**Course Name: Essentials of Topology**
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Welcome to Lecture 54 on Essentials of Topology.

Continuing with the concept of compactness, in this lecture, we will study the extreme value theorem along with some results in metric spaces. Begin with what this extreme value theorem is. Actually, one can say that this theorem is a topologically based theorem used in calculus. It is to be noted that we had already studied one more theorem when we studied the concept of connectedness, the intermediate value theorem; this theorem is for continuous real-valued functions on a connected domain. The question is how this extreme value theorem differs from the intermediate value theorem. Actually, the extreme value theorem is for continuous real-valued function, but instead of a connected domain, here the domain is compact. Let us see the theorem.

The proof of this theorem is based on this lemma. So, let us first prove this lemma. What does this lemma state? It states that if $A$ is a compact subset of the set of real numbers $\mathbb{R}$, then there exist $m, M \in A$ such that $m \leq a \leq M$, for all $a \in A$. What will we do? Let us take $A \subseteq \mathbb{R}$. Given that $A$ is compact. Now, our motive is to show the existence of two real numbers, $m$, and $M$. So, what are we going to show? We are going to show the existence of this $M \in A$ only, and in the same way, one can talk about the existence of $m \in A$. Now, if $A$ is a compact subset of $\mathbb{R}$, what can we conclude? From here, we can conclude two things, that is, $A$ is closed and $A$ is bounded, which we have already studied in the previous lecture. Now, if $A$ is bounded, we can conclude $A$ is bounded above, and if it is bounded above, what we can say is that the least upper bound, that is, $l.u.b.$, exists. So, let us take this $M$ as the least upper bound of $A$.

Now, our motive is to justify that $M \in A$. Let us see it by contradiction. So, what we are assuming is that, if possible, let us take that $M$ is not a member of $A$. Now, if $M$ is not a member of $A$, we can conclude that $M$ is not a limit point of $A$. Why? Because $A$ is closed, we know that if $A$ is closed, $A' \subseteq A$. If $M$ is not an element of $A$, what can we conclude from here? In that case, there exists an open set, or even let us take there exists $\epsilon > 0$ such that we can talk about this interval, that is, $(M - \epsilon, M + \epsilon)$, containing $M$ such that $(M - \epsilon, M + \epsilon) \cap A = \emptyset$. From here, what can we conclude? We can conclude that this $M - \epsilon/2$ is also an upper bound of $A$. It is to be noted that this $M - \epsilon/2 < M$, and what is this $M$? That is the least upper bound. So, we reached a contradiction. Therefore, $M$ will always be an element of $A$. As stated earlier, we can show that $m$ is also an element of $A$.

Moving ahead, let us see the general version of the extreme value theorem. This theorem is stated as: let $(X, \mathcal{T})$ be a compact topological space and $f : X \to \mathbb{R}$ be a continuous function. Then $f$ takes on a maximum value and a minimum value on $X$, i.e.,

$$\text{there exists } a, b \in X \text{ such that } f(a) \leq f(x) \leq f(b), \text{ for all } x \in X.$$

Let us prove this theorem. Note that the function $f : X \to \mathbb{R}$ is continuous. Also, what this $X$ is? This space is compact. So, we are assuming that this is compact, and when we are talking about the set of real numbers, this $\mathbb{R}$ is endowed with standard topology. Now, if $X$ is compact, we can say that $f(X) \subseteq \mathbb{R}$ is also compact. Now, if this is compact, then what property will $f(X)$ have? Just see this lemma. If $f(X)$ is a compact subset of $\mathbb{R}$, there exist $m, M \in f(X)$ such that $m \leq f(x) \leq M$, for all $x \in X$. But it is to be noted that $m$ and $M$, both are members of $f(X)$. If both are the members of $f(X)$, we can conclude that there exist $a, b \in X$ such that $m = f(a)$ and $M = f(b)$. So, we can conclude from here that $f(a) \leq f(x) \leq f(b)$, for all $x \in X$. It is to be noted that what $a$ and $b$ are? $a$ and $b$ are members of $X$, that is the proof of this extreme value theorem.

