Course Name: Essentials of Topology
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Welcome to Lecture 40 on Essentials of Topology.

In this lecture, we will continue the study of the theory of connected topological spaces. In the previous lecture, we have studied the connectedness of the set of real numbers and its subsets. In this lecture, we will use these concepts to discuss the nonexistence of non-constant continuous functions and non-homeomorphic spaces.

The first concept that we will discuss is the nonexistence of non-constant continuous functions because we already know that constant functions are continuous. In order to discuss this theory, we already have studied that if we have a continuous function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, and if the space $(X, \mathcal{T})$ is connected, then its image, that is, $f(X)$ is also connected. Now, let us take a function $f : (\mathbb{R}, \mathcal{T}_e) \to (Y, \mathcal{T}')$. Note that $(\mathbb{R}, \mathcal{T}_e)$ is connected, and the subsets of $\mathbb{R}$ which are connected are intervals. If we construct a continuous function $f : (\mathbb{R}, \mathcal{T}_e) \to (Y, \mathcal{T}')$, what features should this function f have? This function should map the intervals to connected subsets of Y; that is, if we are taking an interval I, $f(I) \subseteq Y$ should always be connected. In case $f(I)$ is disconnected, the function f cannot be continuous.

With this idea, begin with this example: that any continuous function $f : \mathbb{R} \to \mathbb{Q}$ must be a constant function. It means that we cannot construct any non-constant continuous function $f : \mathbb{R} \to \mathbb{Q}$. Whenever we are talking about $\mathbb{R}$, note that $\mathbb{R}$ is with the Euclidean topology, and $\mathbb{Q}$ is a subspace of $\mathbb{R}$. Now, note that constant maps are always continuous. So, there is no problem regarding the statement. But the question is, we cannot construct any other map; why? If we take a non-constant map $f : R \to Q$, what about $f(\mathbb{R})$? Obviously, $f(\mathbb{R}) \subseteq \mathbb{Q}$ and $|f(\mathbb{R})| \geq 2$. Recall the fact that $\mathbb{Q}$ is totally disconnected, and if this is totally disconnected, only $\{x\}$, $x \in \mathbb{Q}$ are connected. It means that $f(\mathbb{R})$ is disconnected, while we have shown that $\mathbb{R}$ is connected. So, what is happening here? We have assumed a non-constant function f; if this is continuous, the image of $\mathbb{R}$ under f should also be connected. But note that if the function is not constant, the image of $\mathbb{R}$ under f is disconnected. Thus, we cannot construct a non-constant continuous function from the set of real numbers to the set of rational numbers. This concept can be extended to other topological spaces. For example, the question is, can we construct a non-constant continuous function from the real line with Euclidean topology to the real line with lower limit topology? The answer is no. We cannot construct any such continuous function. Why? We have already seen that the real line with lower limit topology is totally disconnected. Even if we are talking about $f : (\mathbb{R}, \mathcal{T}_e) \to (\mathbb{R}, \mathcal{T})$, where $\mathcal{T}$ is discrete, the question is whether f is a non-constant continuous function. The answer is no. Why? Because $(\mathbb{R}, \mathcal{T})$ is totally disconnected.

Moving ahead, we have two questions with us. The first is, how to check that the given topological spaces are homeomorphic. It can be answered easily. What we have to do is that if we can construct a homeomorphism from a topological space $(X, \mathcal{T})$ to another topological space $(Y, \mathcal{T}')$, we say that the topological space $(X, \mathcal{T})$ is homeomorphic to the topological space $(Y, \mathcal{T}')$. The second question is, how can we check that the given topological spaces are not homeomorphic? It is not easy to justify that any homeomorphism does not exist from one space to another. The question is, in that case, what to do? The connectedness helps us at some point. How does connectedness help? Let us see an example. Here, we are taking the set $X = [0, 3]$ and $Y = [0, 1] \cup [3, 4]$. These are subspaces of $(\mathbb{R}, \mathcal{T}_e)$. It is to be noted that X is connected. What about Y? This is not connected. Why? Because this is not an interval. So, if there is any homeomorphism from X to Y, Y should be connected. But it is not, in this case. Therefore, we cannot construct any homeomorphism from X to Y.

