Course Name: Essentials of Topology
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Welcome to Lecture 51 on Essentials of Topology.

In this lecture too, we will continue the study of the notion of compactness. Herein, we will study some concepts associated with compact topological spaces by using the notion of Hausdorff topological spaces. In the previous lecture, we have already seen that an arbitrary subspace of a compact topological space may not be compact. So, what is the precise answer that is given here? The answer is: every closed subspace of a compact topological space is compact. Let us see the proof of this result.

Begin with, let us take a compact topological space $(X, \mathcal{T})$. Also, let us take a subset $A \subseteq X$ such that this $(A, \mathcal{T}_A)$, which is obviously a subspace of $(X, \mathcal{T})$, and this A is closed. What our motive is, our motive is to show that A is compact, too. Now, in order to justify that $(A, \mathcal{T}_A)$ is compact, let us take an open cover $\mathcal{C} = \{G_i : i \in I\}$, $G_i \in \mathcal{T}$, of A. Now, if this is an open cover of A, we can write $A \subseteq \cup\{G_i : i \in I\}$. Also, as $X = A \cup A^c$, we can write this X as a union of two sets, that is, $(\cup\{G_i : i \in I\}) \cup A^c$. What about this A^c? Note that this is open. Why? Because A is closed. So, from here, what can we conclude? If we are taking this $\mathcal{C}^* = \mathcal{C} \cup \{A^c\}$, then what this $\mathcal{C}^*$ is? This is an open cover of X. If this is an open cover of X, by using the compactness of $(X, \mathcal{T})$, $\mathcal{C}^*$ is reducible to a finite subcover of $\mathcal{C}$, meaning is, there exist, $G_{i_1}, G_{i_2}, ..., G_{i_k} \in \mathcal{C}$ such that $X = (G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_k}) \cup A^c$. But it is to be noted that what A is? A is a subset of X, and also, $A \cap A^c$ is an empty set. Thus we can conclude that A is nothing but a subset of $G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_k}$, that is there exists a collection $\mathcal{C}' = \{G_{i_1}, G_{i_2}, ..., G_{i_k}\}$, and this is a finite subcover of $\mathcal{C}$. Thus, we have taken an arbitrary open cover of A, and we have shown that this open cover has a finite subcover. Therefore, $(A, \mathcal{T}_A)$ is compact.

The question is again, what about the converse of this result? Note that converse is not necessarily true, and the statement is: a compact set in a topological space is not necessarily closed. Why? The example is, let us take co-finite topology on the set of real numbers $\mathbb{R}$. If we are taking set A as the set of rationals. We have already studied that subspaces of a co-finite topological space are also co-finite. Because the topology on $\mathbb{Q}$ is co-finite, therefore, $\mathbb{Q}$ is also compact, as we have already seen that co-finite topological spaces are compact. But it is to be noted that this $\mathbb{Q}$ is not closed. Why? Because $\mathbb{Q}$ is not finite. The question will be, can we put some restrictions when the compact sets become closed? The answer is yes. Instead of taking an arbitrary topological space, if we are taking a topological space of particular type, and that is known as a Hausdorff topological space. In the case of Hausdorff topological spaces, we can show that a compact set is also closed.

Here, we are going to see some of the concepts associated with Hausdorff topological spaces. Specifically, we will just see the definition alongwith some of the examples and a single result on Hausdorff spaces. We will again discuss it when we will study the concept of separation axioms. Begin with what a Hausdorff topological space is. A topological space $(X, \mathcal{T})$ is called Hausdorff if, for every pair of distinct points $x, y \in X$, there exist open sets G and H such that one open set contains x, another open set contains y, and the open sets G and H are disjoint. It means whenever we are saying that a topological space $(X, \mathcal{T})$ is Hausdorff, let us take this is X, and if we are taking any two elements of this space, that is, this is x, and this is y, we say that this space is Hausdorff if we can construct two open sets, one is containing x, and the other is containing y, and they are disjoint too. It means that each point in a pair of points can be housed off from the other by using open sets, and the open sets should be disjoint; this is the easiest way to remember this concept.

