Course Name: Essentials of Topology
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Welcome to Lecture 50 on Essentials of Topology.

In the previous lecture, we studied the concept of an open cover and the compactness of a topological space. In this lecture, our focus will be on the study of the compactness of subspaces and continuous images of compact topological spaces.

Starting with, let us discuss the compactness of a subset of X, that is, let $(X, \mathcal{T})$ be a topological space, and let us take a subset A of X. We say that A is compact in the topological space $(X, \mathcal{T})$ if $(A, \mathcal{T}_A)$ is compact. Meaning is to say that if we are taking an open cover $\mathcal{C} = \{H_i : i \in I\}$ of A, where $H_i \in \mathcal{T}_A$, and if we can find some members in this open cover $\mathcal{C}$, that is, $H_{i_1}, H_{i_2}, ..., H_{i_k} \in \mathcal{C}$ such that $A \subseteq H_{i_1} \cup H_{i_2} \cup ... \cup H_{i_k}$, then we say that $(A, \mathcal{T}_A)$ is compact.

Whenever we are discussing about the compactness of $A \subseteq X$, note that we are using the relative topology on A. A natural question arises: can we define compactness by taking an open cover, where the open sets are members of the topology $\mathcal{T}$ on X. That is, instead of taking these H_i from this $\mathcal{T}_A$, if we are taking H_i from $\mathcal{T}$, can we discuss the compactness of this A? The answer is yes. So, before taking any example, let us prove this result, and after that, we will take some of the examples. What this result says, this result says that if we are taking a topological space $(X, \mathcal{T})$ and $A \subseteq X$, then A is compact in topological space $(X, \mathcal{T})$ if and only if every $\mathcal{T}$-open cover of A has a finite subcover. Let us prove one by one. So, first, we are assuming that A is compact in $(X, \mathcal{T})$, and when we are assuming that A is compact in $(X, \mathcal{T})$, meaning, we are saying that $(A, \mathcal{T}_A)$ is compact. What is our motive to justify? We have to show every open cover and note that here, the open sets are with respect to topology $\mathcal{T}$, has a finite subcover. In order to show it, let us take a $\mathcal{T}$-open cover of A, that is, begin with this $\mathcal{C} = \{G_i : i \in I\}$, $G_i \in \mathcal{T}$, and what we are taking that $A \subseteq \cup\{G_i : i \in I\}$. What is our motive?

We have to justify that it has a finite subcover. Note that when we have taken $\mathcal{C}$ as an open cover with respect to topology $\mathcal{T}$, we know that if we are taking $A \cap G_i$, this $A \cap G_i \in \mathcal{T}_A$. So, let us take another set $\mathcal{C}^* = \{A \cap G_i : G_i \in \mathcal{C}\}$. Trivially, this A can also be expressed as the union of these $A \cap G_i$, where $G_i \in \mathcal{T}$, and this is possible because $A \subseteq \cup\{G_i : i \in I\}$. But it is to be noted that this A is compact in $(X, \mathcal{T})$; that is, $(A, \mathcal{T}_A)$ is compact. Therefore, $\mathcal{C}^*$ is reducible to a finite subcover. That is, we can find some members in $\mathcal{C}^*$, and those members are, let us take $A \cap G_{i_1}$, $A \cap G_{i_2}$,..., $A \cap G_{i_k}$, these are in C^* such that $A \subseteq (A \cap G_{i_1}) \cup (A \cap G_{i_2}) \cup ... \cup (A \cap G_{i_k})$. From here, we conclude that $A \subseteq G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_k}$. So, what we have shown is that if $\mathcal{C}$ is an open cover of A, we can find a finite subcover of it, and that subcover is nothing, but this is $\{G_{i_1}, G_{i_2}, ..., G_{i_k}\}$. Thus, every $\mathcal{T}$-open cover of A has a finite subcover.

