Course Name: Essentials of Topology
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Welcome to Lecture 5 on Essentials of Topology. In the previous lecture, we studied some concepts of associated set functions. Continuing from there, in this lecture, we will explore more results on image and inverse images of sets under a given function. For a given function $f : X \to Y$ and two subsets A_1 and A_2 of X, if A_1 is a subset of A_2, then $f(A_1)$ is a subset of $f(A_2)$. To justify this one, let us take $y \in f(A_1)$, then this y can be written as $f(x)$, for some $x \in A_1$, or that y can be written as $f(x)$, for some $x \in A_2$, and this is because A_1 is a subset of A_2, or $y \in f(A_2)$, and therefore $f(A_1)$ is a subset of $f(A_2)$.

Moving ahead, let us take the second result: $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. If we want to prove this result, we have to justify that $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$ and $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$. So, in order to prove this result, we have to prove these two results. It is to be noted here that the second one is trivial because A_1 is always a subset of $A_1 \cup A_2$. Similarly, A_2 is also a subset of $A_1 \cup A_2$, and therefore, from the previous result, $f(A_1)$ is a subset of $f(A_1 \cup A_2)$, and $f(A_2)$ is a subset of $f(A_1 \cup A_2)$. Combining these two results, we can conclude that $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$. Coming to the first one and for its justification, let us take a $y \in f(A_1 \cup A_2)$. If $y \in f(A_1 \cup A_2)$, then y can be written as $f(x)$, for some $x \in A_1$ or $x \in A_2$. It means that $y \in f(A_1)$ or $y \in f(A_2)$. That is, $y \in f(A_1) \cup f(A_2)$, and thus $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$, and hence $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Moving ahead, similar to the previous one, we can justify that $f(A_1 \cap A_2)$ is a subset of $f(A_1) \cap f(A_2)$. Even this is trivial because we know that $A_1 \cap A_2$ is a subset of A_1. Also, $A_1 \cap A_2$ is a subset of A_2, and therefore, $f(A_1 \cap A_2)$ is a subset of $f(A_1)$ as well as $f(A_1 \cap A_2)$ is also a subset of $f(A_2)$, and combining these two, we can get this result. But again, the question comes under which condition does equality hold? First, let us take an example and try to justify that, in general, equality may not hold. For example, if we take the function $f : \mathbb{R} \to \mathbb{R}$ again, such that $f(x) = x^2$, A_1 as a singleton set $\{3\}$, and A_2 as a singleton set $\{-3\}$. Then, it is clear that $A_1 \cap A_2$ is an empty set; therefore,

$f(A_1 \cap A_2)$ is also an empty set. But it is to be noted here that $f(A_1)$ is a singleton set $\{9\}$, $f(A_2)$ is also a singleton set $\{9\}$, and therefore, $f(A_1) \cap f(A_2)$ is also a singleton set $\{9\}$. So from this example, it is clear that $f(A_1 \cap A_2)$ may not equal to $f(A_1) \cap f(A_2)$. So, under which condition does equality hold? The answer is here that $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ if and only if f is injective.

To justify this result, let f be injective. Then we have to show that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ and $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$. We have already seen that for every subset A_1 and A_2 of X, this result holds. So we only need to justify that $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$. For it, let $y \in f(A_1) \cap f(A_2)$. Then of $y \in f(A_1)$, and $y \in f(A_2)$, or that y can be written as $f(x_1)$, for some x_1 in A_1, and y can also be written as $f(x_2)$, for some x_2 in A_2. But note that $y = f(x_1)$ and $y = f(x_2)$, it means that $f(x_1) = f(x_2)$, and therefore, by using the injectivity of the function, x_1 is equal to x_2. Therefore, we conclude that $x_1 \in A_1 \cap A_2$, or that y, which is nothing but $f(x_1)$, belongs to $f(A_1 \cap A_2)$. So, from here, we conclude that $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$.

Coming to the converse of this result, let $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. Then, we have to prove that f is injective. In order to justify that f is injective, let two elements x_1 and x_2 in X such that $x_1 \neq x_2$. Also, let us assume that A_1 is a singleton set $\{x_1\}$ and A_2 is a singleton set $\{x_2\}$. Then, it is clear that $A_1 \cap A_2$ is an empty set, or $f(A_1 \cap A_2)$ is an empty set. Since $f(A_1 \cap A_2)$ is same as $f(A_1) \cap f(A_2)$. Therefore, $f(A_1) \cap f(A_2)$ is also an empty set, and as A_1 is a singleton set $\{x_1\}$ and A_2 is a singleton set $\{x_2\}$, from here, we can conclude that $f(x_1)$ is not equal to $f(x_2)$, which justifies that f is an injective function.

