**Course Name: Essentials of Topology**
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Welcome to Lecture 42 on Essentials of Topology.

Continuing from the previous lecture herein, first, we will discuss the connectedness of some of the subsets of $\mathbb{R}^2$, and thereafter, we will study the well-known Intermediate Value Theorem.

Begin with what we have seen in the previous lecture, which is that the unit circle is connected. Let us see another example. The circle $C = \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 = r^2\}$ with standard topology as a subspace of $\mathbb{R}^2$ is connected. In order to show that the circle $C$ is connected, let us define a function $f : [0, 2\pi] \to \mathbb{R}^2$ such that $f(t) = (a + r\cos t, b + r\sin t), t \in [0, 2\pi]$. Actually, $f(t) = (f_1(t), f_2(t))$, where $f_1 : [0, 2\pi] \to \mathbb{R}$ such that $f_1(t) = a + r\cos t$, and $f_2 : [0, 2\pi] \to \mathbb{R}$ such that $f_2(t) = b + r\sin t$. Note that both $f_1$ and $f_2$ are continuous. Therefore, $f$ is also continuous. If $f$ is continuous, let us find the image of the interval $[0, 2\pi]$. Note that $f([0, 2\pi]) = C$. Thus $C$ is a continuous image of the interval $[0, 2\pi]$. Note that $[0, 2\pi]$ is connected. So, we conclude that $C$ is also connected.

Similar to the example of this circle, let us take another example, that is, the example of a parabola, which is given as $P = \{(x, y) \in \mathbb{R}^2 : y^2 = 4ax\}$, with standard topology as a subspace of $\mathbb{R}^2$. We can show that $P$ is also connected. If we want to see the connectedness of this parabola, let us define a function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(t) = (t^2/4a, t), t \in \mathbb{R}$. Just like the previous case, we can conclude that the function $f$ is continuous. Also, $f(R) = P$. Therefore, the parabola $P$ is a connected subset of $\mathbb{R}^2$.

Let us take the next example; this is the line $L = \{(x, y) \in \mathbb{R}^2 : y = ax + b\}$, which is a subset of $\mathbb{R}^2$. We can show that this line $L$ is also connected. For which, define a function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(t) = (t, at + b)$. Now, what about $f(\mathbb{R})$, that is nothing but $L$. Again, because this $f$ is a continuous function, and we know that $\mathbb{R}$ is connected, we can conclude that $L$ is also connected. Why? For the same reason, we have used $L$ is a continuous image of a connected set. Moving ahead, let us take another example that is inspired from the previous one. If we are taking the union of sets $L_1$ and $L_2$, where $L_1 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ and $L_2 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ with standard topology as a subspace of $\mathbb{R}^2$, we can justify that $L_1 \cup L_2$ is also connected. If we see what $L_1$ is, this set is nothing but this is the $y$-axis. What about $L_2$? This is nothing but the $x$-axis. So, what precisely are we saying? That is, the union of $x$-axis and $y$-axis is connected. Why is it so? Note that $L_1$ and $L_2$ both are connected, and $L_1 \cap L_2 \neq \emptyset$, because they are intersecting at the origin. Therefore, $L_1 \cup L_2$ is connected.

Moving to the next one, $L = \{(x, y) \in \mathbb{R}^2 : x = a\} \cup \{(x, y) \in \mathbb{R}^2 : x = b\}$ is disconnected. Actually, what are we taking here? We are taking a set represented by line $x = a$, and another set is represented by line $x = b$. Note that, here, we are proving for $a < 0 < b$. Now, let us see how is $L = \{(x, y) \in \mathbb{R}^2 : x = a\} \cup \{(x, y) \in \mathbb{R}^2 : x = b\}$ disconnected. Note that $L$ is a union of $L_1 = \{(x, y) \in \mathbb{R}^2 : x = a\}$ and $L_2 = \{(x, y) \in \mathbb{R}^2 : x = b\}$. If we find $H_+ \cap L$, note that $H_+ \cap L = L_1$. Also, $L_2 = H_- \cap L$. Meaning is to say that $L_1$ and $L_2$ both are open in $L$. Even if we are finding the complement of $L_1$, that is nothing but $L_2$. From here, we can conclude that $L_1$ is open as well as closed, $L_1$ is a proper subset of $L$, and this is nonempty, too. Therefore, we can conclude that this $L$ is disconnected because we have shown that there exists a nonempty proper subset of $L$, which is both open and closed.

