Course Name: Essentials of Topology
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Welcome to Lecture 49 on Essentials of Topology.

In this lecture, we will initiate the study of the concept of compactness of a topological space. Begin with what we know. Actually, in topology, three $C's$ play a key role. What are these three $C's$? The first one is the concept of continuous functions. The second is the concept of the connectedness of topological spaces, and the third is the concept of compactness. We have already studied the concept of continuous functions and connectedness. Even, we have seen that their definitions are natural. When we come to the concept of compactness, form analysis what do we know? We know that if we are taking a subset A of $\mathbb{R}^n$, A is compact if A is closed as well as bounded. But the question is, in the case of topology, when we are beginning with a topological space $(X, \mathcal{T})$, this X is an arbitrary nonempty set. So, if it is an arbitrary set, we cannot talk about the boundedness. Then, what to do? We cannot use the definition that is given in terms of closedness and boundedness. We have to opt for some other way, and the idea is the concept of cover, precisely, the concept of open cover, which we use to discuss the concept of compactness of a topological space.

Coming to the concept of cover, what is it? If we are having a topological space $(X, \mathcal{T})$, let us take any subset A of X, and $\mathcal{C}$ as a collection of subsets of X. We say that this collection $\mathcal{C}$ is a cover of A if A is contained in the union of sets in $\mathcal{C}$. Meaning is to say that if we are taking $\mathcal{C}$ something like $\{G_i : i \in I\}$, $G_i \subseteq X$, we say that this $\mathcal{C}$ is cover of $A \subseteq X$, if $A \subseteq \bigcup\{G_i : i \in I\}$. Let us take the example of $(\mathbb{R}, \mathcal{T}_e)$. Take a set $A = (0, \infty)$ and also take $\mathcal{C} = \{(0, n) : n \in \mathbb{N}\}$. Then, $\mathcal{C}$ is a cover of A because A is included in the union of intervals of the form $(0, n)$, which are members of $\mathcal{C}$.

Moving ahead, let us talk about an open cover. If $\mathcal{C}$ is a cover of A and each member of $\mathcal{C}$ is open, that is, we are taking the members of $\mathcal{C}$ from topology; in this case, we call $\mathcal{C}$, an open cover of A. Coming back to the previous example, what $\mathcal{C}$ have we taken? $\mathcal{C}$ is a collection of open intervals of the form $(0, n), n \in \mathbb{N}$, and we know that such intervals are always members of Euclidean topology. Therefore, the cover $\mathcal{C}$ is an open cover.

Moving ahead, let us talk about another notion that is known as the subcover of a cover. What is it? If $\mathcal{C}$ is a cover of A, that is, $\mathcal{C}$ covers A, let us take a sub-collection $\mathcal{C}'$ of $\mathcal{C}$ that also covers A, then we call $\mathcal{C}'$, a subcover of $\mathcal{C}$. In the previous example, we have taken the real line with Euclidean topology and $A = (0, \infty)$. Now, let us take $\mathcal{C} = \{(0, n) : n \in \mathbb{N}\} \cup \{(-1, n) : n \in \mathbb{N}\}$. Note that this is a cover of A. If we are taking a part of it, that is $\mathcal{C}' = \{(0, n) : n \in \mathbb{N}\}$, we have already seen that this is a cover of A, but in this case, this $\mathcal{C}'$ is a sub-collection of $\mathcal{C}$. So, this $\mathcal{C}'$ is a subcover of $\mathcal{C}$.

Let us take some more examples. The set of real numbers with Euclidean topology and $\mathcal{C}_1 = \{..., (-1, 1), (0, 2), (1, 3), ...\}$. So, what we can say is that this $\mathcal{C}_1$ is an open cover for the set of real numbers because $\mathbb{R}$ can be expressed as the union of these intervals. We can take another open cover of $\mathbb{R}$. Let us take $\mathcal{C}_2 = \{(-n, n) : n \in \mathbb{N}\}$. What can we do? We can write $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$, that is $\mathbb{R}$ can be expressed as a union of members of $\mathcal{C}_2$. Therefore, $\mathcal{C}_2$ is also an open cover. Now, we are taking another open cover of this $\mathbb{R}$. Let us take this is $\mathcal{C}_3 = \{(-\infty, 1), (-1, \infty)\}$. Note that this $\mathcal{C}_3$ is having only two elements, one is $(-\infty, 1)$, and the second is $(-1, \infty)$. It can be seen that $\mathbb{R}$ can be written as $(-\infty, 1) \cup (-1, \infty)$. So, this $\mathcal{C}_3$ is also an open cover of $\mathbb{R}$. The idea is, one may take a number of open covers, at least three open covers we have seen here for the set of real numbers. One more thing we need to mention here, that is, basis is always an open cover of X. If we are taking any topological space $(X, \mathcal{T})$, let us take its basis $\mathcal{B}$, what we know that X can always be written as a union of members of $\mathcal{B}$. So, it follows from this fact, and this is one of the simplest examples of an open cover.

Moving ahead, let us see the concept of compactness of a topological space. A topological space $(X, \mathcal{T})$ is said to be compact if every open cover of X has a finite subcover. Meaning is to say that if we are having a topological space $(X, \mathcal{T})$ and an arbitrary open cover $\mathcal{C} = \{G_i : i \in I\}$ of it, where $G_i \subseteq X$. If $\mathcal{C}$ is an open cover, X can be written as a union of these G_i's. What is our interest? Our interest is to search some $G_{i_1}, G_{i_2}, ..., G_{i_n}$ in this $\mathcal{C}$, that is in the given open cover so that X can also be written as

$G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_n}$. If we are taking real line with Euclidean topology and the open covers $\mathcal{C}_1 = \{..., (-1, 1), (0, 2), (1, 3), ...\}$ and $\mathcal{C}_3 = \{(-\infty, 1), (-1, \infty)\}$. Note that if we are looking at these two open covers of $\mathbb{R}$, the first one has infinite members in it, while there are only finite members in $\mathcal{C}_3$. Our interest is not in open covers like $\mathcal{C}_3$. Our interest is, begin with an infinite open cover, that is open cover having an infinite number of members in it, and try to find finite members from there so that X can be expressed as a union of these members of the open cover. In case the open covers contain only a finite number of members, there is no need to discuss further.

