Course Name: Essentials of Topology
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Welcome to Lecture 56 on Essentials of Topology.

In this lecture, we will also continue the study of the concepts of compactness. Specifically, from this lecture onwards, we will focus on studying different notions of compactness.

Begin with one such notion, which is known as Limit Point Compactness. In literature, this concept is also known as the Bolzano-Weierstrass property. So, let us see what the concept is. A topological space $(X, \mathcal{T})$ is said to be limit point compact if every infinite subset of X has a limit point. The question is: how is this concept related to the concept of compactness, which is defined in terms of open cover? Before establishing any relationship, let us discuss some examples. A well-known example is, if we are taking an infinite set X with cofinite topology $\mathcal{T}_{cf}$. We have already seen that every cofinite topological space is a compact space. If we take any infinite subset $A \subseteq X$, $x \in X$, and an open set G, which is containing x. What we can justify is that $(G - \{x\}) \cap A \neq \emptyset$. The question is, why? What will happen if this is empty? If possible, let $(G - \{x\}) \cap A = \emptyset$. From here, what can we conclude? We can say that $A \subseteq (G - \{x\})^c$, but G^c is finite. Then, we can conclude that this $(G - \{x\})^c$ is also finite. If this is finite, what about this A? This also becomes a finite set, which is a contradiction. Therefore, $A' \neq \emptyset$. Thus, $(X, \mathcal{T}_{cf})$ is limit point compact.

Let us take another example. We have already seen this example in the case of compact topological spaces. Let us take a set $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ as a subspace of $\mathbb{R}$ with standard topology. This is limit point compact. How is it possible? If we take any infinite subset $B \subseteq A$, what will be the structure of this B? Note that B will contain the real numbers of form $\frac{1}{n}$, which is for arbitrarily large n. If it is so, what can we conclude? We can conclude that the set of limit points of B cannot be empty because 0 will always be the limit point of B. Thus, every infinite subset of A has a limit point, and therefore, this space is a limit point compact topological space. So, we have seen two examples. Both examples are examples of compact topological spaces as well as limit point compact topological spaces.

Now, let us take another example. Why not let us take this set of real numbers with discrete topology? Note that this space is not compact because we can construct an open cover that does not have any finite subcover. Even this is not limit point compact. Why? If we are taking subset $A \subseteq R$, what about A'? Note that $A' = \emptyset$; hence, this is not limit point compact. So, till now, in the examples that we have seen, the spaces were either both compact and limit point compact, or one example is here, this is neither compact nor limit point compact. The question arises: what is the difference between both concepts? The answer is here, and this is a theorem. The theorem is stated as follows: every compact topological space is limit point compact, but the converse is not necessarily true. Let us see one by one.

Begin with a topological space $(X, \mathcal{T})$. Assume that this topological space is compact. Our motive is to show that this is limit point compact. How do you show it? Begin with an infinite subset $A \subseteq X$. We have to show that $A' \neq \emptyset$; that is, the set of limit points of A is not an empty set. Let us see by contradiction. So, let us assume that, if possible, $A' = \emptyset$. Then what will happen? It means that for all $x \in X$, x cannot be an element of this A', i.e., $x \notin A'$. What does it mean? It means that for all $x \in X$, there exists an open set; let us denote this by G_x. This open set contains x with the property that $(G_x - \{x\}) \cap A = \emptyset$. One thing we can conclude from here is the relationship between this G_x and A in terms of the number of elements available in A. So, from here what we can conclude that A can contain at most one element of G_x, and this is an important one that we will use. Now, let us see the structure of these G_x. What have we taken? We have shown that an open set G_x exists for every $x \in X$. So, why not let us take the collection of these open sets, that is, $C = \{G_x : x \in X\}$. We can conclude that this is an open cover of X. If this is an open cover of X, now use the fact that $(X, \mathcal{T})$ is compact; that is, by using the compactness, what can we conclude? So, if we are using the notion of compactness, we can say that $X = G_{x_1} \cup G_{x_2} \cup ... \cup G_{x_k}$. What are these G_{x_i}? $G_{x_1}, G_{x_2}, ..., G_{x_k}$ are members of open cover $\mathcal{C}$, but at the same time, it is to be noted that what A is, $A \subseteq X$. So, from here we can conclude that $A \subseteq G_{x_1} \cup G_{x_2} \cup ... \cup G_{x_k}$. Now, use this fact that A can contain at most one element of G_x, and note that A is also a subset of these finite G_{x_i}'s. So, from here, we can conclude that A is a finite set, and this is a contradiction. Therefore, what we have assumed is wrong. Therefore, A', which is the set of limit points of A, is non-empty. Hence, this topological space $(X, \mathcal{T})$ is limit point compact.

