

**Course Name: Essentials of Topology**  
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Welcome to Lecture 43 on Essentials of Topology.

In this lecture, we will study the concept of path connectedness. Begin with, we have already studied the concept of connectedness of a topological space. The question is what path connectedness is and how it differs from the concept of connectedness of a topological space. We have already seen the concept of connectedness. What connectedness means is that a topological space cannot be separated into distinct pieces. Meaning is to say that if we are taking a topological space  $(X, \mathcal{T})$ , we cannot make a separation of it; that is, no separation is possible. We have already seen the meaning of separation. Similarly, when we are looking at the concept of path connectedness, it means that we can take a continuous walk in the space between any two points in it. Meaning is to say that if we are taking any space, let us take two points  $x$  and  $y$  in it. If we can move from  $x$  to  $y$  by using a path, then we say that this space is path connected. So, the first and natural question will come. How are both the connectedness related, or what are the differences between these two connectedness? The answer is, if a space is path connected then this will always be connected. That is, path connectedness implies connectedness. We will also see that the converse may not be true. That is, if a space is connected, it is not necessary that the space is path-connected. We will see these concepts in details.

Begin with when we are talking about the concept of path connectedness, the concept is in terms of a continuous walk, and that is given in terms of a path. So, let us see what this concept is. The concept of the path is here. That is, if we are having a topological space  $(X, \mathcal{T})$ . Let us take the closed interval  $[0, 1]$  with standard topology as a subspace of  $\mathbb{R}$ . A path in this topological space is a continuous function  $p : [0, 1] \rightarrow X$ . Meaning is to say that if we are taking this as closed interval  $[0, 1]$ . Let us take the space. Let us take this as  $X$ . A path is a continuous function from this closed interval to this space  $X$ . Let us take this function as  $p$ . Let us take this as  $p(0)$ , and this as  $p(1)$ . So,

we can talk about the image of this closed interval  $[0, 1]$  in  $X$  and it will look like in this way. This  $p(0)$  is called the initial point of the path, while this  $p(1)$  is called the terminal point of the path. That is, the path begins at  $p(0)$  and ends at  $p(1)$ .

When we are defining the concept of path, we are taking the domain as closed interval  $[0, 1]$ . The question is, why not let us take an arbitrary closed interval here? Is it possible? The answer is yes. This is also possible, and one may take this closed interval  $[a, b]$  instead of taking this closed interval  $[0, 1]$ . Even several authors follow it. We can justify that both are equivalent. In order to justify that both concepts are equivalent, we have already seen that two closed intervals are homeomorphic to each other. So, first thing is that this closed interval  $[0, 1]$  and closed interval  $[a, b](a < b)$  are homeomorphic. Let us take a path  $p : [0, 1] \rightarrow X$ . Path begins at this  $p(0)$ , and the path ends at  $p(1)$ . Now, let us see the relationship between this closed interval  $[0, 1]$  and this closed interval  $[a, b]$ . What can we define? We can define a function  $f : [0, 1] \rightarrow [a, b]$  such that  $f(t) = (1 - t)a + tb, t \in [0, 1]$ . It is clear from the definition that this function  $f$  is bijective, and  $f$  and  $f^{-1}$  are continuous. Now, what can we do? We can define a function from this closed interval  $[a, b]$  to this space  $X$ . Let us take this is closed interval  $[a, b]$ , this is closed interval  $[0, 1]$ , and this is  $X$ . We already have a continuous function  $p : [0, 1] \rightarrow X$ . Also, we have another function that is  $f^{-1} : [a, b] \rightarrow [0, 1]$ . Let us take  $g = p \circ f^{-1}$ . Now,  $g(a) = p(f^{-1}(a)) = p(0)$ , that is the initial point. Also,  $g(b) = p \circ f^{-1}(b) = p(1)$ . So, what can we conclude from here that instead of this continuous function  $p : [0, 1] \rightarrow X$ , if we are taking a function  $g$  from this closed interval  $[a, b]$  to  $X$ . This function is also continuous as it is a composition of two continuous functions with the same initial and terminal points; that is,  $g(a) = p(0)$ , and  $g(b) = p(1)$ .

Let us take some of the examples of paths. The first example is from the set of real numbers. Obviously, we are talking  $\mathbb{R}$  with Euclidean topology, and this closed interval is  $[0, 1]$ . We have already seen that  $[0, 1]$  is a subspace of  $\mathbb{R}$ . Now, let us take  $x, y \in \mathbb{R}$ , and this is our closed interval  $[0, 1]$ . We can define a continuous function  $p : [0, 1] \rightarrow \mathbb{R}$  such that  $p(t) = (1 - t)x + ty, t \in [0, 1]$ . It is clear that  $p(0) = x$  and  $p(1) = y$ . So, this  $p$ , which we are defining here, is a path. Let us take another example. This is a function  $p : [0, 1] \rightarrow \mathbb{R}^2$ . Obviously,  $\mathbb{R}^2$  is with Euclidean topology, and the function is defined as

$p(t) = (\cos 2\pi t, \sin 2\pi t), t \in [0, 1]$ . This function is a continuous function, and this  $p$  is a path. What does it represent? Let us take the  $[0, 1]$ , and the image of this function is nothing but a circle. So, this is actually  $p$ , and in this case, it is to be noted that  $p(0)$  and  $p(1)$  will always be the same because  $p(0) = (\cos 0, \sin 0) = (1, 0)$ , and  $p(1) = (1, 0)$ .

