

Course Name: Essentials of Topology
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Welcome to Lecture 59 on Essentials of Topology.

Continuing with the concept of compactness, in this lecture, we will study the notion of one-point compactifications. Begin with what we have studied. If we are taking a closed interval $[0, 1]$, this is a compact subset of the set of real numbers. If we are taking the semi-open interval $(0, 1]$, this is a subset of this closed interval $[0, 1]$ but not compact. The interesting thing is that the difference between these two intervals is only a singleton set. Meaning is, this closed interval $[0, 1]$ can be written as this $(0, 1] \cup \{0\}$. What does it mean? If we are taking a set X , and let us add a new element in it. By adding this element, the non-compact set becomes compact. Can we do this in general? The answer is given by the concept of one-point compactification. What precisely are we talking about? Let (X, \mathcal{T}) be a Hausdorff topological space and $Y = X \cup \{\infty\}, \infty \notin X$. Further, let $\mathcal{T}' = \{G \subseteq X : G \in \mathcal{T}\} \cup \{Y - K : K \text{ is a compact subset of } X\}$. Then \mathcal{T}' is a topology on Y . If this \mathcal{T}' is a topology, what this Y is? Note that Y is nothing but $Y = X \cup \{\infty\}$. That is, what $Y - X$ is? The difference is a singleton set $\{\infty\}$. So finally, we say that this (Y, \mathcal{T}') is a one-point compactification of the topological space (X, \mathcal{T}) . But before coming to any conclusion, let us justify that this \mathcal{T}' is a topology on Y .

Note that the empty set is always a member of \mathcal{T}' because it is a member of \mathcal{T} . Also, the question is whether Y is a member of \mathcal{T}' . The answer is yes. Why? Because Y can be written as $Y - \emptyset$, it is to be noted that the empty set is a compact subset of X . Why? Because this is a finite set, and we know that all finite subsets of X are compact.

Moving ahead, let us take $G_1, G_2, \dots, G_n \in \mathcal{T}'$. What is our motive to justify? Our motive is to justify that $G_1 \cap G_2 \cap \dots \cap G_n \in \mathcal{T}'$. What exactly will we do here? We will assume only two elements from \mathcal{T}' . So, let us take $G_1, G_2 \in \mathcal{T}'$ and try to show that $G_1 \cap G_2 \in \mathcal{T}'$. So, we have to justify it. If we can justify it, by using the concept of induction, one can prove that

$G_1 \cap G_2 \cap \dots \cap G_n \in \mathcal{T}'$. Now, whenever we are taking this $G_1, G_2 \in \mathcal{T}'$, there will be different cases.

- If $G_1, G_2 \in \{G \subseteq X : G \in \mathcal{T}\}$. Then $G_1, G_2 \in \mathcal{T}$, whereby $G_1 \cap G_2 \in \mathcal{T}$. Thus $G_1 \cap G_2 \in \mathcal{T}'$.
- If $G_1, G_2 \in \{Y - K : K \text{ is a compact subset of } X\}$. Then $G_1 = Y - K_1$ and $G_2 = Y - K_2$. What K_1 and K_2 are? These K_1 and K_2 are compact subsets of X . Now, $G_1 \cap G_2 = (Y - K_1) \cap (Y - K_2) = Y \cap (K_1^c \cap K_2^c) = Y \cap (K_1 \cup K_2)^c = Y - (K_1 \cup K_2)$. Now, it is to be noted that if K_1 and K_2 are compact subsets of X , therefore $K_1 \cup K_2$ is also a compact subset of X . If $K_1 \cup K_2$ is a compact subset of X , what can we conclude? We can conclude that $G_1 \cap G_2 \in \mathcal{T}'$.
- If $G_1 \in \{G \subseteq X : G \in \mathcal{T}\}$ and $G_2 \in \{Y - K : K \text{ is a compact subset of } X\}$. Then, $G_2 = Y - K$, for some compact subset K of X . Now, $G_1 \cap G_2 = G_1 \cap (Y - K) = G_1 \cap (X - K)$, because this intersection will never contain ∞ , and that is only the difference between Y and X . Now, what this K is? It is to be noted that $K \subseteq X$ is compact. Further, what (X, \mathcal{T}) is? This is Hausdorff. If (X, \mathcal{T}) is Hausdorff, we can say that K is a closed subset of X . If K is a closed subset of X , what about its complement in X ? This is an open subset of X . If this is an open subset of X , G_1 is already in \mathcal{T} , which is an open subset of X . Therefore, this $G_1 \cap G_2 \in \mathcal{T}$, or $G_1 \cap G_2 \in \mathcal{T}'$.

Moving ahead, let us see the last one, that is if we are taking an arbitrary family $\{G_i : i \in I\}$, $G_i \in \mathcal{T}'$. So, if we are assuming it, what is our motive to justify it? We have to justify that $\cup\{G_i : i \in I\} \in \mathcal{T}'$. Again, there will be different cases. Let us see one by one.

- If $G_i \in \mathcal{T}'$ such that $G_i \in \mathcal{T}$, for all $i \in I$. So, what can we conclude? If $G_i \in \mathcal{T}$, for all $i \in I$, we can conclude that $\cup\{G_i : i \in I\} \in \mathcal{T}$, because \mathcal{T} is a topology. From here, we can conclude that $\cup\{G_i : i \in I\} \in \mathcal{T}'$.
- Let us take $G_i \in \mathcal{T}'$ such that every G_i is of the form $Y - K_i$, where K_i is a compact subset of X . Now, $\cup\{G_i : i \in I\} = \cup\{Y - K_i : i \in I\} = Y - \cap\{K_i : i \in I\}$. As K_i is a compact subset of X , for all $i \in I$, what about $\cap\{K_i : i \in I\}$? Because the topological space (X, \mathcal{T}) is Hausdorff, we can conclude that $\cap\{K_i : i \in I\}$, is also a compact subset of X . If this is a compact subset of X , we can write $\cup\{G_i : i \in I\}$ as $Y - K$,

$K \subseteq X$, and this is compact. If this is the case, we can conclude that $\cup\{G_i : i \in I\} \in \mathcal{T}'$.

