Course Name: Essentials of Topology
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Welcome to Lecture 58 on Essentials of Topology.

Continuing with the concept of compactness, in this lecture, we will study another version of compactness called local compactness. Begin with the concept of local compactness. If we are having a topological space $(X, \mathcal{T})$, we say that this topological space is locally compact at a point $x \in X$ if it has a compact neighborhood of x. Meaning is to say that for all $x \in X$, there exists a subset $N \subseteq X$, which is a compact neighborhood of x. It is to be noted that if N is a compact neighborhood of x, being a neighborhood, there exists an open set G such that $x \in G \subseteq N$. Further, we say that $(X, \mathcal{T})$ is locally compact if it is locally compact at each of its points.

From the definition, we can deduce that every compact topological space is locally compact; that is, compactness implies local compactness. It can be justified easily because if we are taking a topological space $(X, \mathcal{T})$, let us take that this topological space is compact. Further, let us take $x \in X$. What we can say is that X is open, and we know that every open set is a neighborhood of each of its points. So, what features are available with X? This is a neighborhood of x. Further, we have already assumed that $(X, \mathcal{T})$ is compact. Thus, we can conclude that $(X, \mathcal{T})$ is locally compact. The question is, what about the converse of it, that is, whether local compactness implies compactness? The answer is not necessarily. Why? Let us see some examples.

Our first example is: let us take the discrete topology $\mathcal{T}$ on an infinite set X. We know that infinite set with discrete topology is not compact. Note that this $(X, \mathcal{T})$, is locally compact. The question is: how to justify it? The answer is: if we are taking any $x \in X$, we know that this singleton set $\{x\}$ is open in the discrete topology. Therefore, we can write $x \in \{x\} \subseteq \{x\}$. It is to be noted that $\{x\}$ is a neighborhood of x. Further, being a singleton set, $\{x\}$ is compact in discrete space. Therefore, each element of X has a compact neighborhood. That's why this $(X, \mathcal{T})$ is locally compact.

Let us take another example, that is, $(\mathbb{R}, \mathcal{T}_e)$. For all $x \in \mathbb{R}$, what can we write? We can write that $x \in (x-1, x+1) \subset [x-1, x+1]$. So, from the definition itself, it is clear that $[x-1, x+1]$ is a neighborhood of x. Further, we already know that this is compact. So, all the elements of $\mathbb{R}$ have a compact neighborhood. Therefore, $\mathbb{R}$ with standard topology is locally compact.

Moving ahead, why not let us take $\mathbb{R}^n$ with standard topology, that is, the Euclidean topology on it. If we are taking an element $x \in \mathbb{R}^n$, that is, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, what we can do that, we can construct an open set containing this x, and therefore $x \in (x_1 - 1, x_1 + 1) \times (x_2 - 1, x_2 + 1) \times ... \times (x_n - 1, x_n + 1) \subset [x_1 - 1, x_1 + 1] \times [x_2 - 1, x_2 + 1] \times ... \times [x_n - 1, x_n + 1]$. We know that the Cartesian product of these closed intervals is compact because this is finite product of compact sets. Further, from the definition itself, it is clear that this is a neighborhood of x. So, each element of $\mathbb{R}^n$ has a compact neighborhood. Therefore, $\mathbb{R}^n$ with standard topology is locally compact.

Moving ahead, let us see a characterization of a locally compact topological space, and this characterization is when the given space is Hausdorff. The result is stated as: if we are having a Hausdorff topological space $(X, \mathcal{T})$, then this space is locally compact if and only if for each $x \in X$, there is an open set G containing x such that $\overline{G}$ is compact. In order to justify it, let us first assume that $(X, \mathcal{T})$ is locally compact. Now, let us take $x \in X$. By definition, there exists a compact neighborhood, say N of x. Because N a compact neighborhood, we can say that there exists an open set G such that $x \in G \subseteq N$. Note that $N \subseteq X$. What $(X, \mathcal{T})$ is? This $(X, \mathcal{T})$ is Hausdorff and what N is? This is compact. So, what can we deduce about this N? We can deduce that this N is closed; if this N is closed, then $\overline{N} = N$. Now, as $G \subseteq N$, $\overline{G} \subseteq \overline{N}$. It is to be noted that N is compact, and what is $\overline{G}$? This is a closed set. We have already studied that a closed subspace of a compact space is compact. Therefore, $\overline{G}$ is compact. That's the proof of this part.

Now, let us prove the converse part. In order to prove the converse part, let us assume that for each $x \in X$, there is an open set G containing x such that $\overline{G}$ is compact. From here, we can deduce that $(X, \mathcal{T})$ is locally compact because for every $x \in X$, we are saying that there exists an open set G and what G is? G is containing this x. It is to be noted that $x \in G \subseteq \overline{G}$ and this is given that $\overline{G}$ is compact. Therefore, each element of X has a compact neighborhood, and hence, $(X, \mathcal{T})$ is locally compact.

