Course Name: Essentials of Topology
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Welcome to Lecture 48 on Essentials of Topology.

In this lecture, we will study the theory of local path connectedness, which is going to be the last lecture on connectedness. Begin with what we have studied. Just recall the concept of components. This concept was based on the concept of connectedness, and the second concept was path components, which was based on the notion of path connectedness. There are some questions that need to be answered. The first question is: what have we seen in the case of components? We have seen that this is closed, and after that, we have seen that, under some restrictions, it has become open, too. But till now, we have not seen the nature of path components, whether they may be closed or open. So, the first question is whether the path components can be clopen under some restrictions. The second question is whether components and path components of a topological space can be the same. Then one more result that we have studied is that path connectedness, implies connectedness. Also, we have seen that connectedness does not imply path connectedness. Then, the question arises: whether it is possible that connectedness implies path connectedness under some conditions. The answer to all these three questions is yes. When do these three results hold? The answer is that when a topological space is locally path-connected, we will get the answer to these questions. These are the main points that we will discuss in this lecture.

Let us see the definition of locally path-connected topological spaces. A topological space $(X, \mathcal{T})$ is locally path connected at $x \in X$ if for each open nbd N of x there exists a path connected open set G such that $x \in G \subseteq N$. Further, the topological space $(X, \mathcal{T})$ is locally path connected if it is locally path connected at each of its points. Having this idea in mind, if we are thinking about the concept that we have studied in the case of local connectedness, we can define the concept of locally path connectedness in terms of a basis. That is inspired by the fact that if we are taking the topological space $(X, \mathcal{T})$, what we are saying that for all $x \in X$, if we are taking this open neighborhood

N of x, we are saying that there exists a path connected open set G such that $x \in G \subseteq N$. Let us take this $\mathcal{B}$ as a collection of all those subsets G of X such that this G is open and path connected in $(X, \mathcal{T})$. So, what is this $\mathcal{B}$? This $\mathcal{B}$ is a basis for this topology, and if this is the basis for this topology, then the required properties will be satisfied. So, similar to the concept of local connectedness, we can say that a topological space $(X, \mathcal{T})$ is locally path-connected if it has a basis consisting of path-connected open sets.

Now, moving ahead, as the definition is similar to the concept of local connectedness, what are we doing, we are replacing the open connected sets by open path-connected sets. So, the natural question arises whether this local path connectedness implies local connectedness and conversely. The first and obvious observation is that local path connectedness implies local connectedness. The question is how? The answer simply follows from the definition of local path connectedness because if we are beginning with a topological space $(X, \mathcal{T})$, take $x \in X$, and let us take $N \subseteq X$; this is an open neighborhood of x. So, the condition is that for all $x \in X$, we say that $(X, \mathcal{T})$ is locally path connected if there exists an open set G which is also path connected such that this $x \in G \subseteq N$. Note that this G is path-connected. Based on what we have studied, G is also connected. If this is connected, it means that corresponding to neighborhood N of x, that is open, there exists a connected set that is open, too, such that $x \in G \subseteq N$. Therefore, $(X, \mathcal{T})$ is locally connected.

Now, let us see some of the examples. So, the first one is, and that is well-known, why not let us take discrete topological space $(X, \mathcal{T})$. Now, what can we do? Let us take this $\mathcal{B} = \{\{x\} : x \in X\}$. We know that $\{x\}$ is open and path-connected. So, this $\mathcal{B}$ is the required basis for discrete space $(X, \mathcal{T})$. Therefore, $(X, \mathcal{T})$ is locally path connected. Moving ahead, why not let us discuss the space $\mathbb{R}$ with Euclidean topology? We know that $\{(a, b) : a, b \in \mathbb{R}, a < b\}$; this is a basis for Euclidean topology. Also, the intervals are path-connected and open; therefore, $(\mathbb{R}, \mathcal{T}_e)$ is locally path-connected. Moving ahead, we can justify that $\mathbb{R}^n$ with standard topology is also locally path-connected.

Let us take one more example. If we are taking the set $X = [0, 1] \cup [2, 3] \subset \mathbb{R}$ alongwith the topology on X as a relative topology with respect to the Euclidean topology on $\mathbb{R}$. We can deduce that this is locally path-connected.