Let us see some of the examples of Hausdorff topological spaces. Begin with, let us take the discrete topological space $(X, \mathcal{T})$. Now, if we are taking two distinct points x and y of X, then $x \in \{x\}$, $y \in \{y\}$ and $\{x\} \cap \{y\} = \emptyset$, where $\{x\}, \{y\} \in \mathcal{T}$. Therefore, discrete topological spaces are always Hausdorff. Moving to the next example, the set of real numbers with standard topology is also Hausdorff. How is it possible? The answer is here. Let us take $x, y \in \mathbb{R}$ be two distinct real numbers such that $|x - y| = \epsilon$. Then there exits two disjoint open sets $G = (x - \epsilon/2, x + \epsilon/2)$ and $H = (y - \epsilon/2, y + \epsilon/2)$ such that $x \in G$ and $y \in H$. So, $\mathbb{R}$ with standard topology is Hausdorff.

Moving ahead, we can justify that the set of real numbers with co-finite topology is not Hausdorff. Let us justify by contradiction. If possible, let $\mathbb{R}$ with co-finite topology be Hausdorff. Now, if this is Hausdorff, what will happen? Let us take $x, y \in \mathbb{R}$ be two distinct real numbers. Then there exist two open sets G and H such that $x \in G,\ y \in H$, with $G \cap H = \emptyset$. From here we can conclude $G \subseteq H^c$. It is to be noted that the topology is co-finite and if the topology is co-finite, H^c is a finite set, and if H^c is finite, what about G? G is also finite. Further, note that G is open, and if G is open, what about its complement? This G^c is closed, and if G^c is closed in co-finite topology, this G^c is finite. Now, what we finally have with us, that is, $\mathbb{R} = G \cup G^c$. Note that G and G^c, both are finite. It means that $\mathbb{R}$ is finite too, and this is a contradiction. Therefore, our assumption is wrong and hence $\mathbb{R}$ with co-finite topology is not Hausdorff.

Before returning back to the concept of compactness, let us see a simple result for Hausdorff spaces. This result is, if the topological space $(X, \mathcal{T})$ is Hausdorff, then every single point subset of X will always be closed. Let us see the justification. So, what we are assuming or what is given to us, let us take that this $(X, \mathcal{T})$ is Hausdorff. Our motive is to show that for all $x \in X$, the singleton set $\{x\}$ is closed, or we have to show that its complement is open. In order to show that $X - \{x\}$ is open, what we have to justify, let us show that this set is a neighborhood of each of its points. So, let us take $y \in X - \{x\}$. It means that x and y are two distinct points of X, and if x and y are two distinct points, by using the Hausdorffness, what we can conclude, there exist open sets G and H such that $x \in G,\ y \in H$ and $G \cap H = \emptyset$. It is to be noted from here that $H \subseteq G^c$, and therefore, x cannot be an element of H. From here, what can we conclude? We can conclude that $y \in H$, and as x is not an element of H, this is a subset of this $X - \{x\}$, that is, there exists an open set H such that $y \in H$, which is a subset of $X - \{x\}$, that is $X - \{x\}$ is a neighborhood of each of its points. If this is a neighborhood of each of its points, we can conclude that $X - \{x\}$ is open, or we can say that this $\{x\}$ is closed.

With these ideas in mind, let us come back to the concept of compactness and see the result regarding subspaces. So, we have a theorem here and this theorem states that every compact subset of a Hausdorff topological space is closed. In order to prove it, let us take this topological space $(X, \mathcal{T})$, this is Hausdorff, and let us take a subset $A \subseteq X$, which is compact. Our motive is to prove that this A is closed. If we want to show that A is closed, try to prove that its complement is open. Now, in order to justify that A^c is open, let us take an element in A^c. So, let us take some $x \in X - A$, and what we have to show that A^c is a neighborhood of x. It means that we have to show that there exits an open set, let us take that open set G such that this $x \in G$, which is a subset of $X - A$. The question is, how do we justify it? Note that we have chosen x as an element of $X - A$. Now, let us take an arbitrary element $a \in A$. What will happen? Note that x and a, both are the elements of X, and they are different too. If they are different elements, by using the Hausdorffness of the topological space, we can conclude that there exist open sets G_a and H_a such that $x \in G_a$ and $a \in H_a$, and $G_a \cap H_a = \emptyset$. Again, for $b \in A$, we can find open sets G_b and H_b such that $x \in G_b$, $a \in H_b$, and $G_b \cap H_b = \emptyset$. Having this idea in mind, what can we do? If we are taking the collection $\mathcal{C} = \{H_a : a \in A\}$. Then $\mathcal{C}$ is an open cover of A.