Moving ahead, let us take one more example, and after that, we will deduce some more results. The example that we are going to discuss here is the set $\mathbb{Q} \subseteq \mathbb{R}$ in the standard topology is not locally compact. Let us justify it by contradiction. So, if possible, let this $\mathbb{Q}$ be locally compact. Now, if this is locally compact, then what will happen for all $x \in \mathbb{Q}$? There exists a subset $N \subseteq \mathbb{Q}$. What is this N? This is a compact neighborhood of x. So, what we can say is that there is an open set $(a, b) \cap \mathbb{Q}$ such that $x \in (a, b) \cap \mathbb{Q} \subseteq N$. Further, what is the relationship of N with $\mathbb{R}$, that is, $N \subseteq \mathbb{Q} \subseteq \mathbb{R}$. We know that $(\mathbb{R}, \mathcal{T}_e)$ is Hausdorff. Now, if $(\mathbb{R}, \mathcal{T}_e)$ is Hausdorff, what about this N? This N, we are taking, is compact. So, what can we conclude? We can conclude that N is a closed set and if this is closed set, it will contain all of its limit point. If we are saying that N is containing all of its limit points, that is, this $N' \subseteq N$, then what will happen? There exists some $z \in \mathbb{R} - \mathbb{Q}$ such that $z \in N$. But it is to be noted that $N \subseteq \mathbb{Q}$. It means that $z \in \mathbb{Q}$. So, what this z is? z is an irrational number, and that is a member of the set of rational numbers, this is a contradiction. So, our assumption is not correct, and therefore $\mathbb{Q}$, that is, the set of rationals, is not locally compact.

From above, we can deduce one more thing. It is to be noted that this $\mathbb{R}$ with standard topology is locally compact, and what $\mathbb{Q}$ is? $\mathbb{Q} \subseteq \mathbb{R}$ and we have shown that this is not locally compact. It means that a subspace of a locally compact space may not be locally compact. So, our conclusion is: a subspace of a locally compact topological subspace is not necessarily locally compact, and this result is similar to the concept that we have already studied in the case of compactness. The question arises: what restrictions do we have to put on the subspace? The natural choice is, why not use the same concept what we have already used in the case of compactness, and that is given in this result. The result is: every closed subspace of a locally compact topological space is locally compact. In order to prove this result, let us take a topological space $(X, \mathcal{T})$. This is given that this topological space is locally compact. Also, let us take a subset $Y \subseteq X$. This is given that this Y is closed. What we have to justify, our motive is to justify that this is locally compact. Now, in order to prove that $(Y, \mathcal{T}_Y)$ is locally compact, let us take $y \in Y$. Note that if $y \in Y$, what Y is, $Y \subseteq X$. So, $y \in X$. Thus, we can say that there is a $\mathcal{T}$- open set G such that $y \in G$ and $\overline{G}$ is compact. Note that $Y \cap G$ is $\mathcal{T}_Y$-open and it contains y. Also, $Y \cap G \subseteq Y \cap \overline{G}$. So, from here, we can say that $Y \cap \overline{G}$ is a neighborhood of y. The question is whether $Y \cap \overline{G}$ is compact. The answer is yes, and that we can justify. It is to be noted that $Y \cap \overline{G} \subseteq \overline{G}$. Further, what this $\overline{G}$ is? This is compact, that we have already seen. If we are looking for $Y \cap \overline{G}$, it is to be noted that $\overline{G}$ is a closed subset of X, and Y is also closed. So, we can say that $Y \cap \overline{G}$ is a closed subset of $\overline{G}$, and because $\overline{G}$ is compact, we can conclude that $Y \cap \overline{G}$ is also compact. Thus, this $Y \cap \overline{G}$ is not only a neighborhood, but we can say that this is a compact neighborhood of y. Therefore, $(Y, \mathcal{T}_Y)$ is locally compact.

Moving ahead, let us discuss the continuous image of a locally compact topological space. The question is: whether the continuous image of a locally compact topological space is locally compact. The answer is: not necessarily. Let us try to justify it by a counterexample and after that we will try to find out some conditions under which the continuous image of a locally compact topological space turns out to be locally compact. So, for a counterexample, let us take $f : (\mathbb{Q}, \mathcal{T}) \to (\mathbb{Q}, \mathcal{T}')$ such that $f(x) = x, x \in \mathbb{Q}$, where $\mathcal{T}$ is the discrete topology and $\mathcal{T}'$ is the relative topology w.r.t. the standard topology on $\mathbb{R}$. It is to noted that f is continuous, $(\mathbb{Q}, \mathcal{T})$ is locally compact but $(\mathbb{Q}, \mathcal{T}')$ is not locally compact. Thus, the continuous image of this locally compact topological space is not a locally compact space. So, what to do? The answer is here, and the answer is given in the form of this theorem. This theorem states that the continuous open image of a locally compact topological space is locally compact. So, let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a continuous, open and surjective function alongwith $(X, \mathcal{T})$ as a locally compact space. Then we have to justify that this $(Y, \mathcal{T}')$ is locally compact. Let us see it.

In order to justify that $(Y, \mathcal{T}')$ is locally compact, why not let us take $y \in Y$. Then, because f is a surjective function, we can say that there exists $x \in X$ such that $f(x) = y$. Now, as $x \in X$ and $(X, \mathcal{T})$ is locally compact, there exists a compact neighborhood N of x in $(X, \mathcal{T})$. Because N is a neighborhood of x, what we can say that there exists a $\mathcal{T}$-open set G such that $x \in G \subseteq N$. Further, from here, what we can conclude that $f(x) \in f(G) \subseteq f(N)$, or that $y \in f(G) \subseteq f(N)$. Now, what conclusion can we draw about this $f(G)$. Note that this is $\mathcal{T}'$-open. Why is this $\mathcal{T}'$-open? Because this function f is open. Further, what about this $f(N)$? $f(N)$ is compact. Why? Because f is continuous. Also, the continuous image of a compact set is compact. So, from here we can deduce two things. The first one is that $f(N)$ is compact, and the second one is that $f(N)$ is a neighborhood of y. Thus, $f(N)$ is a compact neighborhood of y in $(Y, \mathcal{T}')$. Therefore, $(Y, \mathcal{T}')$ is locally compact.

These are the references.

That's all from this lecture. Thank you.