Course Name: Essentials of Topology
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Welcome to Lecture 47 on Essentials of Topology.

In this lecture, we will continue the study of the concept of local connectedness. Begin with what we have studied in the previous lecture. We have seen that local connectedness does not imply connectedness. It means a space that is locally connected may not be connected. Now, what we are going to justify, a connected topological space may not be locally connected. Begin with this example. For each integer $n > 0$, let L_n be the line segment in $\mathbb{R}^2$ joining the origin to point $(1, 1/n)$, and $L_0 = \{(x, 0) : 0 \leq x \leq 1\}$. For example, if we are looking for L_1, this is joining the origin to $(1, 1)$. Similarly, L_2, this is joining the origin to $(1, \frac{1}{2})$, L_3 is joining the origin to $(1, \frac{1}{3})$. We can justify that $X = \cup_{n \geq 0} L_n \subseteq \mathbb{R}^2$ is connected, but it is not locally connected. Let us see the first one: Note that for $n \geq 0$, the line segments L_n are connected and all these segments has origin as a common element. Thus, $X = \cup_{n \geq 0} L_n$ is connected

Now, our motive is to justify that this space is not locally connected. So, what we will do is take an element; for example, we are taking an element $x = (1/2, 0)$. What exactly we will do is show that there does not exist any connected open set containing this x, and if we can justify it, obviously, this space will not be locally connected. The question is, how do we show it? It is to be noted here that if we are taking an open ball, it is clear that this open ball will intersect with some of the line segments. If we are taking any open ball, there will not be any problem. This is the key idea that will help us to show that this space is not locally connected. What exactly is the idea? Let us take an element; that element is $x = (1/2, 0)$. Now, if we are taking an open set G in X. How will the open set look? This will be something like $G = X \cap B$, B is open in $\mathbb{R}^2$. Now, if this B is open in $\mathbb{R}^2$, why not let us take this as an open ball. It will be the best choice because every open set in $\mathbb{R}^2$ can be expressed as a union of open balls. Now, note that $L_n \cap B \subseteq X \cap B = G$ and $L_n \cap B$ is nonempty. The question is, how is this possible? The simple

thing is to choose n so that $1/n$ is less than the radius of ball B. Now, how does this $L_n \cap B$ look like? See, when we are constructing this open ball, note that the line segments are intersecting, or are passing from this open ball, so if we are taking an open ball consisting of this element, what exactly are we saying that this will contain some L_n? Let us take what we are saying: this is our L_n. What will happen about this one? If we are discussing this one, $L_n \cap B$ is nothing but this particular line segment. So, note that this line segment is connected in G; it is not only connected, but it is also a component. Even one more thing, this is closed and open both. So, we have shown that there exists a proper subset $L_n \cap B$, this is non-empty and clopen. It shows that this G is disconnected. This is because of this segment, that is, $L_n \cap B$. So, we can say that there exists an element, that element we have taken as simply $(1/2, 0)$, and we have shown that there does not exist any connected open subset containing it. If this is the case, what is our motive? Our motive is to show that the space is not locally connected. What we have to do is that if we are taking any open neighborhood containing the element, there should be some G, this should be open plus connected, but we have shown that there does not exist G of this form, hence the space that we have taken, this X as a subspace of $\mathbb{R}^2$ is locally disconnected.

Moving ahead, let us discuss some of the results related to locally connected topological spaces. The first theorem is: a topological space $(X, \mathcal{T})$ is locally connected if and only if the components of each open subset of X are open. It will also lead us to deduce that components are also open sets, which we discussed earlier. Now, let us assume that $(X, \mathcal{T})$ is locally connected. What is our motive? Our motive is to show that the components of each open subset of X are open. So, let us take an open subset N of X, and a component $C = C(x) \subseteq N$. Our motive is to justify that this C is open. Now, in order to justify that C is open, we will show that C is a neighborhood of each of its points. In order to justify that C is a neighborhood of each of its points, let us take $y \in C$. Now, if $y \in C$, $y \in N$. What is N? N is an open neighborhood of y, and if this is an open neighborhood of y, there exists an open connected subset G such that $y \in G \subseteq N$. Now, it is to be noted that this G is connected. Being a component, C is also connected. What about this $C \cap G$? This is non-empty. If their intersection is non-empty, we can say that this $C \cup G$ is also connected. Now, what this $C \cup G$ is? This is connected and also containing this C. What is this C? This is a component, and being a

component, this is maximally connected. If this is maximally connected, we can conclude that C is nothing but $C \cup G$. Therefore, we can conclude that $y \in G \subseteq C$, or C is a neighborhood of each of its points, therefore, C is open. This is the proof of the first part.

In order to justify the converse of this theorem, let us assume that the components of each open subset of X are open. Our motive is to show that the topological space $(X, \mathcal{T})$, is locally connected. For it, let us take an open neighborhood $N \subseteq X$ of $x \in X$. Now, we are taking a component $C = C(x)$ of N. So, $x \in C \subseteq N$. This is also given to us that every component is open. So, this C is open. Also, being a component, this is connected. Thus, we can conclude that for all x in X, and for all open neighborhoods N of x, there exists G, this G is an open and connected set such that $x \in G \subseteq N$. Therefore, $(X, \mathcal{T})$, is locally connected.