Let us see the converse of this result. What we have to do is let us assume that every $\mathcal{T}$-open cover of A has a finite subcover, and prove that A is compact in $(X, \mathcal{T})$; that is, our motive is to prove that $(A, \mathcal{T}_A)$ is compact. In order to justify it, we have to show that every open cover has a finite subcover and that openness is defined with respect to the relative topology $\mathcal{T}_A$. So, let us take an open cover of A. We are taking this as $\mathcal{C} = \{H_i : i \in I\}$, $H_i \in \mathcal{T}_A$. Note that this H_i can be written as $A \cap G_i$, where $G_i \in \mathcal{T}$. Now, if A is a subset of $\cup\{H_i : i \in I\}$, obviously, $A \subseteq \cup\{G_i : i \in I\}$. Note that $G_i \in \mathcal{T}$. Meaning is to say that if we are taking this collection $\mathcal{C}^* = \{G_i : i \in I\}, A \cap G_i \in \mathcal{T}_A$, this is an open cover of A, and if this is an open cover of A, what is given to us that every $\mathcal{T}$-open cover of A has a finite subcover. So, from here, there exist some finite members, that is, $G_{i_1}, G_{i_2}, ..., G_{i_k} \in \mathcal{C}^*$ so that $A \subseteq G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_k}$. If A can be expressed in this form, obviously $A \subseteq (A \cap G_{i_1}) \cup (A \cap G_{i_2}) \cup ... \cup (A \cap G_{i_k})$. Let us rename these sets and let us name the sets as $A \cap G_{i_1} = H_{i_1}, A \cap G_{i_2} = H_{i_2}, ... A \cap G_{i_1} = H_{i_k}$. Note that these $H_{i_1}, H_{i_2}, ..., H_{i_k}$ are members of C. Thus, the open cover of A, what we have taken as $\mathcal{C}$, is reduced to a finite subcover, that is, $\mathcal{C}' = \{H_{i_1}, H_{i_2}, ..., H_{i_k}\}$. Therefore, A is compact in topological space $(X, \mathcal{T})$.

Now, let us take some of the examples. The first example is in terms of sequence. Let $(X, \mathcal{T})$ be a topological space and (x_n) be a convergent sequence in X converging to x. Then $\{x\} \cup \{x_n : n = 1, 2, 3, ...\}$ is a compact subset of X. How is it possible? Let us see it. So, what are we taking? We are taking the set $\{x\} \cup \{x_n : n = 1, 2, 3, ...\}$ as A. Now, recall the concept of the convergence of a sequence. What we know is that a sequence (x_n) converges to x if, for all open sets G containing x, there exists a positive integer, that is, n_0 such that this $x_n \in G, \forall n \geq n_0$. This is the key idea to justify that A is compact. What can we do now? Let us take any open cover, that is this $\mathcal{C} = \{G_i : i \in I\}$, $G_i \in \mathcal{T}$, of A. Now, as $x \in A$, x belongs to at least one G_k, for some $k \in I$. Thus there exists a positive integer, $n_0 \in N$ such that $x_n \in G_k, \forall n \geq n_0$. Now, if $x_n \in G_k, \forall n \geq n_0$, we can conclude that only finite members of the sequence will be outside of this G_k. So, let us take a natural number m such that $m < n_0$. What we can do that we can write $A \subseteq G_k \cup G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_m}$. What are the sets $G_{i_1}, G_{i_2}, ..., G_{i_m}$? These sets are from $\mathcal{C}$; that is, these are also open, and these sets contain elements which are outside G_k and m in number. Thus, we can say that if we are taking a collection, that is $\mathcal{C}' = \{G_k, G_{i_1}, G_{i_2}, ..., G_{i_m}\}$. Then $\mathcal{C}'$ is a subcover of C. Thus, each open cover $\mathcal{C}$ of A is reducible to a finite subcover. It is to be noted that $\mathcal{C}'$ is not only a subcover, this is also a finite subcover. Therefore, the set A is compact. By using this result, we can conclude that the set $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact because if we are taking this sequence $(1/n)$, this sequence is converging to 0.

Let us take $\mathbb{R}$ with Euclidean topology and $A = (0, 1]$. Note that A is not compact. In order to justify that this set is not compact, it is simple to show. Try to find an open cover of this set and show that the open cover does not have a finite subcover. For which, let us take $\mathcal{C} = \{(1/n, 1] : n \in \mathbb{N}\}$; which is an open cover of A. But there does not exist any finite subcover of $\mathcal{C}$. What is the problem? If possible, let $A = (1/n_1, 1] \cup (1/n_2, 1] \cup ... \cup (1/n_k, 1]$. Then take a natural number m such that m is greater than the maximum of these $n_1, n_2, ..., n_k$. So, what will happen? This $1/m$ is an element of A, but note that this $1/m$ cannot be a member of $(1/n_1, 1] \cup (1/n_2, 1] \cup ... \cup (1/n_k, 1]$. So, there is a contradiction. Therefore, the semi-open interval $(0, 1]$, is not compact.

