Course Name: Essentials of Topology
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Welcome to Lecture 41 on Essentials of Topology.

In this lecture too, we will continue the study of the concept of connectedness. Specifically, we will discuss that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$. Further, we will discuss the connectedness of some subsets of $\mathbb{R}^2$.

We have seen that the set of reals with the Euclidean topology is connected. Even we have also seen that $\mathbb{R}^2$ is connected. A natural question is whether $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}$. The answer is no. Before going to the precise justification, let us see some of the theories behind the concepts that we have to use. The first one is, $\mathbb{R}^2$ is homeomorphic to open half-plane $H_+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. The question is, how are we saying that H_+ is open? The answer is simple. Let us define a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x, y) = x$. Note that this function is continuous. Now, let $G = (0, \infty)$ be an open subset of $\mathbb{R}$. Then $f^{-1}(G) = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in G\} = \{(x, y) \in \mathbb{R}^2 : x \in G\} = \{(x, y) \in \mathbb{R}^2 : 0 < x < \infty\} = H_+$. Thus, H_+ is open in $\mathbb{R}^2$. Similar to H_+, we can also talk about the open left half plane $H_- = \{(x, y) \in \mathbb{R}^2 : x < 0\}$. Now, we can show that $\mathbb{R}^2$ is homeomorphic to H_+. How do you justify it? The answer is here. We can define a function $f : \mathbb{R}^2 \to H_+$ such that $f(x, y) = (e^x, y)$. What exactly is this function doing? Actually, (i) the left half plane is mapped to the strip in H_+, where $0 < x < 1$, (ii) the y-axis is mapped to the line $x = 1$, and (iii) the right half is mapped to the region in H_+, where $x > 1$. From the definition of function, it is clear that the function f is bijective, and f and its inverse are both continuous. Therefore, we can say that $\mathbb{R}^2$ is homeomorphic to H_+.

Moving ahead, let us discuss the connectedness of $\mathbb{R}^2 - \{(0, 0)\}$. We have already seen that $\mathbb{R}^2$ is connected, but the question is, what about the connectedness of $\mathbb{R}^2 - \{(0, 0)\}$? We have already seen that $\mathbb{R}^2$ is homeomorphic to H_+. As $\mathbb{R}^2$ is connected, H_+ is also connected. Similarly, we can say that $\mathbb{R}^2$ is homeomorphic to H_-, and this is connected, too. So, what exactly do we have with us? We have divided the plane into two parts. One part is H_-, and the second part is H_+. Now, what are the limit points of H_+? Note that all the points lying on the y-axis are the limit points of this right half plane. As H_+ is connected, the closure of H_+, i.e., $H_+ \cup \{(x, y) : x = 0\}$ is also connected. Now, $H+ \subset H_+ \cup (\{(x, y) : x = 0\} - \{(0, 0)\}) \subset H_+ \cup \{(x, y) : x = 0\}$. Thus, $A_1 = H_+ \cup (\{(x, y) : x = 0\} - \{(0, 0)\})$ is connected. With the same justification, if we are taking a set $A_2 = H_- \cup (\{(x, y) : x = 0\} - \{(0, 0)\})$, then A_2 is also connected. Note that $A_1 \cap A_2 \neq \emptyset$. Therefore, we can say that $A_1 \cup A_2$ is also connected. If this is connected, what exactly $A_1 \cup A_2$ is? This is nothing but $\mathbb{R}^2 - \{(0, 0)\}$. Therefore, $\mathbb{R}^2 - \{(0, 0)\}$ is connected. Now, let us use the connectedness of $\mathbb{R}^2 - \{(0, 0)\}$ to justify that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. If possible, let us assume that $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}$. Then there exists a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}$. If f is a homeomorphism, $\mathbb{R}^2 - \{(0, 0)\}$ is homeomorphic to $\mathbb{R} - \{x\}$, where $x = f((0, 0))$. Note that $\mathbb{R}^2 - \{(0, 0)\}$ is connected. Also, $\mathbb{R} - \{x\}$ is disconnected. Therefore, we reach a contradiction. So, the assumption which we have taken is not correct, and hence $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.

Moving ahead, in order to discuss the connectedness of subsets of $\mathbb{R}^2$, we will use two concepts. The first one is that if we have a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ and the topological space $(X, \mathcal{T})$ is connected, then $f(X)$ is connected. The second concept we will use is from the product space. If we are having two functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$, construct a function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(x) = (f_1(x), f_2(x))$, for all $x \in \mathbb{R}$. Then, f is continuous if and only if f_1 and f_2 are continuous. With these concepts in mind, let us discuss some of the examples.

Our first example is the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, which is a subset of $\mathbb{R}^2$, with the standard topology as a subspace of $\mathbb{R}^2$ is connected. In order to discuss the connectedness of S^1, let us have a continuous function $f : [0, 2\pi) \to S^1$ such that $f(\theta) = p_\theta$, for all $\theta \in [0, 2\pi)$. The question is, what p_θ is? It is given as in the Figure 1. We can justify that the function f is continuous. In order to show that f is continuous, what exactly will we do? We will try to show that the inverse image of elements of basis $\mathcal{B}$ for the topology on S^1 is open subsets of the interval $[0, 2\pi)$. The question is, what is the basis element for the topology on S^1? That will be a collection of the points, that is, p_θ, where $a < \theta < b$. Note that a and b are real numbers.

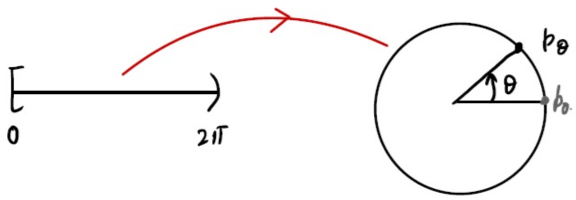

Figure 1: For description of p_θ.

We have already seen it because the topology on the circle is induced by the Euclidean topology on $\mathbb{R}^2$. So, that will look like an arc on this circle. Now, let us take a point p_0, which is also a member of S^1.

Now, we are going to take two cases. Let us take any $B \in \mathcal{B}$. It may happen that $p_0 \notin B$, or $p_0 \in B$. Let us compute the inverse image of this B, when $p_0 \notin B$. So, $f^{-1}(B) = (c, d), c, d \in [0, 2\pi)$. Note that the interval (c, d) is an open subset of $[0, 2\pi)$. Now, in the case this $p_0 \in B$, what $f^{-1}(B)$ is? Actually, this will be the union of two semi-open intervals. This will look like $[0, e) \cup (g, 2\pi)$, which is also open in $[0, 2\pi)$. So, what have we shown? We have shown that in any case, $f^{-1}(B)$ is open in $[0, 2\pi)$. Therefore, f is continuous. Also, from the definition, it is clear that $f([0, 2\pi)) = S^1$. As we know, the interval $[0, 2\pi)$ is connected, and what is S^1? This is a continuous image of the interval $[0, 2\pi)$. Therefore, S^1 is also connected.

This is the first example of a subset of $\mathbb{R}^2$, which is connected. In the next lecture, too, we will continue to discuss some more examples.

These are the references.

That's all from this lecture. Thank you.