How is this possible? We have already seen in this case, if we are visualizing the open sets in relative topology on X, they will look like of the form of some intervals, and all the intervals are path-connected. So, there will not be any problem. One can construct a basis that will consist of open sets, and all the open sets will always be path-connected. Therefore, this is a path-connected space. It is to be noted that this space, that is, X with this relative topology, is not path-connected. The question is, why? One can think that if this is path connected, then this will also be connected, but this is not connected because X is the union of two disjoint open sets in the relative topology. So, this is one of the examples that justifies that a locally path-connected space may not be path connected. The question is, can we construct an example to show that path connectedness does not imply this local path connectedness? The answer is yes. Just think about it. We have already discussed a number of examples. Whether the Topologist's sine curve can serve our purpose, this I am leaving.

Now, similar to the concept of locally connected spaces, let us see a theorem related to the local path connectedness of a topological space. The theorem is that a topological space $(X, \mathcal{T})$ is locally path connected if and only if the path components of each open subset of X are open. In the case of locally connected spaces, what have we taken? We have taken their components; here, we are taking the path components. In order to justify it, let us assume that $(X, \mathcal{T})$ is locally path connected. Also, because we require path components, let us take this as $P = P(x)$. Again, let us take $N \subseteq X$ as an open set. In order to justify that P is open, show that it is a neighborhood of each of its points. Note that $P \subseteq N$. Now, if we are taking $y \in P$, $y \in N$. If $y \in N$, because $(X, \mathcal{T})$ is locally path connected, there exists a path connected open set; let us take this as G, so that $y \in G \subseteq N$. Now, G is containing y and P is also containing y, that is, $G \cap P$ is non-empty. Further, this G is path-connected, being component, P is also path-connected. Therefore, we can conclude that this $G \cup P$ is path-connected. Further, $G \cup P$ is a superset of P. But note that P is the path component. If this is a path component, this is path-connected and maximal, too. Thus, we can conclude that $P = G \cup P$, or we can say that $G \subseteq P$. So, we can conclude that $y \in G \subseteq P$, that is, P is a neighborhood of each of its points, and if it is so, P is open. The converse is a simple one, like what we have discussed in the case of locally connected topological spaces.

Moving ahead, if we are taking a locally path connected topological space $(X, \mathcal{T})$, what will happen? Each path component $P(x)$ of $(X, \mathcal{T})$ is clopen. We have already seen that in the case of a general topological space, the path component may neither be closed nor open, but when the space is taken as locally path connected, it turns out to be closed as well as open both. In order to justify it, let us use the previous theorem. Note that X is open. What is this $P(x)$? This is a subset of X and $x \in X$. We have already seen that the path component of an open set is open, so $P(x)$ is open. Meaning is, each path component of a locally path-connected topological space is open. Now, let us take this $P(x)$ as P. What about P^c? P^c is nothing but a union of path components of $(X, \mathcal{T})$, except one, that is P. Now, take the union of the rest of the path components. This is possible because we know that path components partition the space. So, what are we doing? If this is P, leave this P and take the rest of the path components. But what are path components? We have already seen that these are open sets. We know that an arbitrary union of open sets is open; therefore, P^c is open, and if this P^c is open, P is closed. Thus, we have shown that the path component $P(x)$ is clopen.

Moving ahead, and let us take one more result. The first statement says that in the case of locally path-connected topological spaces, the components and path components both coincide. This is one of the interesting results. The second one is much more interesting because what we have studied is that path-connected topological spaces are connected, but a connected topological space may not be path-connected. But this is a statement, and it says that in the case of locally path-connected, that is, if space is connected and locally path-connected, then that will always be path-connected. Let us see the justifications one by one. If we are looking for the first one, take $(X, \mathcal{T})$ as a locally path-connected topological space, $C = C(x)$ as a component, and $P = P(x)$ as a path component of it. Note that if P is a path component, then it is path-connected, and if P is path-connected, this will be connected too. Also, we have studied that connected subsets are always contained in one component, but as x is common to both this C and P, we can conclude that $P \subseteq C$. Is it possible that this is a proper inclusion? If it is proper, what would be the problem? Note that if this is proper, then as P is non-empty, $P \subset C$, and P is clopen, C becomes disconnected. So, this is a contradiction because C is a component, and being a component, it is always connected. Therefore, the inclusion cannot be proper, that is, $P = C$, or path components and components of this locally path-connected topological space $(X, \mathcal{T})$ will coincide.

Now, coming to the second one, if $(X, \mathcal{T})$ is connected, what will happen is this X is the only component. Now, what this $(X, \mathcal{T})$ is? Note that this is locally path-connected. If we are combining these two, we can conclude that X is the only path component. Note that path components are path-connected. Therefore, $(X, \mathcal{T})$ is path-connected. Finally, we conclude the theory of connectedness here. From the next lecture, we will study the concept of compactness.

These are the references.

That's all from this lecture. Thank you.