Course Name: Essentials of Topology
Professor Name: S.P. Tiwari
Department Name: Mathematics & Computing
Institute Name: Indian Institute of Technology(ISM), Dhanbad
Week: 10
Lecture: 01

Welcome to Lecture 55 on Essentials of Topology.

Continuing with the concept of compactness, in this lecture, we will study the well-known Lebesgue number lemma along with the concept of uniform continuous functions. Begin with a result which we will require to prove this lemma. So, before stating and proving the lemma, let us prove a result from metric spaces. What does this result say? This result states that if we have a metric space (X, d), and a subset A of X, then the function $f : X \to \mathbb{R}$ such that $f(x) = d(x, A)$ is continuous, where $d(x, A)$ is the distance of point x from the fixed set A.

In order to prove the continuity of this function, let us use the notion of the $\epsilon - \delta$ concept. So, what exactly do we have to justify here is for given $\epsilon > 0$, let us take any $x \in X$, there exists some $\delta > 0$ such that whenever $d(x, y) < \delta$, $|f(x) - f(y)| < \epsilon$, where $y \in Y$. Note that $f(x) = d(x, A)$ and $f(y) = d(y, A)$. Now, as we have to justify that $|f(x) - f(y)| < \epsilon$, what we can do is begin with this $d(x, A)$. Note that $d(x, A)$ will always be less than or equal to $d(x, a)$, $\forall a \in A$, as $d(x, A)$ is the infimum of such real numbers. So, by triangle inequality, we can write that $d(x, A) \leq d(x, y) + d(y, a)$, or $d(x, A) - d(x, y) \leq d(y, a)$. Note that this inequality is true for all $a \in A$. So, what we can do is take infimum over a on the right-hand side, and this is possible because $d(x, A) - d(x, y) \leq d(y, a)$, and this is true $\forall a \in A$. Therefore, what we can conclude from here is $d(x, A) - d(x, y) \leq d(y, A)$, or we can say that $d(x, A) - d(y, A) \leq d(x, y)$. Now, interchange the role of x and y, we can also write that $d(y, A) - d(x, A) \leq d(x, y)$. Now, combining these two, what can we conclude? We can say that $|d(x, A) - d(y, A)| \leq d(x, y)$. Now, let us assume that $d(x, y) < \delta$. Also, take $\epsilon = \delta$. So, we can conclude that $|d(x, A) - d(y, A)| < \epsilon$, or what we can say that $|f(x) - f(y)| < \epsilon$. That's the proof of this result.

Moving ahead, let us see the main result, the Lebesgue number lemma. The

lemma is stated as: let (X, d) be a compact metric space, and $\mathcal{C}$ be an open cover of X. Then there is a $\lambda > 0$ such that for all $x \in X$, there exists a $G \in \mathcal{C}$ satisfying $B(x, \lambda) \subseteq G$. Now, let us see the proof. Begin with, let us take this collection $\mathcal{C}$, which is an open cover of X. The question is, whether X is a member of this $\mathcal{C}$? If X is a member of this open cover, what will happen? For this $\lambda > 0$, and for all $x \in X$, it is to be noted that $B(x, \lambda) \subseteq X$. Therefore, this lemma can be proved trivially. So, what are we going to take? We are taking this open cover $\mathcal{C}$ such that X is not a member of it. Further, if this is an open cover of X, as there exists some finite subcover, we can say that there exists some $G_1, G_2, ..., G_n$ in $\mathcal{C}$ such that this $X = G_1 \cup G_2 \cup .. \cup G_n$. Now, let us take the complement of these G_i. So, what we are doing is we are constructing new sets. Let us take F_i, which is the complement of these G_i. It is to be noted that this $1 \leq i \leq n$. What about these F_i's? These will be non-empty because none of the G_i is equal to X, and these will be closed, too.

Moving ahead, let us prepare a platform for using the extreme value theorem, which we are going to use to prove this lemma. So, we are defining a function $f : X \to \mathbb{R}$ such that $f(x) = (1/n) \sum_{i=1}^{n} d(x, F_i), x \in X$. So, what is this function? This is nothing but the average of distances from this x to F_i. It is to be noted that from the previous result, if we are defining a function that is sending each x to $d(x, F)$, where F is a fixed subset of X, then this function is continuous. So, what is this $f(x)$? This $f(x)$ is a finite sum of continuous functions multiplied by some real numbers. Therefore, this f is also continuous. So, we have shown that this function is continuous. Further, it is to be noted that what X is? This is compact. Now, if this X is compact, use the extreme value theorem. Therefore, by the well-known extreme value theorem, we can conclude that this f takes on a minimum value. Let us take that minimum value to be this λ. It is to be noted that when we are saying that this f takes on a minimum value λ, meaning is, this $\lambda \leq f(x), \forall x \in X$. Even, this λ will be something like $f(x_0)$, where $x_0 \in X$. Thus, we have prepared a foundation for proving this Lebesgue number lemma.

