Course Name: Essentials of Topology
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Welcome to Lecture 46 on Essentials of Topology.

In this lecture, we will study the concept of path components and initiate the study of local connectedness. Begin with the concept of path components. We have already studied the concept of components using the notion of connectedness. A natural question arises: can we use the concept of path connectedness for similar study? Can we discuss a concept similar to the notion of components? The answer is yes, and that we call path components. What is it? Let $(X, \mathcal{T})$ be a topological space and R be a relation on X such that xRy if there exists a path from x to y in X. Note that we can show that this relation is an equivalence relation, and after showing the relation as an equivalence relation, we can define the concept of path components.

In order to show that this relation is equivalence, let us take any $x \in X$. What we can say is that xRx, and this is possible because we can define a continuous function $p : [0, 1] \to X$ such that $p(t) = x$, for all $t \in [0, 1]$. Such paths are also known as constant paths. Therefore, the relation R is reflexive. We can also justify that this relation is symmetric. So, if we are assuming that xRy, what does it mean? It means that there exists a continuous function $p : [0, 1] \to X$ such that $p(0) = x$ and $p(1) = y$. Now, by using this function, we can construct another function $p' : [0, 1] \to X$ such that $p'(t) = p(1-t)$, for all $t \in [0, 1]$. Now, this p' is continuous because of the continuity of p, and this we can check directly. Also, $p'(0) = p(1) = y$, and $p'(1) = p(0) = x$. Thus, there exists a continuous function $p' : [0, 1] \to X$ satisfying $p'(0) = y$, and $p'(1) = x$, that is, there exists a path from y to x, and hence we can deduce that yRx. Thus, this relation is symmetric.

Further, the transitivity of R is simple because we have already discussed such a concept. If we are taking xRy and yRz, then there exists continuous functions $p_1 : [0, 1] \to X$ such that $p_1(0) = x$ and $p_1(1) = y$, and $p_2 : [0, 1] \to X$ such that $p_2(0) = y$ and $p_2(1) = z$. Now, we can construct a continuous func-

tion $p : [0, 1] \to X$, which we have already seen, that is defined as $p(t) = p_1(2t)$, if $0 \le t \le 1/2$, and $p(t) = p_2(2t - 1)$, if $1/2 \le t \le 1$. Such constructions we have already seen using the concept of Pasting Lemma. Also, $p(0) = x$, and $p(1) = z$. So, xRz. Thus, what have we shown here? We have shown that this relation R is an equivalence relation. Now, begin with this equivalence relation, we can define the concept of path components.

What path components are actually? If we are having a topological space $(X, \mathcal{T})$ and let R be the equivalence relation on X such that xRy if there exists a path from x to y in X. Then the equivalence classes for R are called the path components of X. For an element $x \in X$, if we are talking about this $[x]$, we call this the path component of $x \in X$. We shall also denote this $[x]$ by $P(x)$, too. It is to be noted that the path components are equivalence classes, therefore, for $x, y \in X$, either $P(x) = P(y)$ or $P(x) \cap P(y) = \emptyset$.

Moving ahead, let us see some of the results related to path components. Begin with a topological space $(X, \mathcal{T})$. Our first result is that the path components of a topological space $(X, \mathcal{T})$ partition the set X. This is simple to justify because we have already shown that the path components are nothing but equivalence classes, and the rest follows from the properties of equivalence classes. Coming to the second one, the path components of a topological space are path-connected. This directly follows from the definition of path components, but still, we can see the justification. For example, if we are taking a path component $P(x)$ and two elements $y, z \in P(x)$. Then there exists a path connecting this y to x, and there exists another path connecting x to z. If we have two paths, one from y to x and one connecting x to z, we can construct another path connecting this y to z. Again, recall the construction. This is the result that we are using again and again, so there is no need to write it. Thus, $P(x)$ is path-connected, and that's the proof of this result.