Let us take some of the examples. The first example is, the topological space $(X, \mathcal{T})$, where $\mathcal{T}$ is the indiscrete topology. This is compact, that is indiscrete topological spaces are always compact. Why? This is possible because, in the case of indiscrete topological spaces, there are only two possible open covers. The first one is, this is containing X, or the second is, that is containing empty set and X. But as we discussed previously, these are already finite in nature. So, no need to do anything. Thus, trivially, this topological space is compact.

Moving ahead, let us discuss about finite topological space, that is, a topological space $(X, \mathcal{T})$, where X has finite members. This will always be compact. Why? The answer is that there are finitely many open sets. If there are finitely many open sets, what about the open cover? Obviously, whenever we are going to take any open cover so that open cover is finite, and if the open cover is finite, there is no need to move further because this will always or can be reduced to a finite subcover, that is itself. Therefore, finite topological spaces are always compact.

Moving ahead, the discrete topological spaces, where we are taking X as infinite sets, these topological spaces are not compact. Why? The problem is, in the case of discrete topology, we know that singleton sets are always members of it. Even we can write X as a union of singleton sets $\{x\}$, $x \in X$. If we are taking this $\mathcal{C}$ as $\{\{x\} : x \in X\}$, note that this is an open cover. Now, whether we can construct a finite subcover of $\mathcal{C}$. The answer is no. For example, if possible there exist $\{x_1\}, \{x_2\}, ..., \{x_n\}$ in $\mathcal{C}$ such that $X = \{x_1\} \cup \{x_2\} \cup ... \cup \{x_n\}$. So, X becomes $\{x_1, x_2, ..., x_n\}$. Meaning is, X is a finite set, this is a contradiction. Therefore, we cannot construct a finite subcover of this $\mathcal{C}$, which covers X, and thereby, the discrete topological spaces, when X is infinite, are not compact.

Moving ahead, let us take another example, the co-finite topological space $(X, \mathcal{T})$, we are taking X as an infinite set. This will always be compact. How? Let us see it. If we are taking an open cover, let us take this as $\mathcal{C} = \{G_i : i \in I\}$. Thus, $X = \bigcup\{G_i : i \in I\}$. Our motive is to justify that $\mathcal{C}$ can be reduced to a finite subcover, that is, try to construct a finite subcover so that it should also cover X. What will we do? Let us take an element $k \in I$. If we are fixing this, take the corresponding G_k. What about $G_k \cup G_k^c$ complement? This will be nothing but X. What about G_k^c? Note that the topology is co-finite; the members inside G_k^c are only finite in number. So, if the members outside G_k are finite, what can we do? Because G_k^c is finite, therefore, this X can be expressed as G_k union a finite set; let us take that finite set as $\{x_1, x_2, ..., x_n\}$. Also, it is to be noted that $X = \bigcup\{G_i : i \in I\}$, so what will happen, these $x_1, x_2, ..., x_n$ will be the members of some of the G_i from $\mathcal{C}$. Therefore, let us take this x_1 is a member of G_{i_1}, x_2 is a member of G_{i_2}, ..., x_n is a member of G_{i_n}. Thus, what we can do that this X can also be written as $G_k \cup G_{i_1} \cup G_{i_2} \cup ... \cup G_{i_n}$. So, if we are beginning with an open cover $\mathcal{C}$, we can construct a subcover. This subcover is nothing but this is $\{G_k, G_{i_1}, G_{i_2} ..., G_{i_n}\}$. Therefore, the co-finite topological space $(X, \mathcal{T})$ is compact.

Moving ahead, let us take another example. This is the well-known set of real numbers, along with the Euclidean topology. Note that this topological space is not compact. In order to justify that this $\mathbb{R}$ is not compact, let us construct an open cover that doesn't have any finite subcover. So, we are taking an open cover $\mathcal{C} = \{(n, n+2) : n \in \mathbb{Z}\}$. Note that $\mathbb{R} = \bigcup\limits_{n \in \mathbb{Z}} (n, n+2)$ and therefore, $\mathcal{C}$ is an open cover of $\mathbb{R}$. The question is, can we find a finite subcover for this open cover? The answer is no. Why? Because if $\mathbb{R}$ can be written as something like, $\mathbb{R} = (n_1, n_1+2) \cup (n_2, n_2+2) \cup ... \cup (n_k, n_k+2)$, that is there exist $n_1, n_2, ... n_k \in \mathbb{Z}$ such that $\mathbb{R} = (n_1, n_1+2) \cup (n_2, n_2+2) \cup ... \cup (n_k, n_k+2)$, then what is the problem? The problem is here. Let us take a real number m which is greater than the maximum of these $n_1+2, n_2+2, ... n_k+2$. It is to be noted that this $m \in \mathbb{R}$, but this m cannot be an element of $(n_1, n_1+2) \cup (n_2, n_2+2) \cup ... \cup (n_k, n_k+2)$. Thus, we have shown that if we are beginning with an open cover and that open cover is given as $\mathcal{C}$, we cannot find any finite subcover of this open cover. Therefore, the set of reals with standard topology is not compact.

These are the references.

That's all from this lecture. Thank you.