $\mathcal{T}'$-closed, it means that $\{y_1\}$ and $\{y_2\}$, both are $\mathcal{T}'$-closed. Now, because $Y - \{y_1\}$ and $Y - \{y_2\}$ are $\mathcal{T}'$-open, it is to be noted that $y_1 \in Y - \{y_2\}$, while y_2 is not a member of this open set. Also, what about y_2? y_2 is a member of $Y - \{y_1\}$, while y_1 is not a member of this open set. Thus, we obtained two open sets such that one is containing y_1 but not y_2, while another containing y_2 but not y_1. Therefore, $(Y, \mathcal{T}')$ is a T_1-space. Further, from here, we can conclude that the property of being a T_1-space is topological because we have justified that the closed image of a T_1-space is T_1.

Moving to the next one. These two are simple results that one can easily justify. The first one is: a subspace of a T_1-space is also T_1; and the second result is: for T_1-spaces $(X_1, \mathcal{T}_1)$, and $(X_2, \mathcal{T}_2)$, the product space is also T_1. Even this result can be generalized for arbitrary products. We are leaving the proof here, but we will prove similar results for T_2-spaces.

These are the references.

That's all from this lecture. Thank you.