Moving ahead, let us take another example of a disconnected subset of $\mathbb{R}^2$. Here, we are taking this as a rectangular hyperbola $H_r = \{(x, y) \in \mathbb{R}^2 : xy = 16\}$. To show the disconnectedness, let $H_{r_+} = H_r \cap H_+$ and $H_{r_-} = H_r \cap H_-$. So, from here, we can conclude that $H_{r_+}$ and $H_{r_-}$, both are open in $H_r$. Also, what about the complement of $H_{r_+}$, that is nothing but $H_{r_-}$. Meaning is to say that $H_{r_+}$ is nonempty. This is a proper subset of $H_r$. Actually, $H_{r_+}$ is both open and closed. It justifies that $H_r$ is disconnected. It is to be mentioned here that we have studied a number of theorems, number of results related to connectedness that were obviously theoretical, but we have used each and every result at some point to discuss the connectedness of $\mathbb{R}$ or the subsets of $\mathbb{R}$, and also, we have used to discuss the connectedness or disconnectedness of subsets of $\mathbb{R}^2$

Moving ahead, let us discuss the Intermediate Value Theorem on a closed interval $[a, b]$, which is well-known in calculus. The statement of this theorem is here: Let $f : [a, b] \to \mathbb{R}$ be continuous, and assume that $r$ lies between $f(a)$ and $f(b)$. Then there exists at least one $c \in [a, b]$ such that $f(c) = r$. One of the well-known applications of this Intermediate Value Theorem is given here, that is, let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a)$ and $f(b)$ have opposite signs. Then, the equation $f(x) = 0$ has a solution between $a$ and $b$. This result has already been used to find out the solution of $f(x) = 0$.

Let us discuss a general version of the well-known Intermediate Value Theorem in the framework of topology. Even this theorem was first encountered in calculus, but after studying this result from the topology, one can see that the Intermediate Value Theorem that we are using in calculus is a particular version of the concept from topology, and this concept is based on the theory of connected topological spaces. The general version of the Intermediate Value Theorem is stated here: Let $(X, \mathcal{T})$ be a connected topological space and $f : (X, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_e)$ be a continuous function. If $p, q \in f(X)$ and $p \leq r \leq q$, then $r \in f(X)$. In order to prove this theorem, begin with $p \leq r \leq q$. Note that if $p = r$, or $q = r$, then $r \in f(X)$, because $p$ and $q$ are members of $f(X)$. Now, let us take $p < r < q$. Our motive is to justify that $r \in f(X)$. Let us try to prove the requirement by contradiction. If possible, let us assume that $r \notin f(X)$. Now, if we are taking two sets, one is $(-\infty, r)$, and another is $(r, \infty)$, note that both are open in $\mathbb{R}$. Also, let $A_0 = (-\infty, r) \cap f(X)$ and $B_0 = (r, \infty) \cap f(X)$. From here, note that $A_0$ and $B_0$, both are nonempty, as $A_0$ contains $p$ and $B_0$ contains $q$. The second one, $A_0$ and $B_0$ are open in $f(X)$. Further, $A_0 \cap B_0 = \emptyset$ and $A_0 \cup B_0 = f(X)$. Thus, if $r$ is not an element of $f(X)$, we have created a separation of $f(X)$, and it led us to conclude that $f(X)$ is disconnected. But it is to be noted that what $(X, \mathcal{T})$, we have assumed. This is connected. What is the function we have taken? The function is continuous. By continuity of the function $f$, $f(X)$ is connected. Thus, we reach a contradiction. Thus, $r \in f(X)$. That's the proof of this general version of the Intermediate Value Theorem.

Now, as we know, $[a, b]$ is connected. So, let $X = [a, b]$, $f(a) = p$, and $f(b) = q$. Also, let $p \leq r \leq q$. Then $f(a) \leq r \leq f(b)$. By using the above theorem, $r \in f(X)$, that is, there exists $c \in X = [a, b]$ such that $f(c) = r$. This is proof of the Intermediate Value Theorem in calculus.

These are the references.

That's all from this lecture. Thank you.