Moving ahead, let us see the converse of this theorem. We have already stated that the converse is not necessarily true. So, it is sufficient to give an example. Let us construct an example for it. Take a set X, which is nothing but the complement of the set of integers in $\mathbb{R}$, i.e., $X = \mathbb{R} - \mathbb{Z}$. Also, let us take a collection $\mathcal{B} = \{(n-1, n) : n \in \mathbb{Z}\}$. Note that $X = \bigcup_{n \in \mathbb{Z}} (n-1, n)$. Also, all the intervals are disjoint. Therefore, this $\mathcal{B}$ satisfies the properties to be a basis. So, why not let us take a topology $\mathcal{T}$ generated by basis $\mathcal{B}$. So, we have a topological space $(X, \mathcal{T})$. One thing is clear from here that this topological space is not compact. Why? This basis $\mathcal{B}$ itself is an open cover of X with no finite subcover. What we can show is that this is limit point compact. The question is, how do we justify that this is a limit point compact? So, why not let us take an infinite subset $A \subseteq X$. Now, let us take an element $x \in A$. Because x is an element of A, this will be an element of some of the open intervals, as $A \subseteq X$, and X is covered by intervals. So, there exists an open interval; let us take that open interval as $(m-1, m)$, which contains x. Now, let us take an element, that is, y, which is an element of this interval, but y is not equal to x. What can we justify? We can justify that y is a limit point of A. The question is, how do we justify it? The answer is simple. Let us take any open set containing y. Obviously, that open set will contain the interval that contains this y. So, what can we write? We can write that the open interval $((m-1, m) - \{y\}) \cap A \neq \emptyset$ because both intervals contain at least one element, that is, x itself. Therefore, we conclude that A' is not an empty set. Thus, $(X, \mathcal{T})$ is limit point compact.

Moving ahead, what we have seen in the case of compact topological spaces. We have seen that if we are taking a compact topological space $(X, \mathcal{T})$, and let us take another topological space $(Y, \mathcal{T}')$, and a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, the idea is that the image of X is also compact. The question is whether a similar result holds for the concept of limit point compactness. The answer is no, and it is stated here that a continuous image of a limit point compact topological space need not be limit point compact.

The question is, again, why? Let us take an example. Why not let us take the topological space, which we have already shown that this is limit point compact. So, we are taking that topological space $(X, \mathcal{T})$. Remember that what X is? $X = \mathbb{R} - \mathbb{Z}$. What is the topology? The topology is generated by $\mathcal{B} = \{(n-1, n) : n \in \mathbb{Z}\}$. Now, let us take another topological space $(Y, \mathcal{T}')$. What is this $(Y, \mathcal{T}')$ precisely? We are taking this $Y = \mathbb{Z}$. What is this $\mathcal{T}'$? This is the discrete topology. Now, define a function f from $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$, and how are we defining this function? This is defined as $f(x) = n$, where $n - 1 < x < n$. Moving ahead, if this is the function, it is clear that f is a surjective function. If f is a surjective function, we can say that $f(X) = Y$. Further, f is continuous. Why is this f continuous? This is simple to deduce because what is the basis for this topology $\mathcal{T}'$, that is a collection of singleton sets. Now, if we are taking the inverse image of any singleton set, which is a member of the basis for the topology $\mathcal{T}'$ that will be of the form $(n-1, n)$, and that is $\mathcal{T}$-open. Thus, f is a continuous function. Note that what this $(X, \mathcal{T})$ is? We have already seen that this $(X, \mathcal{T})$ is limit point compact. But what can we make a conclusion about this topological space $(Y, \mathcal{T}')$, that is precisely Y with discrete topology. Note that this is not limit point compact because of topology, that is, discrete. Thus, we can conclude that the continuous image of a limit point compact topological space need not be limit point compact.

Now, a question will arise. Can we put some restrictions on this limit point compact topological space, or if we are adding some more features here so that this continuous image becomes a limit point compact? The answer is yes. We can do it, and that is the notion of T_1-ness, that is, the concept of T_1-spaces. We will study this in detail next week. Still, as we are going to use this concept, let us see what this concept is. Again, it is to be noted that in detail, we will study it later.

So, we say that a topological space $(X, \mathcal{T})$ is called a T_1- space if, for each pair of distinct points x and y in X, there is an open set containing x but not y, and at the same time, there exists another open set containing y but not x. At this moment, one can think about how it differs from Hausdorff spaces, which we have seen. We have not seen much of the properties of that space, but we will study it later on.

