

Course Name: Essentials of Topology
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Week: 09
Lecture: 04

Welcome to Lecture 52 on Essentials of Topology.

Continuing with the concept of compactness, in this lecture, we will study a characterization of compactness in terms of closed sets. Further, we will also study the finite product of compact topological spaces.

Begin with what exactly we will use to characterize the concept of compactness in terms of closed sets. We have already studied that if we are taking a compact topological space (X, \mathcal{T}) , what compactness means that every open cover of X has a finite subcover. So, let us take an open cover $\mathcal{C} = \{G_i : i \in I\}$, where $G_i \in \mathcal{T}$, of X . If (X, \mathcal{T}) is compact, one can construct a finite subcover of \mathcal{C} . What does it mean that if $X = \cup\{G_i : i \in I\}$, finite subcover says that X can also be expressed as $G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_n}$. Now,

- $X = \cup\{G_i : i \in I\} \Rightarrow X^c = \cap\{G_i^c : i \in I\} \Rightarrow \cap\{G_i^c : i \in I\} = \emptyset,$
- $X = G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_n} \Rightarrow X^c = G_{i_1}^c \cap G_{i_2}^c \cap \dots \cap G_{i_n}^c \Rightarrow G_{i_1}^c \cap G_{i_2}^c \cap \dots \cap G_{i_n}^c = \emptyset.$

Again, what \mathcal{C} is? This is an open cover. If every G_i is an open set, from here, we can construct a family of closed sets, that is $\{G_i^c : i \in I\}$. What this G_i^c is? This is a closed set. What are the properties associated with this family? Note that in this family, $\cap\{G_i^c : i \in I\} = \emptyset$, and the intersection of finite members of this family is also empty. Let us think differently. If we are looking in a different way, that is, if we are taking the intersection of finite members as nonempty, whether the intersection of all members will also be nonempty. This is the idea, and by using this idea, we will characterize the concept of compactness.

By using the concept of closed sets, if we are taking a family of closed sets and if the finite intersection of members of \mathcal{F} is nonempty, it implies that

the intersection of members of \mathcal{F} is also nonempty, compactness can be characterized. So, if the finite intersection of members is nonempty, does such a family exist? This concept is known as finite intersection property. What exactly is this property? A collection \mathcal{F} of subsets of a nonempty set X is said to have the finite intersection property (FIP) if for every finite subcollection $\{F_1, F_2, \dots, F_n\}$ of \mathcal{F} , $F_1 \cap F_2 \cap \dots \cap F_n$ is nonempty.

Let us take the first example. Take this X as the set of integers, and $\mathcal{F} = \{F_n : n \in \mathbb{Z}\}$. What F_n is? F_n is $\{n\}$, if n is odd and $\{n-1, n, n+1\}$, if n is even. Now, if we are taking two sets, for example, F_3 and F_7 , what F_3 is? That is $\{3\}$. What F_7 is? That is nothing but $\{7\}$. What $F_3 \cap F_7$ is? That is an emptyset. Meaning is, \mathcal{F} does not possess finite intersection property. Moving ahead, if we are taking another example, let us take X as the set of natural numbers and let us take $\mathcal{F} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \dots\}$. Note that this collection has FIP, that is the finite intersection property. Why? Because if we are taking any finite number of members from \mathcal{F} and if we are taking their intersection, that will be nonempty. So, what have we seen? We have taken two examples. In between these two examples, the second family has FIP.