Now, what is its counterpart in calculus, that is, the extreme value theorem on a closed interval $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $f$ takes on a maximum value and a minimum value on $[a, b]$. This is a simple to justify from the previous theorem. If we replace $X$ by $[a, b]$, note that what we have taken there. We have taken this $X$ as a compact space. Note that $[a, b]$ is also a compact subset of $\mathbb{R}$. So, it follows from the theorem which we have proved for a compact topological space. Even from here, one question arises: if we are taking this function $f : [a, b] \to \mathbb{R}$, can we make a guess on the image of this closed interval $[a, b]$ under $f$? What can we justify? We can justify that the image of this closed interval $[a, b]$ under $f$, that will be some closed interval $[c, d]$, i.e. $f([a, b]) = [c, d]$, where $c$ and $d$ are some real numbers. The question is how? The answer is simple. Just see the nature of this interval. Note that $[a, b]$ is connected and compact. If this interval is connected, it guarantees that the image of $[a, b]$ is connected, and this is a connected subset of $\mathbb{R}$. We know that the connected subsets of $\mathbb{R}$, are intervals, which means that $f([a, b])$, is an interval. But at the same time, if we are using the power of this closed interval, because it is compact what we can conclude that $f([a, b])$ is compact too. If this is compact, meaning is, this is a closed and bounded set, and the closed and bounded set should be an interval, meaning that this $f([a, b])$ is a closed interval, or we can say that $f([a, b])$ is a closed and bounded interval, that's the proof of this result that the image of a closed and bounded interval $[a, b]$ is a closed and bounded interval $[c, d]$, for some real numbers $c$ and $d$.

Moving ahead, let us use the notion of compactness in the case of metric spaces. For that, we require some results or basic concepts from metric spaces. The first one that we require is the notion of distance between two subsets of $X$. So, what is stated here, that is if we are having a metric space $(X, d)$, and two nonempty subsets $A, B \subseteq X$, the distance between $A$ and $B$ is $d(A, B)$, given by $d(A, B) = inf\{d(a, b) : a \in A, b \in B\}$. For example, if we are talking about the set of real numbers with $d$ as a discrete metric. This is the simplest example. If we are taking two distinct real numbers $x, y \in \mathbb{R}$ and if we are looking for what is $d(\{x\}, \{y\})$, it is nothing but $d(x, y)$, and note that $x$ and $y$ are not equal, so $d(x, y) = 1$. The question is, if we are taking $A$ and $B$, two subsets of $\mathbb{R}$, and if $A \cap B = \emptyset$ , what about this $d(A, B)$? Just think about it. In case, if $A$ and $B$ are not disjoint sets, what will be the distance between these two sets? Just think about it.

Moving ahead, here we are taking two sets, $A$ and $B$. Now, let us take $A = \{x\}$, that is, $x$ is an element of $A$. Then $d(\{x\}, B) = inf\{d(x, b) : b \in B\}$. Even, we can also denote this one by $d(x, B)$, and we call this distance of a point $x$ from the set $B$. Now, let us see a result that we have to use, and the concept that we have already studied is the notion of metric, which is the distance function. What we can justify is that if we are taking any metric space $(X, d)$, then the distance function $d : X \times X \to \mathbb{R}$ is continuous. When we are talking about the continuity of this function, note that this $\mathbb{R}$ is equipped with this Euclidean topology. When $(X, d)$ is a metric space, we can talk about the metric topology on $X$. What about this $X \times X$? $X \times X$ is equipped with the product topology induced by the metric topology on $X$. In order to prove that this function is continuous, what do we have to show? We have to show that for each $G$ open in $\mathbb{R}$, $d^{-1}(G)$ is an open subset of $X \times X$. Begin with, if $G$ is an open subset of $\mathbb{R}$, and $d^{-1}(G)$ is an empty set, the proof is trivial. If $d^{-1}(G)$ is non-empty, let us take an element here, $(x, y) \in d^{-1}(G)$. So, what will happen? We can say that $d(x, y) \in G$. Now, let us take $d(x, y) = a$. So, what is here, that is $a \in G$. Note that what this $G$ is? $G$ is an open subset of $\mathbb{R}$. So, we can say that there exists $\epsilon > 0$ such that $a \in (a-\epsilon, a+\epsilon) \subseteq G$. Now, in order to show that $d^{-1}(G)$ is an open subset of $X \times X$, what exactly do we have to show? We have to show that if we are taking an arbitrary element $(x, y) \in d^{-1}(G)$, there should exist something in between $(x, y)$ and $d^{-1}(G)$, and whatever something is coming here, let us take this is $H$, this should be open in $X \times X$. The question is, how do we find out this $H$?