What precisely do we have to justify? We have to justify two things, (i) $\lambda > 0$, and (ii) $B(x, \lambda) \subseteq G$, where $G \in \mathcal{C}$. So, first, we are going to justify that $\lambda > 0$. In order to show that this $\lambda > 0$, we will justify that this $f(x)$ will always be strictly greater than 0, for all $x \in X$. If we can justify that this $f(x) > 0$, for all $x \in X$, from here we can conclude that $\lambda > 0$. Now, in order

to show that $f(x) > 0$, let us take any arbitrary $x \in X$. Note that if x is in X, what X is? X is nothing but the union of G_i's, $1 \leq i \leq n$. So, $x \in G_k$, for some $k \in \{1, 2, ..., n\}$. Note that (X, d) is a metric space. So, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq G_k$. Thus, we can conclude from here is that if we are trying to find $d(x, F_k) \geq \epsilon$, i.e., $d(x, y) \geq \epsilon$, for all $y \in F_k$. The question is, how can we do it? If for some $y \in F_k$, this $d(x, y) < \epsilon$, what will happen? Note that if $d(x, y) < \epsilon$, $y \in B(x, \epsilon)$ and $B(x, \epsilon) \subseteq G_k$. It is to be noted that $y \in F_k$. What F_k is? F_k is a complement of G_k, so this is not possible, and therefore, $d(x, F_k) \geq \epsilon$. Now, let us see the nature of our function. The function that we have defined, that is, is defined as $f(x) = (1/n) \sum_{i=1}^{n} d(x, F_i)$. Note that this can be written as something like $(1/n)[d(x, F_1) + ... + d(x, F_i) + ...d(x, F_n)]$. As $d(x, F_k) \geq \epsilon$, so, we can conclude that $f(x) \geq \epsilon/n$. It is to be noted that $\epsilon > 0$. Therefore, we can conclude that this $f(x) > 0$, and if $f(x) > 0$, we can conclude that $\lambda > 0$. So, we have proved this first part.

Now, let us try to justify that the open ball centered at x and radius λ is included in some of the members of open cover $\mathcal{C}$. Actually, we can justify that for every $x \in X$; $B(x, \epsilon) \subseteq G_m$, for some $m \in \{1, 2, ..., n\}$. The question is, how is it possible? Let us take $d(x, F_m)$, which is a maximum of these $d(x, F_i)$'s, where $1 \leq i \leq n$. Now, what can we do? Let us see this $f(x)$, which is always greater than or equal to λ, and this is nothing but $(1/n) \sum_{i=1}^{n} d(x, F_i) \leq 1/n \sum_{i=1}^{n} d(x, F_m)$, and if we are finding the sum, this is precisely this $d(x, F_m)$, that is we found two bounds for this $f(x)$, that is $f(x) \geq \lambda$, and this is less than or equal to this is $d(x, F_m)$. From here, we can claim that $B(x, \lambda) \subseteq G_m$. Because, if possible, there exists some $y \in B(x, \lambda)$ such that y is not a member of G_m, what will happen? Note that F_m is a complement of G_m, so $y \in F_m$, and if $y \in F_m$, we can conclude that $d(x, y) \geq \lambda$. But at the same time, it is to be noted that what y is? y is a member of this open ball, that is, this $B(x, \lambda)$, and if this is a member of this open ball, we can conclude that $d(x, y) < \lambda$. So, we reach a contradiction. Hence, what do we conclude that $B(x, \lambda) \subseteq G_m$, and what is this G_m? G_m is a member of this open cover, which is proof of this well-known Lebesgue number lemma. It is to be noted here that this lambda also has a special name. Actually, this lambda is called the Lebesgue number of this open cover; what we have taken here, is $\mathcal{C}$.

Now, let us use the concept of this Lebesgue number lemma to discuss the uniform continuity in metric spaces. What we can justify is that if met-

ric spaces are compact, the continuous function turns out to be a uniform continuous function. Before coming to this concept, let us recall the concept of uniform continuity, that is, a function f from a metric space (X, d) to the metric space (Y, d') is said to be uniformly continuous if for a given $\epsilon > 0$ there exists a $\delta > 0$ such that for every pair of points $x_1, x_2 \in X$, $d(x_1, x_2) < \delta \Rightarrow d'(f(x_1), f(x_2)) < \epsilon$. Having this idea in mind, let us prove a simple result, which is the relationship between continuous functions. It states that if we have two metric spaces (X, d), and (Y, d'), and these are compact, then the continuous function $f : X \to Y$ is uniformly continuous.

In order to prove this result, let us take or begin with this continuous function $f : X \to Y$. Also, it is to be noted that X and Y both are compact, and being metric spaces, we can talk about the metric topology. So, let us take an open cover of this Y. So, what are we taking? We are taking $\{B(y, \epsilon/2) : y \in Y\}$ as an open cover of Y. If this is an open cover of Y, $\mathcal{C} = \{f^{-1}(B(y, \epsilon/2)) : y \in Y\}$ is an open cover of X. If this is an open cover of X, let us take the Lebesgue number, that is, λ of this open cover $\mathcal{C}$. So, if we are taking this Lebesgue number as λ, corresponding to this $\epsilon > 0$, we can say that there is a δ, and this δ is nothing but this Lebesgue number. Let us take this $\delta = \lambda$. Now, we are taking any two elements x_1 and x_2 in X such that the $d(x_1, x_2) < \delta$. Then what will happen? This two points set $\{x_1, x_2\}$ will be a member of an element of this open cover. So, $\{x_1, x_2\} \subseteq f^{-1}(B(y, \epsilon/2))$, or $f(\{x_1, x_2\}) \subseteq B(y, \epsilon/2)$, that is, $\{f(x_1), f(x_2)\} \subseteq B(y, \epsilon/2)$, or we can conclude that this $d'(f(x_1), f(x_2)) < \epsilon$. So, whenever $d(x_1, x_2) < \delta$, this $d'(f(x_1), f(x_2)) < \epsilon$. What exactly we have taken is that for any arbitrary $\epsilon > 0$, this δ is nothing but the Lebesgue number of the open cover that we have constructed here. Therefore, the continuous function f is uniformly continuous.

These are the references.

That's all from this lecture. Thank you.