Let us discuss the continuous image of a compact topological space. This theorem says that the continuous image of a compact topological space is compact. For this, begin with a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, and let us take $(X, \mathcal{T})$ as compact space. So, we have two things with us. We have to justify that $f(X)$, which is a subset of Y, is also compact. In order to justify that $f(X)$ is compact, let us take an open cover of this $f(X)$.

So, let us take $\mathcal{C} = \{G_i : i \in I\}$, where $G_i \in \mathcal{T}^{'}$ as an open cover of $f(X)$, that is, from here, we can write that $f(X) \subseteq \cup\{G_i : i \in I\}$. If this is the case, then we can write $X \subseteq f^{-1}(\cup\{G_i : i \in I\})$, or simply we can write that this $X \subseteq \cup\{f^{-1}(G_i) : i \in I\}$. From here, what can we conclude? We can conclude that $\mathcal{C}^* = \{f^{-1}(G_i) : i \in I\}$ is an open cover of X. Note that $f^{-1}(G_i)$ is open in X and this is because what have we taken? We have taken $G_i \in \mathcal{T}'$, that is G_i is $\mathcal{T}^{'}$-open, by continuity $f^{-1}(G_i)$ is $\mathcal{T}$-open, that is, $f^{-1}(G_i) \in \mathcal{T}$. Now, by the compactness of $(X, \mathcal{T})$, we can say that there exist some finite members in $\mathcal{C}^*$, that is, $f^{-1}(G_{i_1})$, $f^{-1}(G_{i_2})$, ..., $f^{-1}(G_{i_k})$ in C^* such that $X \subseteq f^{-1}(G_{i_1}) \cup f^{-1}(G_{i_2}) \cup ... \cup f^{-1}(G_{i_k}) = f^{-1}(G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_k})$. Now, use the concept from the set associated functions. What can we conclude? We can conclude that $f(X) \subseteq G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_k}$. So, we find a finite subcover of $\mathcal{C}$, and that subcover is given by $\{G_{i_1}, G_{i_2}, ..., G_{i_k}\}$. From here, we can conclude that $f(X)$ is compact as every open cover of $f(X)$ is reducible to a finite subcover.

By using this theorem, it is obvious to conclude that compactness is a topological property. Also, we can conclude that the quotient space of a compact topological space is compact. For this, let us see it. Let us take a compact topological space $(X, \mathcal{T})$ and quotient map $q : (X, \mathcal{T}) \to (A, \mathcal{T}_q)$. Now, if q is a quotient map, this map is always continuous and surjective. Also, if q is continuous and surjective, what about this $(A, \mathcal{T}_q)$? This is nothing but a continuous image of the topological space $(X, \mathcal{T})$. If q is continuous and surjective, what exactly A is? That is nothing but $q(X)$. What about this $(X, \mathcal{T})$? This is a compact, and the continuous image of a compact topological space is compact. Therefore, $(A, \mathcal{T}_q)$ is compact too.

Now, let us continue with this notion that compactness is a topological property, which we have already studied regarding homeomorphism and compactness of spaces. We have already seen that $\mathbb{R}$, with the standard topology, is not compact. We have also seen that $(0, 1]$ is not compact. Also, what we have seen by using homeomorphism, the open intervals $(a, b), (-\infty, b), (a, \infty)$ are homeomorphic to $\mathbb{R}$. Even we have also seen the homeomorphism for the intervals $[a, b), (a, b], (-\infty, b], [a, \infty)$. What about if $(0, 1]$ is not compact. By using the fact that compactness is a topological property and these spaces are homeomorphic to non-compact topological spaces, we can deduce that all such intervals are not compact.

What we have not taken in consideration till now, that is the closed interval $[a, b]$. Note that this is compact. We will see, even we will prove that the closed intervals in $\mathbb{R}$ are always compact, wait for that. Without proving it, we are using the compactness of this $[a, b]$, let us take that this interval $[a, b]$, which is compact. Note that $(a, b) \subset [a, b] \subset \mathbb{R}$. Among these, $[a, b]$ is compact, what about (a, b)? This is not compact, and what about this $\mathbb{R}$, this is also not compact. It means that the subset and superset of a compact set may not be compact. Now, recall the notion of subspaces. So, what we can conclude from here is that the subspace of a compact topological space is not necessarily compact. The question arises of whether we can put some restrictions on subspaces. If we can put some restrictions in place and they turn out to be compact, what will be the restrictions? We will discuss this in the next lecture.

These are the references.

That's all from this lecture. Thank you.