- If we are taking some of the G_i 's from the first family and some of the G_i 's are from the second family. Then, if we are computing $\cup\{G_i : i \in I\}$, how will it look like? This will be something like $(\{G_j : j \in I_1\}) \cup (\{Y - K_j : j \in I_2\})$, where $I_1, I_2 \subseteq I$. Thus, $\cup\{G_i : i \in I\} = G \cup (Y - K)$. It is to be noted that what G is? $G \in \mathcal{T}$, and what this K is, $K \subseteq X$, and this is compact. Our motive is to justify that this union is also a member of \mathcal{T}' . Now, let us define a subset F of X such that $F = X - G$. Then $G \cup (Y - K) = (X - F) \cup (Y - K) = (Y - F) \cup (Y - K) = Y - (F \cap K)$. Now, let us see what F and K are. Note that $F \subseteq X$ is closed. Also, K being a compact subset of a Hausdorff space, it is closed, too. If F and K , both are closed, what about this $F \cap K$? This is also closed. But one more thing to be noted here is that this $F \cap K$, this is a subset of K . What K is? This K is compact. So, this $F \cap K$ is a closed subset of a compact space. Thus, we can conclude that $F \cap K$ is also compact. If this is compact, then this union can be written as $\cup\{G_i : i \in I\} = Y - K'$. What K' is? K' is precisely $F \cap K$. It is to be noted that this $K' \subseteq X$, and at the same time, this is compact too. Therefore, we can conclude that this $G \cup (Y - K)$, this is a member of \mathcal{T}' , or we can conclude that the $\cup\{G_i : i \in I\} \in \mathcal{T}'$.

Hence, \mathcal{T}' is a topology on Y , and (Y, \mathcal{T}') is called one-point compactification of (X, \mathcal{T}) . It is to be noted that what Y is? Y is nothing but $X \cup \{\infty\}$, where ∞ is not an element of X . In the rest of the lecture, we will always assume (Y, \mathcal{T}') , where $Y = X \cup \{\infty\}$, and $\mathcal{T}' = \{G \subseteq X : G \in \mathcal{T}\} \cup \{Y - K : K \text{ is a compact subset of } X\}$.

Now, there are some questions about (Y, \mathcal{T}') . The questions are: as we are beginning with a Hausdorff topological space (X, \mathcal{T}) , the first question will be whether (Y, \mathcal{T}') is also Hausdorff. The second question is, whether (Y, \mathcal{T}') will always be compact, and even there will be third question too. Note that we are starting from (X, \mathcal{T}) , and now we are constructing this new topological space (Y, \mathcal{T}') . It is to be noted that what X is? $X \subseteq Y$, and if $X \subseteq Y$, we can talk about the relative topology on X . If we can talk about relative topology on X , there is one topology that is already available on X , that is, \mathcal{T} . Can we compare this \mathcal{T} as well as \mathcal{T}'_X ? The answer is yes. What we can justify

is that both the topologies are the same. So, let us begin from here, which is stated in this result. The result is, let (Y, \mathcal{T}') be one point compactification of Hausdorff topological space (X, \mathcal{T}) , then $\mathcal{T}'_X = \mathcal{T}$.

In order to justify it, let us see what this \mathcal{T}'_X is. Note that $\mathcal{T}'_X = \{X \cap G : G \in \mathcal{T}'\}$. First, we will try to justify that \mathcal{T}'_X is coarser than the topology \mathcal{T} , and second, we will justify that the topology \mathcal{T} is coarser than the topology \mathcal{T}'_X . Now, if we want to justify the first one, let us take any $H \in \mathcal{T}'_X$. Then, $H = X \cap G$, where $G \in \mathcal{T}$, or this G can be written as something like $Y - K$, where K is a compact subset of X . Now, if we are taking the first case, that is if $G \in \mathcal{T}$. Then it is clear that $H \in \mathcal{T}$. Now, moving to the second case, when G is of this form, that is $Y - K$. How H can be written as? H can be written as $X \cap (Y - K)$. It is to be noted that ∞ is not an element of $X \cap (Y - K)$. So, $X \cap (Y - K)$ is same as $X \cap (X - K)$, that is $X - K$. Now, see what this K is? $K \subseteq X$, (X, \mathcal{T}) is Hausdorff, and K is compact. Thus, we can conclude that this K is a closed subset of X , or that $X - K$ is \mathcal{T} -open. If $X - K$ is \mathcal{T} open, then $H \in \mathcal{T}$, that is, \mathcal{T}'_X is coarser than \mathcal{T} . Now, let us justify that \mathcal{T} is also coarser than \mathcal{T}'_X . For that, let us take $G \in \mathcal{T}$. Then by the definition of \mathcal{T}' , we can say that $G \in \mathcal{T}'$, and if $G \in \mathcal{T}'$, it is to be noted that G can be written as $X \cap G$. It means that G is a member of \mathcal{T}'_X , that is the topology \mathcal{T} is coarser than the relative topology \mathcal{T}'_X on X . Hence, \mathcal{T}'_X is same as the topology \mathcal{T} .

Two more concepts we have to discuss about the topological space (Y, \mathcal{T}') . Whether this topological space is compact and whether this topological space is Hausdorff. These are the two concepts from where we will start the next lecture.

These are the references.

That's all from this lecture. Thank you.