Moving ahead, let us use this concept to characterize the compactness of a topological space by using the concept of closed sets. The statement is: if we are taking a topological space (X, \mathcal{T}) , then this space is compact if and only if every collection \mathcal{F} of closed sets in (X, \mathcal{T}) having the finite intersection property, the intersection $\bigcap_{F \in \mathcal{F}} F$ of all elements of \mathcal{F} is nonempty. In order to prove it, let us assume that (X, \mathcal{T}) is compact. Also, take a family $\mathcal{F} = \{F_i : i \in I\}$, where F_i is \mathcal{T} -closed and \mathcal{F} has FIP. Now, what exactly do we have to show? We have to show that $\bigcap\{F_i : i \in I\}$ is nonempty. Let us see the proof by contradiction. If possible, let us assume that $\bigcap\{F_i : i \in I\} = \emptyset$. Then $\bigcup\{F_i^c : i \in I\} = X$. Note that every F_i is \mathcal{T} closed, and if F_i is \mathcal{T} -closed, what about F_i^c ? This is \mathcal{T} -open. So, from here, we conclude that $\mathcal{C} = \{F_i^c : i \in I\}$ is an open cover of X . If this is an open cover of X , it is already given that (X, \mathcal{T}) is compact. So, there exists a finite subcover of \mathcal{C} , that is, X can be expressed as $F_{i_1}^c \cup F_{i_2}^c \cup \dots \cup F_{i_n}^c$. Thus $\mathcal{C}' = \{F_{i_1}^c, F_{i_2}^c, \dots, F_{i_n}^c\}$ is a subcover of \mathcal{C} . Now, if $X = F_{i_1}^c \cup F_{i_2}^c \cup \dots \cup F_{i_n}^c$, then $F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_n} = \emptyset$, this is a contradiction as \mathcal{F} has FIP. Hence we conclude that $\bigcap\{F_i : i \in I\} \neq \emptyset$. This is the proof of the first part.

Let us see the converse of this theorem. We are assuming that every collection \mathcal{F} of closed sets in (X, \mathcal{T}) having the finite intersection property, the intersection $\bigcap_{F \in \mathcal{F}} F$ of all elements of \mathcal{F} is nonempty, and we are going to prove that (X, \mathcal{T}) is compact. Again, let us assume that, if possible, this (X, \mathcal{T}) , let us take this is not compact. What does it mean? If (X, \mathcal{T}) is not compact, it means that there exists an open cover of X having no finite subcover. So, let us take that open cover as $\mathcal{C} = \{G_i : i \in I\}$. Note that G_i is \mathcal{T} -open, and \mathcal{C} is having no finite subcover. What does it mean? If \mathcal{C} has no finite subcover, it means that X can be expressed as $\bigcup\{G_i : i \in I\}$, but if we are taking any finite family from \mathcal{C} , let us take that as $\{G_{i_1}, G_{i_2}, \dots, G_{i_n}\}$, their union cannot be X . So, $G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_n} \subset X$. Thus, $G_{i_1}^c \cap G_{i_2}^c \cap \dots \cap G_{i_n}^c \supset \emptyset$, or that $G_{i_1}^c \cap G_{i_2}^c \cap \dots \cap G_{i_n}^c \neq \emptyset$. Therefore, \mathcal{F} is a family of closed sets having FIP. If this is having FIP, then from the assumption $\bigcap\{G_i^c : i \in I\} \neq \emptyset$. If this is nonempty, let us take the complement of both sides. So, what we can conclude is that $\bigcup\{G_i : i \in I\} \neq X$. See what it justifies. It is contradicting the fact that this family \mathcal{C} is a cover of X . So, we reach a contradiction. Therefore, our assumption is wrong, and hence, the topological space (X, \mathcal{T}) is compact.

By using this characterization, we can easily deduce this simple result. Let (X, \mathcal{T}) be a compact topological space, and let $F_1 \supseteq F_2 \dots \supseteq F_n \supseteq F_{n+1} \dots$ be a nested sequence of \mathcal{T} -closed sets. If each F_n is nonempty, then $\bigcap_{n \in \mathbb{N}} F_n$ is nonempty. It is to be noted what have we taken. We have taken (X, \mathcal{T}) as compact. Now, let us take a family $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$. It is given that F_n is nonempty. So, what we can deduce from here is that $\bigcap F_i$ will always be nonempty, $i = 1 \dots n$, or that this \mathcal{F} has FIP. Thus, (X, \mathcal{T}) is a compact topological space, and \mathcal{F} is a collection of closed sets having FIP. Therefore, we conclude that the $\bigcap_{n \in \mathbb{N}} F_n$ is nonempty.