Moving ahead, one more thing we can note from here: if we are talking about this topological space $(X, \mathcal{T})$, why not let us take N equal to X? X is always an open set. If X is open and the space is locally connected, we can deduce from here that the components of X are also open. So, this is a simple conclusion from this theorem, that is, components of X are open. So, our conclusion is that, in general, when we are talking about a topological space, the components of the topological space are always a closed set, but if we are taking a particular type of topological space, that is, if we are talking about locally connected topological spaces, in this case, components are open too.

Moving ahead, we have already studied the concept of continuous image of connected spaces, even continuous image of path-connected spaces. We have shown that the continuous image of connected topological spaces is connected, and the same is true in the case of path-connected topological spaces. The question is whether the similar concept is true in the case of local connectedness. The answer is no. Let us see it. Let us take $X = \mathbb{N} \cup \{0\}$, this X is obviously a subset of $\mathbb{R}$, and $\mathbb{R}$ is with Euclidean topology. So, what will be the topology on X? That will be nothing but a discrete topology. Also, let us take a set $Y = \{0\} \cup \{1/n : n = 1, 2, ...\}$. This example we have already seen. Again, this $Y \subseteq \mathbb{R}$. We have shown that this space is not locally connected. Now, let us define a function $f : X \to Y$ such that $f(0) = 0$, and $f(n) = 1/n$, for all $n \in \mathbb{N}$. Obviously, this f is a bijective function. Even this is contin-

uous, too, because the topology on X is the discrete topology. The question is whether the image of X is locally connected. Again, note that X is locally connected, and Y is locally disconnected, which is already stated here. So, what is going on here is that a continuous image of a locally connected space may not be locally connected. Thus, the question is, what other restrictions can we impose? So, one well-known concept that we require is the concept of homeomorphism.

If we are using a homeomorphism, then there is no problem; that is, local connectedness is a topological property. In order to justify that local connectedness is a topological property, let us take a locally connected topological space $(X, \mathcal{T})$, and another topological space $(Y, \mathcal{T}')$, and a homeomorphism $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$. What is our motive? Our motive is to justify that $(Y, \mathcal{T}')$ is locally connected. In order to justify that $(Y, \mathcal{T}')$ is locally connected, what do we have to show? Just recall the previous theorem. Let us take $N \subseteq Y$, which is $\mathcal{T}'$-open, and let us take a component C of it. If we want to show that $(Y, \mathcal{T}')$ is locally connected, what we have to justify is that C is $\mathcal{T}'$-open. In order to justify that C is $\mathcal{T}'$-open, first, what can we claim, or can we prove that this $f^{-1}(C)$ is $\mathcal{T}$-open? Note that $f^{-1}(C)$ is non-empty. We have to justify that this is open, for which, try to justify that this is a neighborhood of each of its points. So, let us take $x \in f^{-1}(C)$. From here, it is clear that this $f(x) \in C$. But note that this C is also a subset of N. So, we can say that $f(x) \in N$, or this $x \in f^{-1}(N)$. It is to be noted that N is $\mathcal{T}'$-open, and by continuity $f^{-1}(N)$ is $\mathcal{T}$-open. So, what exactly $f^{-1}(N)$ is? $f^{-1}(N)$ is a $\mathcal{T}$-open neighborhood of x, and if this is a neighborhood of x, by using the local connectedness of $(X, \mathcal{T})$, there exists a $\mathcal{T}$-open set G, and note that this is not only open, this is connected too, such that $x \in G \subseteq f^{-1}(N)$, or $f(x) \in f(G) \subseteq N$. Now, $f(x) \in C$ and $f(x) \in f(G)$. Also, G being connected, $f(G)$ is also connected. Thus, we can conclude that this $C \cup f(G)$ is connected. At the same time, $C \cup f(G)$ will always be a superset of C. But what is this C? C is a component. So, by maximality of C, we can conclude that $C = C \cup f(G)$, or this $f(G) \subseteq C$, and if $f(G) \subseteq C$, we can say that $f(x) \in f(G) \subseteq C$, or $x \in G \subseteq f^{-1}(C)$. That's all to justify that $f^{-1}(C)$ is a neighborhood of x. Because this x is arbitrary, therefore the $f^{-1}(C)$ is a neighborhood of each of its points. Thus, $f^{-1}(C)$ is $\mathcal{T}$-open.

Finally, use the power of f. Note that this f is a homeomorphism. Till

now, we have used the continuity of f, and being homeomorphism, this is bijective and closed. Now, as $f^{-1}(C)$ is $\mathcal{T}$-open, we can talk about $X - f^{-1}(C)$, and this is $\mathcal{T}$-closed. What about $f(X - f^{-1}(C))$? This is $\mathcal{T}'$-closed as f is a homeomorphism, f is a closed function. Now, by bijectiveness of f, we can conclude that $f(X - f^{-1}(C)) = Y - C$, which is $\mathcal{T}'$-closed, or C is $\mathcal{T}'$-open, and this is the concept which we required. Therefore, $(Y, \mathcal{T}')$ is locally connected.

These are the references.

That's all from this lecture. Thank you.