Further, moving ahead, the path components of a topological space are the maximal path-connected subsets of X. What we can do here is simple because if we are taking any path component $P(x)$, it can be shown that $P(x) = \cup_{x \in P} P$. Note that P are path-connected sets containing x. The first one is trivial because this $P(x) \subseteq P$, as $P(x)$ is also path-connected, and this contains x. So, $P(x) \subseteq \cup_{x \in P} P$. Further, if we want to show that $\cup_{x \in P} P \subseteq P(x)$, what do we have to justify? Let us take $y \in \cup_{x \in P} P$. As P also contains x, it

means that $x, y \in P$. As P is path connected, there exists a path connecting x to y, and if this is a path connecting x to y, meaning $y \in P(x)$. Thus, $\cup_{x \in P} P \subseteq P(x)$, and hence, this $P(x) = \cup_{x \in P} P$. From this construction, it is clear that a path component is always a maximal path-connected subset of X.

Moving ahead, the next one is that each non-empty path-connected subset of a topological space $(X, \mathcal{T})$ is contained in precisely one path component. Take a set A, this is a subset of X, and we are taking that this is non-empty, and this is path connected. Now, if this is non-empty, we can take an element in this set. Let us take $x \in A$. Now, let us also take the path component $P(x)$ of x in the space. What we can justify is that this $A \subseteq P(x)$. The question is, how is it possible? The answer is, note that this A and $P(x)$ are not disjoint, and if they are not disjoint, we can say that this $A \cup P(x)$ is path connected. Also, $P(x) \subseteq A \cup P(x)$ and $P(x)$ is maximal, too. So, using the maximality of $P(x)$, we can conclude that $P(x) = A \cup P(x)$, and so $A \subseteq P(x)$. This is the proof of this result.

Further, a path component is always contained in a component. This is the relationship between the path component and the component. It is simple to justify. Let us take this is $P(x)$, path component of x. Note that this is path-connected, and if this is path-connected, we have already shown that every path-connected set is connected. So, this will be connected, too. We know the result that every connected set is contained in a component, therefore, this $P(x)$ will also be contained in a component. Finally, we can show that each component is a disjoint union of path components. For it, let us take a connected set C, this connected set C is a subset of a component. This we have already shown. Therefore, this $P(x)$ will always be a subset of a component. Also, each component is a disjoint union of path components. Now, if we are using the previous result, that is, the path components are contained in the component, it is to be noted that path components are equivalence classes. So, there is a disjoint union, which is nothing but a component, that is the justification of this result.

Moving ahead, let us take some of the examples. We have already studied the Topologist's sine curve. In this case, we have taken the set $S = \{(x, \sin(1/x)) : 0 < x \leq 1\}$, and we have shown that $X = S \cup \{(0,0)\}$ is connected in $\mathbb{R}^2$ but not path-connected. Now, if this is the case because this is connected, so we

can say that X is the only component, but note that there are two path components. The path components are S and $\{(0,0)\}$. One thing is clear, the components are always closed sets, but here S is a path component, but S is not closed. So, path components may not be a closed set. Why? This S is not a closed set because it doesn't contain its limit point, that is this $\{(0,0)\}$.

Moving ahead, let us take the set of real numbers with lower limit topology. We know that this is totally disconnected, and if this is totally disconnected, what are the components? That's all singleton sets; that is, singleton sets are both components as well as path components. The question is, whether for $x \in \mathbb{R}$, $\{x\}$ is open in this topology. The answer is no. So, what we conclude from here is that even if components are closed sets, they may not open. What about path components? We have already shown here that a path component may not be closed, and even a path component may not be open. Thus, again, a question arises. Can we talk about another form of connectedness so that the components become open sets? The answer is yes, and we can do it in the framework of the concept of local connectedness.