Now, if $\mathcal{C}$ is an open cover of A, as this A is compact, and if this A is compact, we can find some finite elements in this cover. So, by using the compactness of A, we can deduce that there exist $H_{a_1}, H_{a_2}, ..., H_{a_k} \in \mathcal{C}$ such that $A \subseteq H_{a_1} \cup H_{a_2} \cup ... \cup H_{a_k}$. Let us take this union as H. It is to be noted that corresponding to every H_{a_i} there is a G_{a_i} because that is our construction. So, what we can do that corresponding to H_{a_1}, let us take a set G_{a_1}, also corresponding to the H_{a_2}, let us take a set G_{a_2}, corresponding to this H_{a_k} take the set G_{a_k}, and instead of taking their union, we are taking their intersection, and we are saying that this set is $G = G_{a_1} \cap G_{a_2} \cap ... \cap G_{a_k}$. What these H_{a_i} and G_{a_i} are? Actually, this $H_{a_i} \cap G_{a_i} = \emptyset$. Now, just see the structure of G and H. What H is? H is a finite union of open sets. So, this H is open. Also, what this G is? G is open too. Further, G will contain x because x is an element of $G_{a_1}, G_{a_2}, ..., G_{a_k}$. The question is, what about this $G \cap H$, whether it will be an empty set? The answer is yes. This will always be an empty set. Why? For example, if we are looking at structure $G \cap H$, this is nothing but $(G_{a_1} \cap, G_{a_2} \cap ... \cap G_{a_k}) \cap (H_{a_1} \cup H_{a_2} \cup ... \cup H_{a_k})$. If we are simplifying it, what will be the first term of $G \cap H$, that is nothing but $(G_{a_1} \cap G_{a_2} \cap ... \cap G_{a_k}) \cap H_{a_1}$. Similarly, now let us take union and try to get the rest of the elements. What will happen? If we are computing $(G_{a_1} \cap G_{a_2} \cap ... \cap G_{a_k}) \cap H_{a_1}$, it is to be noted that this $H_{a_i} \cap G_{a_i} = \emptyset$; therefore, $H_{a_1} \cap G_{a_1} = \emptyset$. Therefore, $(G_{a_1} \cap G_{a_2} \cap ... \cap G_{a_k}) \cap H_{a_1}$ will be an empty set. Similarly, other intersections too. Thus, their union will always be an empty set. Finally, what we have with us, that is $G \cap H = \emptyset$ but it is to be noted that what H is? This H is a superset of A, so if $G \cap H$ is an empty set, we can conclude that this $G \cap A$ is also an empty set, or $G \subseteq A^c$, that is $X - A$. Also, as $x \in G$, we reach to the conclusion, that is, for each element x of this $X - A$, there exists an open set G with the property that $x \in G \subseteq A^c$. Therefore, this $X - A$ is a neighborhood of x, and if this is a neighborhood of x, we conclude that this $X - A$ is open or A is closed. So, finally, we reached our goal from where we had started. Our motive was whether compact subspaces may be closed for general topological spaces. The answer was negative, but if we are taking Hausdorff topological spaces, in this case, compact subspaces become closed.

Moving ahead, let us see this theorem. This is interesting one and this theorem states that when a bijective continuous function can be a homeomorphism. The statement of the theorem is: let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a bijective continuous function. Further, let $(X, \mathcal{T})$ be compact and $(Y, \mathcal{T}')$ be Hausdorff. Then f is a homeomorphism. If we want to show that the function f is a homeomorphism, it is sufficient to justify that f is either closed or open or the inverse is continuous. So, what we are going to justify is that f is a closed map, so just show it. Now, if we want to show that f is closed what to do? Let us take a subset $F \subseteq X$ and what we are assuming that this is closed. Note that this is a closed subset of a compact space. Therefore, this will be compact, too. Now, what about the continuous image of a compact set. If F is compact, $f(F)$ is also compact, and this is compact in this $(Y, \mathcal{T}')$, but note that what $(Y, \mathcal{T}')$ is Hausdorff. If $(Y, \mathcal{T}')$ is Hausdorff, and $f(F)$ is a compact subset of Y, what about this $f(F)$? This will be closed. So, the image of the closed subset F of X is a closed subset of Y. Therefore, f is a closed map, and hence, f is a homeomorphism. Let's stop here.

These are the references.

That's all from this lecture. Thank you.