Moving ahead, in the next, we will show that if we are taking two compact topological spaces, their product is, or the product space is also compact. But before proving that the product space is compact, we require a result known as the tube lemma. What is this lemma? Let us first prove this lemma, and after that, we will use this lemma to prove that the finite product of compact topological spaces is compact. The statement of the lemma is: Let (X, \mathcal{T}) and (Y, \mathcal{T}') be topological spaces, and let (Y, \mathcal{T}') be compact. If $x \in X$, and N be an open subset of $X \times Y$ containing $\{x\} \times Y$, then there exists an open subset G of X containing x such that $G \times Y \subseteq N$. Let us try to understand

the statement of this lemma. Take the Cartesian product of these two sets, X and Y . We are taking this as a set X and we are taking this as a set Y . So, this box is nothing but this is representing $X \times Y$. If we are taking an element $x \in X$, so let us take an element x here, that is, corresponding to this one. So, how does $\{x\} \times Y$ will, look like? This will look like this way (ref slides). This is the singleton set $\{x\} \times Y$. Now, let us take an open subset N of $X \times Y$ containing $\{x\} \times Y$. Then, this lemma says that there exists an open subset G of X containing x such that $G \times Y \subseteq N$. Meaning is, if we are finding this $G \times Y$, that is this slice, this slice will be contained in N . The shaded region is $G \times Y$, and that is known as a tube.

Now, in order to prove this lemma, let us begin with a set $N \subseteq X \times Y$, and this is open in the product topology. The topological spaces that we have with us are (X, \mathcal{T}) , and (Y, \mathcal{T}') . If we are taking the Cartesian product of X and Y , let us take the product topology on $X \times Y$ as \mathcal{T}'' . We know that this collection, that is $\mathcal{B} = \{G \times H : G \in \mathcal{T}, H \in \mathcal{T}'\}$, is a basis for the product topology on $X \times Y$. If this is a basis for product topology \mathcal{T}'' on $X \times Y$, it is to be noted that this $\{x\} \times Y \subseteq N$. If this is a subset of N , what can we conclude? We can conclude that for all $y \in Y$, there exist two open sets. One is, let us take $G_y \in \mathcal{T}$, and another is, let us take this is $H_y \in \mathcal{T}'$ such that by using the definition of basis, we can conclude that this ordered pair (x, y) is a member of $G_y \times H_y$, and this is a subset of N . Now, what construction have we taken? We are saying that for all $y \in Y$, there exists $H_y \in \mathcal{T}'$, and if this is for all $y \in Y$, why not let us take a collection $\mathcal{C} = \{H_y : y \in Y\}$. Note that $H_y \in \mathcal{T}'$. Now, this \mathcal{C} is an open cover of Y , but at the same time, it is to be noted that the space (Y, \mathcal{T}') is compact. So, by compactness, what can we conclude? We can conclude that \mathcal{C} has a finite subcover, say, $\{H_{y_1}, H_{y_2}, \dots, H_{y_n}\}$. If this is a finite subcover of \mathcal{C} , corresponding to H_{y_1} , we can find G_{y_1} . What have we seen here? So, corresponding to this H_{y_1} , we can find a G_{y_1} , corresponding to H_{y_2} , we can find G_{y_2} , and corresponding to this H_{y_n} , we can find G_{y_n} . Let us take $G_{y_1} \cap G_{y_2} \cap \dots \cap G_{y_n}$ as G . What the feature of G is? The first feature of G is that this will contain x because $G_{y_1}, G_{y_2}, \dots, G_{y_n}$, these all are containing x . Also, this G is open because this is a finite intersection of open sets. As per the lemma, our motive was to find an open subset G of X . So, we found an open subset G of X and noted that this contains x . What we further want to show is that $G \times Y \subseteq N$. Now, let us compute this $G \times Y$. Let us take $G_{y_i} \times H_{y_i}$, and take their union, where

$i = 1, 2, \dots, n$. We are taking the Cartesian product of G_{y_i} and H_{y_i} . Note that such a Cartesian product is always a subset of N , and therefore, their union is also a subset of N . Further, this will always contain this $G \times Y$. That's the proof of the tube lemma.

