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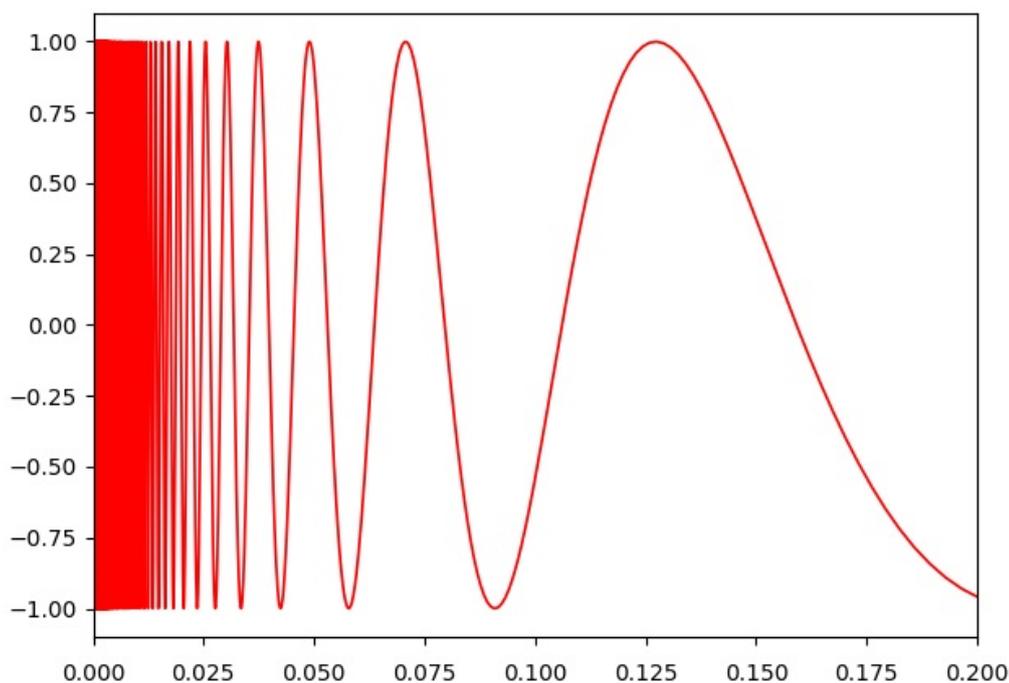
Welcome to Lecture 44 on Essentials of Topology.

In this lecture too, we will continue the study of the concept of path connectedness. Specifically, we will see the possible relationship between the concept of connectedness and path connectedness. Begin with what we noted in the previous lecture: path connectedness implies connectedness, while connectedness does not imply path connectedness. Let us see the justification of both statements.

Beginning with the first one, every path-connected topological space is connected. In order to justify it, let us take a topological space (X, \mathcal{T}) , and assume that this is path-connected. Our motive is to justify that this space is connected, too. Now, let us try to justify by using contradiction. If possible, let us take this space is disconnected. So, we are assuming that this (X, \mathcal{T}) is disconnected, and if it is disconnected, what will happen? There exists some subsets A and B of X such that both are non-empty, A and B are open in X , these will always be disjoint, and their union is nothing but X . It is to be noted here that A and B are non-empty, so what we can do is take some of the elements in these, so let us take $x \in A$ and also let us take $y \in B$. So, we are assuming it. It means that x and y are elements of X . But at the same time, because this space (X, \mathcal{T}) , which is path connected, what will happen? We can talk about a path from x to y ; that is there exists a continuous function $p : [0, 1] \rightarrow X$ such that $p(0) = x$ and $p(1) = y$. Now, as $x \in A$, $p^{-1}(x) \in p^{-1}(A)$. Similarly, as $y \in B$, so $p^{-1}(y) \in p^{-1}(B)$. Thus, we can deduce that $p^{-1}(A)$ and $p^{-1}(B)$ are non-empty. Because this p is a continuous function, $p^{-1}(A)$ and $p^{-1}(B)$ are open in this subspace, that is $[0, 1]$. Also, as $A \cap B = \emptyset$, $p^{-1}(A) \cap p^{-1}(B) = \emptyset$. Finally, $p^{-1}(A \cup B) = p^{-1}(X) = [0, 1]$. Thus, if we are assuming that (X, \mathcal{T}) is disconnected and it has a separation, that is given by a pair (A, B) ; by using this separation, we have constructed a separation $(p^{-1}(A), p^{-1}(B))$ of $[0, 1]$. Therefore, this interval is disconnected. So, we reached to a contradiction. Therefore, our assumption, which we have

taken, that the space (X, \mathcal{T}) is disconnected, is wrong. So, the space (X, \mathcal{T}) is also connected; that is, every path-connected topological space is connected.

Moving ahead, let us see the justification of this statement: a connected topological space may not be path-connected. In order to justify this statement, it is sufficient to provide a counter-example. It is to be noted that there are no trivial examples for it. What we are going to discuss here is what is known as Topologist's sine curve. We will justify that this space is connected, but the same space is not path-connected. What is the structure precisely? Let us take a set $S = \{(x, \sin(1/x)) : 0 < x \leq 1\}$. Note that this is a subset of \mathbb{R}^2 . So, we can talk about the relative topology on S with respect to Euclidean topology on \mathbb{R}^2 . Also, if we are computing the closure of this set, the closure is given by $cl(S) = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$. This set, is known as Topologist's sine curve. The graph of this curve is given here; that is, we have restricted only $0 \leq x \leq 2$.



