Course Name: Essentials of Topology
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Welcome to Lecture 60 on Essentials of Topology.

In this lecture too, we will continue the study of the concept of one-point compactifications. Begin with what we have seen in the previous lecture that if we are beginning with a Hausdorff topological space $(X, \mathcal{T})$, and if we are assuming this $Y = X \cup \{\infty\}$, $\infty \notin X$, then we have seen that $\mathcal{T}' = \{G \subseteq X : G \in \mathcal{T}\} \cup \{Y - K : K \text{ is a compact subset of } X\}$ is a topology on Y. We have called this topological space $(Y, \mathcal{T}')$, one-point compactification of the topological space $(X, \mathcal{T})$. We have also seen that the topology $\mathcal{T}$ is precisely the relative topology with respect to the topology $\mathcal{T}'$. Further, we have stated in the last lecture to study whether $(Y, \mathcal{T}')$ is compact and Hausdorff.

So, let us begin with the first one and show that $(Y, \mathcal{T}')$ is a compact topological space. Let us begin with an open cover $\mathcal{C} = \{G_i : i \in I\}$ of Y. If this is an open cover of Y, what are these $G_i's$? Note that $G_i \in \mathcal{T}'$, for all $i \in I$. It is to be noted that $\infty \in Y$. So, there exists $G_k \in \mathcal{C}$ such that $\infty \in G_k$. The question is, what does this G_k look like? It is to be noted that when this ∞ is a member of G_k, $G_k = Y - K$, where K is a compact subset of X. Now, if K is a compact subset of X, let us try to construct an open cover of this K, and use the compactness for our help. For that, what we are going to do is that because $\mathcal{C}$ is an open cover of Y, with the help of this open cover, let us try to construct an open cover of X. So, let us take the collection $C^* = \{X \cap G_i : G_i \in C, i \in I\}$. Then C^* is an open cover of X. It is to be noted that the sets of the form $X \cap G_i, G_i \in \mathcal{T}'$ will always be an open subset of X because we have already seen that the topology $\mathcal{T}$ is the same as the relative topology computed with respect to $\mathcal{T}'$. Now, if this C^* is an open cover of X, then C^* is also an open cover of K. If C^* is an open cover of K, by the compactness of K, there exist finite members in C^* so that $K \subseteq (X \cap G_{i_1}) \cup (X \cap G_{i_2}) \cup ... \cup (X \cap G_{i_n})$, or $K \subseteq G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_n}$. Further, it is to be noted that what Y is? Y can be expressed as $Y = K \cup (Y - K)$, therefore $Y = G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_n} \cup G_k$.

Thus, we can conclude that $\mathcal{C}' = \{G_{i_1}, G_{i_2}, ..., G_{i_n}, G_k\}$ is a finite sub-cover of $\mathcal{C}$. Hence, $(Y, \mathcal{T}')$, which is a one-point compactification of topological space $(X, \mathcal{T})$, that is compact.

Moving to the next, let us show that $(Y, \mathcal{T}')$ is Hausdorff, too. In order to show that this topological space $(Y, \mathcal{T}')$ is Hausdorff, we require one additional condition on this topological space that this topological space should be locally compact. So, the final result is stated here, and it says that if $(Y, \mathcal{T}')$ is one-point compactification of a locally compact Hausdorff topological space $(X, \mathcal{T})$, then $(Y, \mathcal{T}')$ is Hausdorff. Let us justify it. If we are taking any two elements $x, y \in Y$, it is possible that x and y both are members of X. If both are members of X, it is to be noted that $(X, \mathcal{T})$ is Hausdorff. So, what we can do is that there exist two open sets, G and H, such that $x \in G, y \in H$, and $G \cap H = \emptyset$. One of the other possible cases is that if we are taking $x, y \in Y$, it may happen that $x \in X$ and $y = \infty$. Now, we also have to construct two open sets, G, and H, having the property that $x \in G, y \in H$ and $G \cap H = \emptyset$. In order to find G and H, let us use the local compactness of $(X, \mathcal{T})$. As $(X, \mathcal{T})$ is locally compact, there exists a compact neighborhood N of x. Therefore, there exists an open set G such that $x \in G \subseteq N$. Now, let us take a subset of Y, that is, $H = Y - N$. It is to be noted that N is a compact subset of X. What about $(X, \mathcal{T})$. This is Hausdorff. Now, if $(X, \mathcal{T})$ is Hausdorff, we can say that N will be closed. So, what about this H? This is an open subset of Y, and if this is an open subset of Y, we already have an open subset of X, that is, G. Thus, there exist two open subsets of Y, and these open subsets are G and H such that $x \in G$, $y \in H$, and $G \cap H = \emptyset$. Note that $G \cap H = \emptyset$ as it is given that $x \in G \subseteq N$. Thus, we have shown that for two distinct elements, $x, y \in Y$, there exist two open sets, G and H, so that $x \in G$, $y \in H$, and $G \cap H = \emptyset$. Hence, $(Y, \mathcal{T}')$ is a Hausdorff topological space.