Having the knowledge of the path, let us discuss the concept of path connectedness. A topological space  $(X, \mathcal{T})$  is a path connected if for every  $x, y \in X$ , there exists a path  $p : [0, 1] \rightarrow X$  such that  $p(0) = x$  and  $p(1) = y$ , that is path starts from  $x$ , and it ends at  $y$ . It is a natural question: if we are taking a subset of  $X$ , how do we define the path connectedness of a subset? The answer is here: if  $Y \subseteq X$ , we say that  $Y$  is path connected in topological space  $(X, \mathcal{T})$  if the subspace  $(Y, \mathcal{T}_Y)$  is path connected.

Let us see some of the examples.  $\mathbb{R}$  is path connected. We have already seen that if we are taking the topological space  $(\mathbb{R}, \mathcal{T}_e)$ , we can always define a continuous function  $p : [0, 1] \rightarrow \mathbb{R}$  such that for two elements  $x, y \in \mathbb{R}$ ,  $p(t) = (1 - t)x + ty, t \in [0, 1]$ . We have also seen that  $p(0) = x$  and  $p(1) = y$ . So, for each pair of elements in  $\mathbb{R}$ , there exists a path in  $\mathbb{R}$ .  $\mathbb{R}^2$  is also path connected. This is similar to the concept of this real line with Euclidean topology that if we are taking  $\mathbb{R}^2$  with Euclidean topology or standard topology on it, we can always define a function  $p : [0, 1] \rightarrow \mathbb{R}^2$  such that  $p(t) = (1 - t)x + ty, t \in [0, 1]$ , where  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  are in  $\mathbb{R}^2$ . Note that  $p(t) = ((1 - t)x_1 + ty_1, (1 - t)x_2 + ty_2)$ ,  $p(0) = (x_1, x_2)$  and  $p(1) = (y_1, y_2)$ . This function is a continuous function from this closed interval  $[0, 1]$  to  $\mathbb{R}^2$ , and such functions exist for each pair of elements  $x$  and  $y$  from  $\mathbb{R}^2$ . Therefore,  $\mathbb{R}^2$  is path connected. In general, we can see that  $\mathbb{R}^n$  is also a path connected.

Moving ahead, let us take some more examples. The Sierpinski space is path connected. In order to justify that this space is path connected, let us take  $X = \{a, b\}$ . The topology on  $X$  is  $\{\emptyset, X, \{a\}\}$ . Now, let us define a function  $p : [0, 1] \rightarrow X$ . This function is defined as  $p(t) = a$ , provided  $0 \leq t < 1$ , and  $p(t) = b$ , if  $t = 1$ . We can verify that this function is a continuous function. Note that  $p^{-1}(\emptyset) = \emptyset, p^{-1}(X) = [0, 1]$ , that is open in relative topology. Now, if we are finding the inverse image of  $\{a\}$ , that will be nothing but the semi-open interval  $[0, 1)$ , which is open in closed interval  $[0, 1]$ , as this is a subspace

of the set of reals with Euclidean topology. In the same fashion, we can also prove that the indiscrete topological spaces are path-connected. Just think about it, and this is simple to justify as we know that if we are defining any function from a topological space  $(X, \mathcal{T})$  to topological space  $(Y, \mathcal{T}')$ , and if  $\mathcal{T}'$  is indiscrete, the function will always be continuous. So, use this concept and justify that indiscrete topological spaces are path-connected.

