Course Name: Essentials of Topology
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Welcome to Lecture 57 on Essentials of Topology.

Beginning with the concept of compactness, in this lecture, we will study another version of compactness known as countably compactness. Further, in between, we will also study the notion of Lindelöf topological spaces.

Beginning with the concept of countably compactness, we say that a topological space $(X, \mathcal{T})$, is countably compact if every countable open cover has a finite subcover. The question arises: How is this concept related to the concept of compactness, which is defined in terms of open cover and finite subcover? The answer is simple, and we can deduce that every compact topological space is countably compact because, in the case of compact topological spaces, every open cover has a finite subcover. Therefore, every countably open cover should also have a finite subcover. The question is, what about the converse of it? When we want to justify that countably compactness implies compactness, we have to fill this gap; that is, every open cover should have a countable subcover. The question is, how to move from open cover to countable open cover? The answer is given by the concept of Lindelöf spaces. The question is, what is it? Let us see the definition of Lindelöf space.

We say that a topological space $(X, \mathcal{T})$ is Lindelöf if every open cover has a countable subcover. From here, we can deduce that if we are beginning with a topological space, and this topological space is countably compact, and this is also Lindelöf, then this will imply compactness; that is, the topological space will be a compact topological space. So, for the converse, we got the answer that only countably compactness will not imply the concept of compactness, but the concept of countably compact, along with Lindelöfness, will provide the concept of compactness. Now, when we are discussing the concept of Lindelöf topological spaces, the definition itself is given in terms of covers, so a natural question arises: how is this concept related to the concept of compact topological spaces? Again, the answer is simple; that is, every compact topological space is a Lindelöf space because when we are talking about compactness, that is, the open cover has a finite subcover; therefore, that subcover is countable, too. But again, a question arises: what about the converse of this statement? In order to justify the converse, let us see a result first because the concept of Lindelöf spaces is what we have seen that is given in terms of countable sets, and we have already studied countability axioms such as first countable spaces and second countable spaces. Even, we have also studied the concept of separable spaces.

There is an interesting relationship between Lindelöf spaces and second countable spaces, which is stated here, and the statement is that every second countable topological space is Lindelöf. Let us justify it. In order to justify it, begin with a topological space $(X, \mathcal{T})$, and assume that this space is a second countable space. If this space is a second countable space, this has a countable basis. Let us take that basis be this $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$. What do we have to justify? We have to justify that $(X, \mathcal{T})$ is a Lindelöf space. In order to justify that this space is Lindelöf, why not begin with an open cover of X. So, why not let us take this $\mathcal{C} = \{G_i : i \in I\}$, $G_i \in \mathcal{T}$. So, we are taking this as an open cover of X. Now, if this is an open cover of X, what we can write is that $X = \bigcup_{i \in I} G_i$. Further, if $\mathcal{B}$ is a basis for this space, what we can write is that this $X = \bigcup_{n \in \mathbb{N}} B_n$. The question is, how are these B_n and G_i related? We have already studied that if we are taking any $x \in X$, as X is the union of these open sets, there exists $G_k \in \mathcal{C}$ such that $x \in G_k$. Now, if G_k is an open set containing x, by using the definition of basis, we can conclude that there exists some $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq G_k$. This relationship is important for us. So, what we can do that let us choose for all $n \in \mathbb{N}$, G_{i_n} from $\mathcal{C}$ such that $B_n \subseteq G_{i_n}$. Now, one thing is clear that every G_{i_n} is from $\mathcal{C}$, that's why this is an open set, and if we are taking the collection of these G_{i_n} as $\mathcal{C}'$. What about the union of such G_{in}'s? That will always be equal to X because $X = \bigcup_{n \in \mathbb{N}} B_n$. So, what we can deduce is that $\mathcal{C}'$ is a subcover of $\mathcal{C}$, and this is not only a subcover but is also countable. Thus, every open cover of X has a countable subcover, and therefore, $(X, \mathcal{T})$, is a Lindelöf space. That's the proof of this result.

Now, let us use this result and what we can deduce; we can first deduce this one: a Lindelöf space is not necessarily compact. Why? Let us take an example, and the example is $(\mathbb{R}, \mathcal{T}_e)$. We know that this topological space is not compact. At the same time, we have already studied that this topological space is second countable, and if this space is second countable, from the previous result, this space is Lindelöf. Therefore, from this example, we can conclude that a Lindelöf space is not necessarily compact. But again, there is a question: what we have proved here is that every second countable space is Lindelöf; what about the converse? The converse is stated here, and from this statement, it is clear that the converse is not necessarily true; that is, a Lindelöf space is not necessarily second countable. So, let us take an example of it. The example is, let us take $(X, \mathcal{T}_{cf})$, where X is uncountable. We have already studied that this topological space is not a second countable space. But it is to be noted that we have also studied that co-finite topological spaces are compact and therefore Lindelöf, too.

Moving ahead, what have we seen? We have seen that in the case of Lindelöf spaces, open covers should have a countable subcover, which is similar to the concept of compact topological spaces, where open cover has a finite subcover. The question is, what about the results related to Lindelöf spaces? Can we deduce similar results to those in the case of compactness? The answer is here, and that is yes. The first one is that the continuous image of a Lindelöf topological space is Lindelof. From here, we can also deduce that the quotient space of a Lindelöf topological space is Lindelöf. Even from here, we can deduce that the homeomorphic image of a Lindelöf space is Lindelöf. The question is, what about the subspaces? The answer is similar to what we have studied in compact topological spaces, that is, every closed subspace of a Lindelöf topological space is Lindelöf.