Moving ahead, let us see a result regarding the uniqueness of this $(Y, \mathcal{T}')$ up to homeomorphism. The result is stated here: let $(Y, \mathcal{T}')$ be one-point compactification of locally compact Hausdorff topological space $(X, \mathcal{T})$. Then $(Y, \mathcal{T}')$ is unique up to homeomorphism. The question is, how do we justify it? Let us see the justification of this result. What do we have to do exactly? Let us consider another topological space, that is, $(Z, \mathcal{T}'')$. What are we taking? We are taking that $(Z, \mathcal{T}'')$ is compact and Hausdorff, too. Here, $Z = X \cup \{p\}$, $p \notin X$. What this $(X, \mathcal{T})$ is? This $(X, \mathcal{T})$ is a subspace of $(Z, \mathcal{T}'')$. So, what exactly this $(Z, \mathcal{T}'')$ is? Actually, this is another one-point compactification of $(X, \mathcal{T})$. In order to justify that $(Y, \mathcal{T}')$ is unique up to homeomorphism, we have to construct a homeomorphism from $(Z, \mathcal{T}'')$ to $(Y, \mathcal{T}')$. The question is, how do we construct this unique homeomorphism? The answer is, define a function $f : (Z, \mathcal{T}'') \to (Y, \mathcal{T}')$ such that $f(x) = x$, if $x \in X$ and $f(p) = \infty$. Note that if we are defining f in this way, f is bijective. In order to show that this f is a homeomorphism, it is only sufficient to justify that this f is continuous because what information do we have about $(Z, \mathcal{T}'')$? This space is compact. What about $(Y, \mathcal{T}')$? This is Hausdorff. We have already studied that a bijective continuous function from a compact topological space to a Hausdorff topological space is a homeomorphism. So, let us justify that this f is continuous. In order to justify that f is continuous, let us take any $G \in \mathcal{T}'$. Now, if this $G \in \mathcal{T}'$, how will this G look like? The first one is that either this G is a member of the topology $\mathcal{T}$, or G will look like in the form of $Y - K$, where K is a compact subset of X. It is to be noted that the singleton set $\{p\}$ is a closed subset of Z. Why? Because $(Z, \mathcal{T}'')$ is Hausdorff, and if this is closed here, what can we conclude about X? We can say that X is open. So, X is an open subset of Z. Now, find $f^{-1}(G)$. If $G \in \mathcal{T}$, then $f^{-1}(G) = G \in \mathcal{T}''$. Why? It is to be noted that $(X, \mathcal{T})$ is a subspace of $(Z, \mathcal{T}'')$. Therefore, G can be written as $X \cap G'$, where $G' \in \mathcal{T}''$. But what X is? We have already justified that X is an open subset of Z, and because G' is also an open subset of Z, $G \in \mathcal{T}''$. Finally, if we are taking $G = Y - K$, then how this $f^{-1}(G)$ will look like? Note that $f^{-1}(G) = f^{-1}(Y - K) = Z - K$. Can we conclude that K is a closed subset of Z? Yes. The question is how? It's simple, and just think about it. If K is a closed subset of Z, then $Z - K \in \mathcal{T}''$. So, what we have shown is that the inverse image of an arbitrary $\mathcal{T}'$-open set G is $\mathcal{T}''$-open set; that is, f is a continuous function. Therefore, f is a homeomorphism. The uniqueness of this function is trivial. Hence, $(Y, \mathcal{T}')$ is unique up to homeomorphism.