Now, let us prove the theorem regarding the product of compact topological spaces. This theorem states that if (X, \mathcal{T}) , and (Y, \mathcal{T}') are two compact topological spaces, then the product space, $X \times Y$ with this product topology, that is \mathcal{T}^* , is compact. In order to prove it, let us begin with an open cover of $X \times Y$. So, what are we taking? Let us assume that this $\mathcal{C} = \{G_i : i \in I\}$ is an open cover of $X \times Y$, and our motive is to find a finite subcover of it. In order to find a finite subcover, let us begin with this construction. What we are going to do is let us take any element x of X . What can we do, or what can we visualize that if we are taking this $\{x\} \times Y$, note that this set is homeomorphic to Y . If this is homeomorphic, note that this (Y, \mathcal{T}') is compact. So, what this $x \times Y$ is? This is a compact subset of $X \times Y$. If this is a compact subset of $X \times Y$, we can say \mathcal{C} is an open cover of $\{x\} \times Y$. If \mathcal{C} is an open cover of $\{x\} \times Y$, we can have a finite subcover. So, let us denote the finite subcover of \mathcal{C} , by \mathcal{C}_x , just we are assuming it. Also, let us take a set N_x , which is a union of members of this \mathcal{C}_x . Then, N_x is open in product space. Also, $\{x\} \times Y$ will be a subset of N_x . So, what do we have? We have an open set in the product space containing $\{x\} \times Y$. Therefore, we can now use the tube lemma. By using the Tube lemma, what can we get? There exists an open set $G_x \in \mathcal{T}$, and this G_x is containing x such that $G_x \times Y \subseteq N_x$. So, from here, we can conclude that if we are taking a collection $\{G_x : x \in X\}$ as \mathcal{C}^* . Then what this \mathcal{C}^* is? \mathcal{C}^* is an open cover of X , and if this is an open cover of X , what can we do by using the compactness of the topological space (X, \mathcal{T}) ? As (X, \mathcal{T}) is compact, $X = G_{x_1} \cup G_{x_2} \cup \dots \cup G_{x_k}$. It is to be noted that $G_{x_1}, G_{x_2}, \dots, G_{x_k}$, these all are members of \mathcal{C}^* . Further, what can we conclude about the $G_{x_1}, G_{x_2}, \dots, G_{x_k}$? If we are looking at this structure that $G_x \times Y \subseteq N_x$, we can write that this $G_{x_1} \times Y \subseteq N_{x_1}, G_{x_2} \times Y \subseteq N_{x_2}, \dots$ this $G_{x_k} \times Y \subseteq N_{x_k}$. From here, we can write that this $X \times Y$ can be expressed as a union of $G_{x_i} \times Y$, where i is from 1 to k . Note that this union will always be a subset of $N_{x_1} \cup N_{x_2} \cup \dots \cup N_{x_k}$. But it is to be noted that what N_x is, N_x is nothing but the union of members of \mathcal{C}_x . So, we can say that this is a finite union of members of \mathcal{C}_x because \mathcal{C}_x itself is finite. Thus, N_{x_1} is the union of members of $\mathcal{C}_{x_1}, N_{x_2}$ is the union of members of $\mathcal{C}_{x_2}, \dots, N_{x_k}$ is the

union of members of \mathcal{C}_{x_k} . So, if we are taking $\mathcal{C}' = \mathcal{C}_{x_1} \cup \mathcal{C}_{x_2} \cup \dots \cup \mathcal{C}_{x_k}$, note that $\mathcal{C}_{x_1}, \mathcal{C}_{x_2}, \dots, \mathcal{C}_{x_k}$ are finite. Thus \mathcal{C}' is finite. Further, the union of members of this \mathcal{C}' is equal to $X \times Y$. Thus, we can say that this \mathcal{C}' is a finite subcover of \mathcal{C} . Hence, $X \times Y$ with this product topology, \mathcal{T}^* is compact. This result can be generalized, and we can say that the product of finitely many compact topological spaces is compact.

These are the references.

That's all from this lecture. Thank you.