Moving ahead, now come to the concept of limit point compactness as well as countably compactness. The concept of limit point compactness, we have studied in the previous lecture, and in this lecture, we have started to study the concept of countably compactness. Let us see the possible relationship between these two concepts. So, the first one is that every countably compact topological space is limit point compact. Let us justify it. So, begin with, let us take a topological space $(X, \mathcal{T})$, and assume that this is countably compact. What do we have to justify? We have to justify that this is limit point compact. We are proving by contradiction. If possible, let $(X, \mathcal{T})$ be not limit point compact. Then there exists an infinite subset $A \subseteq X$ so that $A' = \emptyset$.

Now, let us choose $B \subseteq A \subseteq X$, and what is this B we are choosing? The first thing is that this is infinite, and the second thing is that this B is countable. Note that $A' = \emptyset$ and $B \subseteq A$. So, $B' = \emptyset$, that is, B also has no limit point. Now, if this $B' = \emptyset$, what we can conclude is that for all $x \in X$, x cannot be an element of B', that is, for all $x \in X$, there exists an open set G_x, this is containing x such that $(G_x - \{x\}) \cap B = \emptyset$. Now, if $x \in B$, $G_x \cap B = \{x\}$. But if $x \notin B$, we can say that this $G_x \cap B = \emptyset$. From here, we can conclude that this $G_x \subseteq B^c$. It is to be noted that $x \notin B$, meaning that $x \in B^c$. Thus, $x \in G_x \subseteq B^c$, or that B^c is an open set. So, what we can conclude from here is that if we are taking a collection, that is, $\mathcal{C} = \{B^c\} \cup \{G_x : x \in B\}$, we can say that $\mathcal{C}$ is a countable open cover of X. Now, if this is a countable open cover of X, use the concept of countably compactness. Now, if we are using this concept, that is because $(X, \mathcal{T})$ is countably compact, we can conclude that there exist some finite members from $\mathcal{C}$ so that $X = B^c \cup G_{x_1} \cup G_{x_2} \cup ... \cup G_{x_n}$. But there will be a problem at this stage. What is the problem? Actually, the problem is, let us take a $x \in B$, such that $x \notin \{x_1, x_2, ..., x_n\}$. What will happen? The question is whether x belongs to the above union. The answer is no, x cannot be an element of it. Why? Because if so, obviously $x \notin B^c$ as it is an element of B, thus $x \in G_{x_1} \cup G_{x_1} \cup ... G_{x_n}$, therefore $x \in G_{x_k}$, for some $1 \leq k \leq n$. Thus $B \cap G_{x_k} = \{x, x_k\}$, which is not possible as $G_x \cap B = \{x\}$. From here, we can conclude that $\mathcal{C}$ no finite subcover. So, we reached a contradiction. Hence, every countably compact topological space is limit point compact.

Moving ahead, let us see the converse of the previous result. When we are talking about the converse of this result, we require one more concept, which is the concept of T_1-spaces, that we have used in the previous lecture. So, what we can deduce is that every limit point compact topological space, which is also T_1, is countably compact. In order to prove this theorem, let us begin with topological space $(X, \mathcal{T})$, and let us assume that this is limit point compact and T_1. Our motive is to justify that $(X, \mathcal{T})$ is countably compact. In order to prove it, again, let us assume that this $(X, \mathcal{T})$, is not countably compact, and if this is not countably compact, what will be the problem? It means that there exists a countable open cover, say $\mathcal{C} = \{G_n : n \in \mathbb{N}\}$; and this does not have any finite subcover. If this is not having any finite subcover what does it mean? It means that for all $n \in \mathbb{N}$, $X \neq \bigcup\limits_{i=1}^{n} G_i$. Meaning is, $\bigcup\limits_{i=1}^{n} G_i \subset X$, or $X - \bigcup_{i=1}^{n} G_i \neq \emptyset$. Now, let us choose $x_n \in X - \bigcup_{i=1}^{n} G_i$ such that $x_n \neq x_i$, $1 \leq i \leq n-1$. Also, let $A = \{x_n : n \in \mathbb{N}\}$. It is clear that A is an infinite set, and if this set is an infinite set, the question is whether it has a limit point. We can justify that $A' = \emptyset$. If we can justify that $A' = \emptyset$, this will contradict the fact that $(X, \mathcal{T})$ is limit point compact. Now, if $x \in A'$, what does it mean? It means that this $x \in X$, and what is this X? $X = \bigcup\{G_n : n \in \mathbb{N}\}$. Therefore, $x \in G_n$, for some $n \in \mathbb{N}$. Note that this G_n is an open set containing x. Also, $(X, \mathcal{T})$ is a T_1- space, and we have already seen a property that $x \in A'$ if and only if for every open set G containing x, it contains infinitely many elements of A. But the problem is here, whether this G_n contains infinitely many elements of A. The answer is no. Why? Because for all $i > n$, note that $x_i \notin G_n$. Even we know that $x_n \neq x_m$, for $n \neq m$. So, what we have assumed that $x \in A'$, that is not correct, that is, $A' = \emptyset$. Hence, every limit point compact topological space, which is also T_1, is countably compact.

These are the references.

That's all from this lecture. Thank you.