Moving ahead, let us see some of the examples of one-point compactification. The first one is: the one-point compactification of $(0, 1)$ is homeomorphic to the unit circle S^1. Why is it so? The answer is, because there exists a homeomorphism f from the open interval $(0, 1)$ to $S^1 - \{(1, 0)\}$, where $f(t) = (cos2\pi t, sin2\pi t), t \in (0, 1)$. It is to be noted that this unit circle S^1 is compact. Even, we have discussed that the closed interval $[0, 1]$ is also compact, that is, a superset of this open interval $(0, 1)$. But this is not one-point compactification. Finally, we can also justify that one-point compactification of $\mathbb{R}$, the set of real numbers, is homeomorphic to the unit circle S^1 because we know that this open interval $(0, 1)$ is homeomorphic to the set of real numbers.

Moving ahead, let us see one of the characterizations of locally compact topological spaces by using the concept of one-point compactifications. This is a theorem and is stated as: Let $(X, \mathcal{T})$ be a Hausdorff topological space. Then $(X, \mathcal{T})$ is locally compact iff for each $x \in X$, and for all open sets G containing x there exists another open set H containing x with compact closure $\overline{H}$ such that $\overline{H} \subseteq G$. In order to prove it, let us assume that for each $x \in X$ and for all open sets G containing x there exists another open set H containing x with compact closure $\overline{H}$ such that $\overline{H} \subseteq G$. Then, it is trivial to justify that $(X, \mathcal{T})$ is locally compact. The question is how? Why not let us take $G = X$. Now, if we are taking $G = X$, for all $x \in X$, we can talk about $x \in H \subseteq \overline{H}$, and H is an open neighborhood of x. Therefore, $\overline{H}$ is a compact neighborhood of x.

In order to justify the converse, let us assume that $(X, \mathcal{T})$ is locally compact. It is to be noted that this is not only locally compact, this is Hausdorff, too. Now, let us take any $x \in X$, and consider any open set G containing x. Further, what do we have to take? Let us take $Y = X \cup \{\infty\}$, and one point compactification $(Y, \mathcal{T}')$ of $(X, \mathcal{T})$. Now, let us take $F = Y - G$. It is to be noted that what G we have taken, that is a subset of X. Therefore, this will always be a subset of Y. Thus, we can talk about this $Y - G$. It is to be noted that $G \in \mathcal{T}'$, that is G is an open subset of Y. If G is an open subset of Y, we can say that F is a closed subset of Y. Now, if F is a closed subset of Y, again recall the result that we used a number of times. Note that this F is closed. Being one point compactification, we have already seen that this $(Y, \mathcal{T}')$ is Hausdorff, and if this is Hausdorff, we can say that F is compact. It is to be noted that $x \in X$, but x cannot be an element of F. What this F is? This is compact. Note that $x \in Y$. Now, let us recall a result that we have not proved yet, but we will prove it when we discuss the concept of Hausdorff topological spaces. The result is: if we are having a Hausdorff topological space $(X, \mathcal{T})$, and if we are taking an element $x \in X$ alongwith a compact subset $F \subseteq X$ not containing x, then there exist disjoint open sets H and H' such that $x \in H$ and $F \subseteq H'$. With these, in our case, we can say that there exist two open sets H and H' such that $x \in H$, $F \subseteq H'$, and $H \cap H' = \emptyset$. Now, from here what can we conclude? We can say that $H \subseteq H'^c$, or $\overline{H} \subseteq H'^c$. But it is to be noted that H'^c is closed as H' is an open set. Therefore, we can say that

$\overline{H} \subseteq H'^c$, or we can say that $\overline{H} \subseteq Y - H'$. Also, as $F \subseteq H'$, so, $\overline{H} \subseteq Y - F$. It is to be noted that this $Y - F = G$. Thus, $\overline{H} \subseteq G$. Hence, the proof.

These are the references.

That's all from this lecture. Thank you.