Let us use this $\epsilon$ and use the product topology on $X \times X$. Now, if we are taking open balls, let us take two open balls, one is $B(x, \epsilon/2)$ centered at $x$ and radius $\epsilon/2$, and also let us take another open ball, that is $B(y, \epsilon/2)$ centered at $y$ and radius $\epsilon/2$. It is to be noted that these are open in metric topology. Therefore, $B(x, \epsilon/2) \times B(y, \epsilon/2)$ is open in $X \times X$. Also, it is to be noted that this is containing $(x, y)$. The question is whether $B(x, \epsilon/2) \times B(y, \epsilon/2)$ is precisely $H \subseteq d^{-1}(G)$. Let us see it. For it, let us take $(x', y') \in B(x, \epsilon/2) \times B(y, \epsilon/2)$. What will happen? This $x'$ is a member of $B(x, \epsilon/2)$, and what about this $y'$, $y'$ is the member of $B(y, \epsilon/2)$. From here, $d(x', x) < \epsilon/2$ and $d(y', y) < \epsilon/2$. Now, if we are computing $|d(x', y') - d(x, y)|$, then $|d(x', y') - d(x, y)| \leq d(x', x) + d(y', y) < \epsilon/2 + \epsilon/2$, that is $\epsilon$. Thus, $|d(x', y') - a| < \epsilon$, or $d(x', y') \in (a - \epsilon, a + \epsilon)$, and what we have seen that $(a - \epsilon, a + \epsilon) \subseteq G$, so, $d(x', y') \in G$, or we can say that this $(x', y')$ is a member of $d^{-1}(G)$. Thus what we have shown that $(x, y) \in B(x, \epsilon/2) \times B(y, \epsilon/2) \subseteq d^{-1}(G)$. Therefore, $d^{-1}(G)$ is open in $X \times X$, or that $d$ is a continuous function.

With these two concepts, let us move ahead and what we can show that if $(X, d)$ is any metric space and if $A$ and $B$ are disjoint compact subsets of $X$, then $d(A, B) > 0$. In order to prove it, let us use the concept that we have studied, and also, we will use the concept of the extreme value theorem. What is given to us is that $A$ and $B$ are disjoint compact subsets of

$X$. Because these are compact, we can say that $A \times B \subseteq X \times X$, is also compact. Now, $A \times B$ is compact, and we have shown that the function $d$ is continuous, so we can use the concept of the extreme value theorem. Let us use this theorem. So, by the extreme value theorem, there exist $a^* \in A$ and $b^* \in B$ such that $d(a^*, b^*) \leq d(a, b)$, for all $a \in A$, and $b \in B$. Note that $d(A, B) = inf\{d(a, b) : a \in A, b \in B\}$. From here, we can conclude that $d(A, B) = d(a^*, b^*)$. It is to be noted that $A$ and $B$ are disjoint, and what $a^*$ is? $a^* \in A$, and $b^* \in B$, meaning is, $a^* \neq b^*$, and if this is the case, it is to be noted that $d(a^*, b^*) \neq 0$. Therefore, what we can conclude from here, that is $d(A, B) > 0$, that is there is a positive distance between two disjoint compact subsets of $X$.

These are the references.

That's all from this lecture. Thank you.