Moving to the next, we can generalize the previous results for union and intersection to a collection of subsets of X indexed by the set I. That is, one can prove that $f(\cup\{A_i : i \in I\})$, where A_i is a subset of X, is the same as $\cup\{f(A_i) : i \in I\}$. Similarly, $f(\cap\{A_i : i \in I\})$ is a subset of $\cap\{f(A_i) : i \in I\}$. Finally, $f(\cap\{A_i : i \in I\})$ is equal to $\cap\{f(A_i) : i \in I\}$, provided f is injective. Moving to the next, let us prove a result that states that for a given function $f : X \to Y$ and a subset A of X, the complement of $f(A)$ in Y is the same as the image of the complement of A in X, provided f is surjective. So assume that it holds, and let us try to justify that f is surjective.

This is trivial to justify, as this one holds for each subset A of X. So why not let us take A as an empty set. Suppose we are taking A as an empty set. In that case, this can be stated as $Y - f(\emptyset) \subseteq f(X - \emptyset)$, or that Y is a subset of $f(X)$. We know that $f(X)$ is always a subset of Y. Combining these two, $f(X) = Y$; therefore, f is a surjective function. For the converse part, let f be surjective. In order to prove that $Y - f(A)$ is a subset of $f(X - A)$, let any $y \in Y - f(A)$. Then, it is clear that y is not a member of $f(A)$. But note that f is a surjective function; there exists an element $x \in X$ such that $f(x) = y$.

We can claim that x cannot be an element of A. The question is, why? Because if x is an element of A, then $f(x)$ is an element of $f(A)$, or that y is an element of $f(A)$, contradicting the fact that y does not belong to $f(A)$. If x is not an element of A, then obviously x belongs to the complement of A in X, or that $f(x) \in f(X - A)$, or that $y \in f(X - A)$, and this justifies that $Y - f(A)$ is a subset of $f(X - A)$.

Moving to the next result, let us show that $f(X - A)$ is a subset of $Y - f(A)$ if and only if f is injective. In order to prove this result, let us assume that f is injective first and show that $f(X - A)$ is a subset of $Y - f(A)$. To justify this result, let us take y as an element of $f(X - A)$. Then y can be written as $f(x)$, for some $x \in X - A$. As $x \in X - A$, it is clear that x cannot be an element of A. From here, we can conclude that $f(x)$ cannot be an element of $f(A)$. Because if $f(x)$ is an element of $f(A)$, $f(x)$ can be expressed as $f(x')$, for some $x' \in A$, and after that, by injectivity of f, x and x' will be equal, or that x is also an element of A. Now, coming to this fact that $f(x)$ is not an element of $f(A)$, we can say that y is not an element of $f(A)$, or that $y \in Y - f(A)$. Therefore, we can say that $f(X - A)$ is a subset of $Y - f(A)$.

Moving ahead, let us see the converse part of this result. For justifying the converse part, let us assume that $f(X - A)$ is a subset of $Y - f(A)$, and we have to justify that f is injective. To justify that f is injective, let us prove the result by contradiction. That is, let us assume that there exist two elements x_1 and x_2 in X such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Also, take A as a singleton set x_1. Then, it is clear that x_2 cannot be an element of A, or x_2 is an element of $X - A$, or $f(x_2)$ is an element of $f(X - A)$. But $f(x_2)$, which is equal to $f(x_1)$, as per our assumption, this belongs to $f(A)$. Meaning is that $f(x_2) \notin Y - f(A)$. This contradicts the fact that $f(X - A)$ is a subset of $Y - f(A)$, because what we have shown that there exists an element $f(x_2) \in f(X - A)$, but that does not belong to $Y - f(A)$. Therefore, our assumption is wrong; hence, f is an injective function.

Finally, from the previous two results, one was regarding the surjectivity of f, and the second was regarding injectivity of f, we can conclude that $Y - f(A)$ is equal to $f(X - A)$ if and only if f is bijective. Also, similar to the results that we have seen for the image of sets, we can conclude that if we are taking a function $f : X \to Y$, three sets B_1, B_2, and B, subsets of Y and B_i, $i \in I$, is any family of index subsets of Y, indexed by the set I. If B_1 is a subset of B_2, $f^{-1}(B_1)$ is a subset of the $f^{-1}(B_2)$, $f^{-1}(\cup\{Bi : i \in I\})$, is same as $\cup\{f^{-1}(B_i) : i \in I\}$, $f^{-1}(\cap\{Bi : i \in I\})$, is same as $\cap\{f^{-1}(B_i) : i \in I\}$, and $f^{-1}(Y - B)$ is same as $X - f^{-1}(B)$. It is to be noted here that all the results hold here without any condition.

These are the references.

That's all from today's lecture. Thank you.