Begin with the concept of local connectedness. A topological space $(X, \mathcal{T})$ is locally connected at $x \in X$ if for each open nbd N of x there exists a connected open set G such that $x \in G \subseteq N$. The topological space $(X, \mathcal{T})$ is locally connected if it is locally connected at each of its points. From here, we can deduce one thing that if this topological space $(X, \mathcal{T})$ is locally connected, let us take collection $\mathcal{B}$, and this is a collection of connected open sets, then for all open neighborhoods N of x, there exists a connected open set G such that $x \in G \subseteq N$. If this property holds, this $\mathcal{B}$ becomes a basis for this topology. Also, if we are taking a topological space $(X, \mathcal{T})$, and let us take a basis $\mathcal{B}$ for it. What is this $\mathcal{B}$? This is a collection of connected open sets. So, if this is the basis, so for all open neighborhoods N of x, we can say that there exists some $G \in B$ such that $x \in G \subseteq N$, that is, this space $(X, \mathcal{T})$ is locally connected. The idea is that the local connectedness of a topological space can also be characterized in terms of a basis, and that characterization is here; A topological space $(X, \mathcal{T})$ is locally connected if it has a basis consisting of connected (open) sets. Now, we have two notions with us: one is the concept of connectedness, and now we have seen the definition of local connectedness. The question is, is there any relationship between both notions? Whether connectedness implies local connectedness? The answer is no. Again, the question

arises whether local connectedness implies connectedness. The answer is no, and this is an interesting one.

Let us see some of the examples of local connectedness. Begin with the first example that is, indiscrete topological spaces are locally connected because if we have a indiscrete topological space $(X, \mathcal{T})$, $\{X\}$ is a basis for it. Note that this is open as well as connected. Similarly, if we are talking about discrete topological spaces, discrete topological spaces are also locally connected. Why? The answer is simple. We know the nature of the basis for it, that is, $\{\{x\} : x \in X\}$. It is to be noted that the singleton sets are always connected. These are also open because the topology is discrete. Therefore, discrete topological spaces are locally connected. Even the set of real numbers with standard topology is also locally connected because we know that $\{(a, b) : a, b \in \mathbb{R}, a < b\}$ is the basis for standard topology.

Moving ahead, let us take an example of a space that is not locally connected. The example is here that if we are taking the set $X = \{0\} \cup \{1/n : n = 1, 2, \ldots\}$, as a subspace of $(\mathbb{R}, \mathcal{T}_e)$. Note that the set $\{1/n\}$ is both open and closed in X. If we are taking a nbd of 0, that is what we are talking this N, this is an open neighborhood of 0, what will happen? Any open neighborhood of 0 will contain the elements of this form, that is $1/n$. Even if we are taking the set of the form $[0, \epsilon)$, what will happen? By using the Archimedean property, we know that there exists a natural number n such that $1/n < \epsilon$, that is, this $1/n$ is a member of $[0, \epsilon)$, but it is to be noted that this singleton set $\{1/n\}$ is clopen and if this is clopen what about $[0, \epsilon)$? This set is not connected. So, we cannot find any G that is connected, so that $x \in G \subseteq N$. Therefore, X is not locally connected.

Moving ahead, let us take one more example. This is the example that justifies that a locally connected space may not be connected. The space X is $[0, 1/2) \cup (1/2, 1]$, we are taking it as a subspace of the set of real numbers with Euclidean topology. We have already studied this example in the case of connectedness. Actually, we are removing $1/2$ from this set. Therefore, this set is not connected, but it is locally connected. The question is, how is it possible? The answer is simple because if you are taking any element of this set X and let us take any neighborhood, we can always construct a connected open set, that is, G, such that $x \in G \subseteq N$. This construction is possible

because in this space, some of the open sets will be of the form something like this $[0, a)$, this is open interval $(a, 1/2)$. Even that may be of the form (a, b), that may also look like $(1/2, b)$, even that may look like $(b, 1]$. So, the open sets G will always be of the form of such intervals. Thus, we can construct G, and we already know that all of these are connected. Therefore, we can always find such G, and this space will be locally connected.

These are the references.

That's all from this lecture. Thank you.