Having this idea in mind, let us discuss some of the non-homeomorphic topological spaces. But before studying them, let us take a simple result. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a homeomorphism, $a \in X$, $X' = X - \{a\}$ and $Y' = Y - \{f(a)\}$. Then $(X', \mathcal{T}_{X'}) \cong (Y', \mathcal{T}'_{Y'})$. In order to justify it, let $g : (X', \mathcal{T}_{X'}) \to (Y', \mathcal{T}'_{Y'})$ be a function such that $g(x) = f(x)$, for all $x \in X'$. Note that this g is bijective, as f is bijective. Even we can say that the function g is nothing but the restriction of f over X'; that is, $g = f|_{X'}$ is a restricted function. So, g is bijective. We know that restricted functions are continuous, so g is also continuous; g^{-1} is continuous, too, because that can be visualized as a restricted function, that is, $g^{-1} = f^{-1}|_{Y'}$. Therefore, g is a homeomorphism, and hence, $(X', \mathcal{T}_{X'}) \cong (Y', \mathcal{T}'_{Y'})$.

Now, let us use this result to show that some topological spaces are not homeomorphic; that is, no homeomorphism exists between these spaces. We are taking examples of intervals. Let us begin with the first one. We want to show that the open interval (a, b) is not homeomorphic to the semi-open interval $[c, d)$. In order to justify it, let us begin with, if possible, assuming that (a, b) is homeomorphic to $[c, d)$. If it is, then there exists a homeomorphism, $f : [c, d) \to (a, b)$. Now, if f is a homeomorphism, then from the previous result, what we can conclude is that if we are removing the singleton set $\{c\}$ from the interval $[c, d)$, this should also be homeomorphic to the open interval $(a, b) - \{f(c)\}$. Note that this $f(c) \in (a, b)$. Now, $[c, d) - \{c\} = (c, d)$, and $(a, b) - \{f(c)\} = (a, b) - \{y\}$, where $y = f(c)$. Note that (c, d) is connected because this is an interval, but $(a, b) - \{y\}$ is disconnected; this is a contradiction. Therefore, (a, b) cannot be homeomorphic to $[c, d)$.

Even we can show that (a, b) is not homeomorphic to $[c, d]$, where these a, b, c, and d are real numbers with $a < b$, and $c < d$. Similar to the previous one, that is how we have shown that (a, b) is not homeomorphic to $[c, d)$, if we assume that (a, b) is homeomorphic to $[c, d]$. Then, there exists a homeomorphism, say $f : [c, d] \to (a, b)$. Now, if we are removing c from $[c, d]$, then $[c, d] - \{c\}$ should be homeomorphic to $(a, b) - \{f(c)\}$, where $f(c) \in (a, b)$. Note that this $[c, d] - \{c\}$ is nothing but a semi-open interval $(c, d]$, which is homeomorphic to $(a, b) - \{y\}$, where $y = f(c)$. Again, note that $(c, d]$ is connected. What about $(a, b) - \{y\}$? This is disconnected. Why? Because this is not an interval. Therefore, we reach a contradiction again. Thus, (a, b) is not homeomorphic to $[c, d]$.

Finally, let us show that $[a, b)$ is not homeomorphic to $[c, d]$. Begin with, if possible, let us assume that $[a, b)$ is homeomorphic to $[c, d]$. It means that there exists a homeomorphism, say $f : [c, d] \to [a, b)$. Therefore, from the result that we have already discussed if we are removing the singleton set $\{c\}$ from $[c, d]$, $(c, d]$ will be homeomorphic to $[a, b) - \{f(c)\}$, where $f(c) \in [a, b)$, that is $(c, d]$ is homeomorphic to $[a, b) - \{x\}$, where $x = f(c) \in [a, b)$. Now, if $(c, d]$ is homeomorphic to $[a, b) - \{x\}$, it means that there exists a homeomorphism; let us take that homeomorphism as $g : (c, d] \to [a, b) - \{x\}$. Again, what can we conclude that if we remove the singleton set $\{d\}$ from $(c, d]$, (c, d) should be homeomorphic to $((a, b) - \{x\}) - \{g(d)\}$, or (c, d) is homeomorphic to $[a, b) - \{x, y\}$, where $y = g(d)$. It is to be noted here that $x \neq y$. Now, (c, d) is connected, but what about $[a, b) - \{x, y\}$? This will always be disconnected. Therefore, we reach a contradiction. Hence, $[a, b)$ is not homeomorphic to $[c, d]$.

These are the references.

That's all from this lecture. Thank you.