Course Name: Essentials of Topology
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Welcome to Lecture 45 on Essentials of Topology.

In this lecture too, we will study the concept of path-connectedness. In the previous lecture, we have already seen the possible relationship between connectedness and path-connectedness. We have also studied a number of results related to connected topological spaces. Even connectedness doesn't imply path-connectedness. We will study some of the results related to the path-connectedness of a topological space, which is similar to those of connected topological spaces.

Begin with the first one. We have already studied such theorems in the case of connected topological spaces. The statement of this theorem is: Let $(X, \mathcal{T})$ be a topological space and $\{E_i : i \in I\}$ be a collection of path-connected subsets of X such that $\bigcap_{i\in I} E_i \neq \emptyset$. Then $\bigcup_{i\in I} E_i$ is path-connected. In order to justify it, let us begin with the fact that $\bigcap_{i\in I} E_i \neq \emptyset$, so, we can take an element, say, $x_0 \in \bigcap_{i\in I} E_i$. Also, let us take $\bigcup_{i\in I} E_i$ as a set E. Our motive is to show that E is path-connected. If we want to show that this is path-connected, let us take $x, y \in E$. If $x, y \in E$, one of the possibilities is that $x, y \in E_i$, for some $i \in I$. As E_i is path-connected, we can always find a path joining x and y. Now, let us take $x \in E_j$ and $y \in E_k$, for some different $j, k \in I$. It is to be noted that E_j and E_k are path-connected. If $x \in E_j$, note that $x_0 \in E_j$, too. It means that $x_0, x \in E_j$, and similarly, $x_0, y \in E_k$. If $x_0, x \in E_j$, there exists a continuous function $p_1 : [0, 1] \to E_j$ such that $p_1(0) = x$ and $p_1(1) = x_0$. Similarly, there exists a continuous function $p_2 : [0, 1] \to E_k$ such that $p_2(0) = x_0$ and $p_2(1) = y$. So, what we have shown here is that if we are taking this set E, what is happening here is that if we are taking any $x, y \in E$, this is x and y, if we are taking this is x_0 there exists some path joining x to x_0, and also, there exists a path joining x_0 to y. So, use the construction based on Pasting Lemma, which we have already studied in the previous lecture. What we can deduce is that for all $x, y \in E$, there exists a continuous function $p : [0, 1] \to E$ such that $p(0) = x$ and $p(1) = y$. It is to be noted here that I am not providing the details about this function p because of the construction we did in the previous lecture. Thus, we can say that for each x and y in E, there exists a path in E, and hence this E is the path-connected. That's the proof of this theorem.

Moving ahead, let us take another theorem, i.e., the continuous image of a path-connected topological space is path-connected. In order to justify it, let us take a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ and assume that $(X, \mathcal{T})$ is path-connected. We have to justify that $f(X)$ with relative topology is path-connected. Now, in order to justify it, let us take $y_1, y_2 \in f(X)$. Our motive is to justify that there exists a path joining y_1 and y_2 in $f(X)$. As $y_1, y_2 \in f(X)$, obviously, there exists $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Also, note that $x_1, x_2 \in X$, and $(X, \mathcal{T})$ is path-connected. If this is path-connected, we can construct a continuous function $p : [0, 1] \to X$ such that $p(0) = x_1$, and $p(1) = x_2$. So, finally, what we have with us, this is the space X, and let us take this as Y, but we have this $f(X)$ inside Y, that is, this is our X, this is Y, and this is $f(X)$. We have taken two elements y_1 and y_2 in $f(X)$. Corresponding to y_1 and y_2, we have chosen x_1 and x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$; that is, this function is f, this is y_1, and this is y_2. Also, let us take this as $[0, 1]$, and what we have shown is that $p(0) = x_1$, and $p(1) = x_2$. Now, from this picture, it is clear that we can construct a function from this closed interval $[0, 1]$ to $f(X)$. The question is how? The answer is here. Let us take a function $p' : [0, 1] \to f(X)$, such that $p' = f \circ p$. Note that f and p are continuous functions. Therefore, p' is also a continuous function. The question is, what about this $p'(0)$, that is given as $p'(0) = f(p(0)) = f(x_1) = y_1$. Similarly, $p'(1) = f(p(1)) = f(x_2) = y_2$. Thus, there exists a path in $f(X)$ joining y_1 and y_2. Therefore, $f(X)$ is path-connected.

Moving ahead, let us use this theorem. The statement of this theorem is: the quotient space of a path-connected topological space is path-connected. Just recall the concept of quotient spaces. What we have studied is that if a topological space $(X, \mathcal{T})$ is given, the quotient map $q : (X, \mathcal{T}) \to (A, \mathcal{T}_q)$ is a continuous surjective function. Therefore, by using the previous result, we can say that $(A, \mathcal{T}_q)$ is path-connected, that is, the quotient space of a path-connected topological space is path-connected. Even one more result we can deduce from the previous one, that is path-connectedness is a topological property. Just recall the concept of homeomorphism, it's simple to deduce.

Moving ahead, let us discuss the path-connectedness of the product space. Again, this result is similar to the result that we have studied for connected topological spaces. So, let us take two path-connected topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. We can justify that the product space $(X_1 \times X_2, \mathcal{T})$ is path-connected. In order to justify it, let us take $(x_1, x_2), (x_1', x_2') \in X_1 \times X_2$. Our motive is to justify that there is a path joining (x_1, x_2) and (x_1', x_2'). We can conclude from here that $x_1, x_1' \in X_1$, and $x_2, x_2' \in X_2$. Again, it is to be noted that $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are path-connected. If both are path-connected, we can construct some path connecting x_1 and x_1', and a path connecting x_2 and x_2'. So, let us take this continuous function $p_1 : [0, 1] \to X_1$ such that $p_1(0) = x_1$, and $p_1(1) = x_1'$. Similar to it, we can take another continuous function $p_2 : [0, 1] \to X_2$ such that $p_2(0) = x_2$, and $p_1(1) = x_2'$. So, what exactly do we have with us? Let us take this as the closed interval $[0, 1]$, and we have two spaces, one is X_1, and one is X_2, and if we are taking elements here, this is x_1, and let us take this as x_1'. If we are taking here an element x_2, let us take this as x_2'. So, we can construct a function p_1 from this closed interval $[0, 1]$ to X_1, and that function is sending this 0 to x_1 and 1 to x_1'. Similarly, this function p_2 is sending 0 to x_2, and 1 to x_2'. Our motive is to construct a function $p : [0, 1] \to X_1 \times X_2$. The question is, how do we construct this continuous function? Recall the idea we have a continuous function $p_1 : [0, 1] \to X_1$ and another continuous function $p_2 : [0, 1] \to X_2$. Thus, we can construct a continuous function $p : [0, 1] \to X_1 \times X_2$ such that $p(x) = (p_1(x), p_2(x))$, for all $x \in [0, 1]$. Also, $p(0) = (p_1(0), p_2(0)) = (x_1, x_2)$ and $p(1) = (p_1(1), p_2(1)) = (x_1', x_2')$. So, what have we justified? We have justified that for all $(x_1, x_2), (x_1', x_2') \in X_1 \times X_2$, there exists a continuous function $p : [0, 1] \to X_1 \times X_2$ so that $p(0) = (x_1, x_2)$ and $p(1) = (x_1', x_2')$. Therefore, the product space $(X_1 \times X_2, \mathcal{T})$ is path-connected.

These are the references.

That's all from this lecture. Thank you.