Moving ahead, let us see a useful concept that we have to use, and the concept is: if we have a space, let us take this topological space as  $(X, \mathcal{T})$ , and in this space, if we take three elements, that is,  $x, y$ , and  $z$ . If there exists a path from  $x$  to  $y$ , let us connect them by this curve, and also, there exists a path which is connecting  $y$  and  $z$  by using this curve. Then is it possible to find a path from  $x$  to  $z$ ? The answer is yes, we can find that one. Formally, what I want to say is that if we are taking three elements  $x, y, z$  in this  $X$ . There is a path, that is a continuous function  $p_1 : [0, 1] \rightarrow X$  such that this  $p_1(0) = x$  and  $p_1(1) = y$ . Also, there exists another path, that is a continuous function  $p_2 : [0, 1] \rightarrow X$  such that this  $p_2(0) = y$  and this  $p_2(1) = z$ . The question is: can we construct a continuous function  $p : [0, 1] \rightarrow X$  such that  $p(0) = x$  and  $p(1) = z$ . The question is how to achieve the goal. The answer is to use these two functions,  $p_1$  and  $p_2$ , along with the well-known Pasting Lemma. The question is, how will this Pasting Lemma help us? The answer is simple. let us take a set  $A = [0, 1/2]$  and  $B = [1/2, 1]$ . It is to be noted that this  $A \cup B$  is nothing but this closed interval  $[0, 1]$ . Also,  $A$  and  $B$  are closed in this closed interval  $[0, 1]$ . We have two continuous functions, one is  $p_1$  and another is  $p_2$ . But it is to be noted that their domain is closed interval  $[0, 1]$ . What we can do that with the help of these  $p_1$  and  $p_2$ , we can construct another function  $p : [0, 1] \rightarrow X$  such that  $p(t) = p_1(2t)$ , if  $0 \leq t \leq 1/2$ , and  $p(t) = p_2(2t - 1)$ , if  $1/2 \leq t \leq 1$ . It is to be noted that what  $A \cap B = \{1/2\}$ . Also, at  $t = 1/2$ ,  $p_1(t) = p_2(t) = y$ . Further, what are we considering? These are continuous functions in their domain. How are these continuous? The answer is simple. What are we doing? Let us take a function which is sending every  $t$  to this is  $2t$ . Let us take this function as  $f$ , and now if we are taking the function  $p_1$  here, which is sending this  $2t$  to  $p_1(2t)$ . So what do we have? We have a composition of these two functions, that is,  $p_1 \circ f$ , which is sending every  $t$  to  $p_1(2t)$ . Note that  $p_1$  is a continuous function, and  $f$  is also a continuous function. Therefore, their composition, that is,  $p_1 \circ f$ , is also continuous. Similarly, we can justify that this function is also continuous. Finally, if we

are computing the value of  $p(0)$ , that is nothing but  $p_1(0)$ , that is precisely  $x$ , and if we are computing the value of  $p(1)$ , that is  $p_2(1)$ , and that is nothing but  $z$ . So, what have we constructed? Everything is clear from here; that is, we are constructing a continuous function. This  $p$  is continuous, which we have already justified by using the concept of Pasting Lemma, and  $p(0) = x$ ,  $p(1) = z$ ; that is, there exists a path connecting  $x$  to  $z$ . That was our motive.

Now, let us use this result to discuss some examples. The first example that we are going to discuss here is the concept of the parabola. Let us take a set  $P = \{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ . We can see that this  $P$  is a path connected. Let us see what the logic we are going to use here. Let us see the parabola first. We are talking about this structure, that is  $y^2 = x$ . So, let us take this is  $y^2 = x$ . If we want to show that this is path connected, let us take two points on this parabola. So, we are taking any arbitrary points. The first one is this  $(x_1, y_1)$ , and the second one is  $(x_2, y_2)$ . Our motive is to justify that there exists a path connecting these elements, that is  $(x_1, y_1)$  and  $(x_2, y_2)$ . For this, what exactly are we going to do here? We will construct a path connecting the origin to  $(x_1, y_1)$ . In the same fashion, one can construct a path connecting the origin to  $(x_2, y_2)$ . Now, if we are using the previous result, we are having two paths. One is connecting the origin to  $(x_1, y_1)$ , and one is connecting the origin to  $(x_2, y_2)$ . It means that there exists a path connecting  $(x_1, y_1)$  to  $(x_2, y_2)$ , and hence this parabola is path-connected. So, the question is how to construct a path connecting origin to  $(x_1, y_1)$ . If we want to see, it is simple. Let us take a function  $p : [0, 1] \rightarrow P$  such that  $p(t) = (t^2x_1, ty_1)$ ,  $t \in [0, 1]$ . It is to be noted that this element, that is,  $(t^2x_1, ty_1)$ , will always be a member of  $P$ . Also, it is to be noted that both the components are continuous. Therefore,  $p$  is also continuous. Further, if we are finding the value of  $p(0)$ , that is nothing but the origin, and if we are looking for the value of  $p(1)$ , that is nothing but  $(x_1, y_1)$ . So, we have constructed a path connecting this origin to  $(x_1, y_1)$ . Similarly, one can construct a path connecting the origin to this  $(x_2, y_2)$ . Therefore, as per our discussion, there exists a path connecting  $(x_1, y_1)$  to  $(x_2, y_2)$ , and that is for every pair of elements in this  $P$ . Hence this  $P$  is path connected.

Let us take one more example, that is  $\mathbb{R}^2 - \{(0, 0)\}$ . That is, remove the origin from the plane, and we can justify that if this is the set  $X$ , this set is path connected. The question is, what is the justification behind it? The answer is simple because what we are doing here, we are removing this origin

from  $\mathbb{R}^2$ , and we are getting this new set, that is  $X$ . If we are taking any two elements in  $\mathbb{R}^2$ , let us take these two elements. One may think that if we are constructing a path, that path will pass through the origin, and that will not be continuous but recall the previous construction. What we have already seen that let us take another point and what can we do? We can construct a path connecting this first point, that is this is  $P_1$  to  $P_2$ , we can construct another path connecting this  $P_2$  and  $P_3$ . So, finally, using the result that we have already discussed, there is a path from  $P_1$  to  $P_3$ , and therefore, this set, that is,  $X$  is path-connected.

These are the references.

That's all from this lecture. Thank you.