Let us see some of the examples. A well-known example is the real line with

Euclidean topology $(\mathbb{R}, \mathcal{T}_e)$. Note that this is a T_1-space. The question is how? The answer is simple. Let us take $x, y \in \mathbb{R}$ such that $|x - y| = \epsilon > 0$. Now, what we can do is we can construct an open set $(x - \epsilon/2, x + \epsilon/2)$, note that this will contain x, but at the same time, y cannot be an element of $(x - \epsilon/2, x + \epsilon/2)$. Also, there exists another open set $(y - \epsilon/2, y + \epsilon/2)$. It is to be noted that this contains y, but at the same time, this cannot contain x; that is, x is not a member of this open set.

Even we can take one more example. Why not let us take $\mathbb{R}$ with cofinite topology $(\mathbb{R}, \mathcal{T}_{cf})$? Note that this topological space is also T_1. In order to justify that this cofinite topological space is T_1, let us take two real numbers, that is, $x, y \in \mathbb{R}$, $x \neq y$. If we are looking at the behavior of $\mathbb{R} - \{x\}$ and $\mathbb{R} - \{y\}$, note that their complements are finite. So, both are open in the cofinite topology. Thus, both are open, but what is the feature of $\mathbb{R} - \{x\}$? This will contain y, but x cannot be an element of this one, and similarly, $\mathbb{R} - \{y\}$ contain x, but y cannot be an element of this one. Therefore, $\mathbb{R}$ with cofinite topology is T_1.

Now, let us see some of the results on T_1-spaces, which we have to use here, but we are not providing the proof. We will see the proof of these results in the lectures of next week. So, the first one is: a topological space $(X, \mathcal{T})$ is T_1 if and only if every single point subset of X is closed. Even such results we have already seen in the case of Hausdorff topological spaces, but there was only one part, that is, we have not considered if and only if. Further, if we are taking a topological space $(X, \mathcal{T})$, and a subset $A \subseteq X$, then an element $x \in X$ is a limit point of A if and only if each open set containing x contains infinitely many points of A. This is the important one which we are going to use in the next result. So, what exactly the next result is? If we are taking a limit point compact topological space, which is also T_1. Then, we can justify that the continuous image of this topological space is limit point compact.

Let us see the proof of this result. So, what exactly are we taking? We are taking a continuous surjective function f from a topological space $(X, \mathcal{T})$ to $(Y, \mathcal{T}')$. What is $(X, \mathcal{T})$? This is limit point compact and this one is also T_1. What is our motive? Our motive is to justify that the space $(Y, \mathcal{T}')$ is limit point compact. Now, if we want to show that $(Y, \mathcal{T}')$ is a limit point compact, let us take an infinte subset $B \subseteq Y$. By using the surjectivity of f, we can conclude that for all $y \in B$, there exist $x \in X$ such that this $x \in f^{-1}(\{y\})$. So, what we are going to do is that for each $y \in B$, let us choose $x \in f^{-1}(\{y\})$. Now, let us take collection of these x, and denote this collection as A. So, what A will be here? Can we say that this will also be infinite? Yes. It is a simple one, as f is a function from X to Y. See the nature of set B, which we have taken here. Now, A is an infinite subset of X, therefore $A' \neq \emptyset$, whereby let $p \in A'$. What about $f(p)$? Note that $f(p) \in Y$. Now, the question is whether this $f(p)$ is a limit point of B. Let us see it.

As $f(p) \in Y$, let us take a $\mathcal{T}'$-open set G such that $f(p) \in G$. Now, if $f(p) \in G$, what about p? Obviously, $p \in f^{-1}(G)$, and it is to be noted that what $f^{-1}(G)$ is? This is a $\mathcal{T}$-open set containing p; what is p? Note that p is a limit point of A. So, from here, what can we conclude? Because $(X, \mathcal{T})$ is a T_1-space, we can conclude that each open set containing x contains infinitely many points of A, i.e., $f^{-1}(G)$ contains an infinite number of elements from A. If this is the case, we can deduce from here that G contains infinitely many points of B. If G contains infinitely many points of B, what we can say is that $(G - \{f(p)\}) \cap B \neq \emptyset$, that is, $f(p)$ is a limit point of B, that is $f(p) \in B'$. Therefore, $(Y, \mathcal{T}')$ is limit point compact.

These are the references.

That's all from this lecture. Thank you.