It is to be noted that this set S is connected. In order to justify the connectedness of S , let us define a function $f : (0, 1] \rightarrow \mathbb{R}^2$ such that $f(x) = (x, \sin(1/x))$. It is to be noted that if we are taking a function, something like

$f_1(x) = x$ and $f_2(x) = \sin(1/x)$. Then as both the functions are continuous, f is continuous too. So, what is the image of $(0, 1]$? That is nothing but S , that is, S is a continuous image of a connected set, therefore, S is also connected. From this S , we are going to construct a set that is not precisely this closure of S . We are going to construct a different set, and we will show that that set is connected too, but that is not path connected.

Now, coming to our construction. We have shown that the set $S = \{(x, \sin(1/x)) : 0 < x \leq 1\}$ is connected. We also know that if S is connected, its closure is also connected, and not only its closure, if we are taking any set lying in between S and $cl(S)$, that will also be connected. So, we are going to take a set X , and this set X is precisely $S \cup \{(0, 0)\}$. It is clear that this set is connected, but what can we justify? We can justify that X is not path-connected. In order to justify that this is not path connected, we will again show it by contradiction.

Now, let us assume that this X is path-connected. Now, if this is path-connected, what will happen? For all elements $x, y \in X$, there exists a continuous function $p : [0, 1] \rightarrow X$ such that $p(0) = x$ and $p(1) = y$. In order to create some contradiction, let us choose two elements from X , and one natural choice is why not let us take this origin, that is $(0, 0)$, and the second element which we are choosing, that is $(1/\pi, 0)$. Note that both the elements are in X . Because we have assumed that this X is path connected, so by using the definition of path connectedness there exists a continuous function, we are taking that continuous function again as p , this is $p : [0, 1] \rightarrow X$ such that $p(0) = (0, 0)$ and $p(1) = (1/\pi, 0)$.

We have already studied the concept of projection maps. Let us take the projection map $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$, which is given by $\pi_1(x, y) = x$. We know that this function is continuous. Now, we can talk about the composition of functions $p : [0, 1] \rightarrow X$ and $\pi_1 : X \rightarrow \mathbb{R}$ (the restricted function with respect to projection function π_1). Note that both the functions are continuous, so what will happen? Their composition, $\pi_1 \circ p$, is also a continuous function. It is to be noted that $(\pi_1 \circ p)(0) = \pi_1(p(0)) = \pi_1((0, 0)) = 0$. Also, $(\pi_1 \circ p)(1) = \pi_1(p(1)) = 1/\pi$. So, what exactly we have done? We have constructed a continuous function from $[0, 1]$ to \mathbb{R} with $(\pi_1 \circ p)(0) = 0$ and $(\pi_1 \circ p)(1) = 1/\pi$.

Now, we can use the well-known Intermediate Value Theorem. So, if we are using the Intermediate Value Theorem, what can we do? If we are taking any element in between 0 and $1/\pi$. The choice of this point is important. We are choosing the elements of this form, which is $(2n + 1/2)\pi$. Why are we choosing the points of this form? It is to be noted that if we are finding the value of $\sin(1/x)$, then this will be nothing but 1. It means we are choosing the points on which this peak is 1, and this will help us to create or to see that there is some contradiction. Now, if we are taking any point in between 0 and $1/\pi$, then by using this theorem, there exists $x_n \in [0, 1]$ such that $(\pi_1 \circ p)(x_n) = 1/(2n + \frac{1}{2})\pi$. From here, we can conclude that or we can think that about $p(x_n)$. It is to be noted that $p(x_n)$ is nothing but $(1/(2n + \frac{1}{2})\pi, 1)$ because $\sin(1/x)$ at this point will always be 1. Also, we can conclude that this $\{p(x_n)\}$, this is a sequence in X . If this is sequence in X , whether this is converging. The answer is yes. Note that $\{p(x_n)\}$ is a convergent sequence and this sequence is converging to $(0, 1)$. Also, we have taken $x_n \in [0, 1]$. So, what we can see that this $\{x_n\}$ is a sequence in this $[0, 1]$. Now, We can use the concept of analysis here.

By using the well-known Bolzano-Weierstrass theorem, we can conclude that there exists a subsequence. Let us assume that $\{x_{n_k}\}$ and note that this is not only a subsequence of sequence $\{x_n\}$, this is also convergent. Now, if this sequence is convergent, assume that it converges to x_0 . The question is, if this sequence is convergent to this x_0 , whether $x_0 \in [0, 1]$. The answer is yes because this is a closed set. Now, if $\{x_{n_k}\}$ is converging to x_0 , by using the continuity of the function p , what about $\{p(x_{n_k})\}$? $\{p(x_{n_k})\}$ should converge to $p(x_0)$. What about this $p(x_0)$? This belongs to X because of the definition of this p . Now, we have already seen that the sequence $\{p(x_n)\}$ is converging to an element $(0, 1) \in \mathbb{R}^2$. As $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, so if we are talking about the convergence of $\{p(x_{n_k})\}$, this will also converge to $(0, 1) \in \mathbb{R}^2$. It is to be noted here that $(0, 1)$ is not an element of X . At the same time, if $\{x_{n_k}\}$ is converging to x_0 , $\{p(x_{n_k})\}$ will converge to $p(x_0)$, and in this case, $p(x_0)$ is an element of X . So, this is a contradiction. Thus, our assumption that is the space X , which we have assumed, that this space is path connected, is wrong. Hence, this space is not path-connected. So, what exactly have we seen here? We have constructed an example, and this is an example of a space that is connected but not path-connected.

These are the references.

That's all from this lecture. Thank you.